PROFESSOR: OK, earlier lecture introduced the logarithm as the inverse function to the exponential. And now it's time to do calculus, find its derivative. And we spoke about other inverse functions, and here is an important one: the inverse sine function, or sometimes called the arc sine. We'll find its derivative, too.

OK, so what we're doing really is sort of completing the list of important rules for derivatives. We know about the derivative of a sum. Just add the derivatives. f minus g, just subtract the derivatives. We know the product rule and the quotient rule, so that's add, subtract, multiply, divide functions.

And then very, very important is the chain of functions, the chain rule. This is never to be mixed up with that. You wouldn't do such a thing. And now we're adding one more f inverse. That's today, the derivative of the inverse. That will really complete the rules. Out of simple functions like exponential, sine and cosine, powers of x, this creates all the rest of the functions that we typically use.

OK, so let's start with the most important of all: the f of x is e to the x. And then the inverse function, we named the natural logarithm. And notice, remember how I reversed the letters. Here, x is the input and y is the output from the exponential function. So for the inverse function, y is the input. I go backwards to x, and the thing to remember, the one thing to remember, just tell yourself x is the exponent. The logarithm is the exponent.

OK, so a chain of functions is coming here, and it's a perfectly terrific chain. This is the rule for inverse functions. If I start with an x and I do f of x, that gets me to y. And now I do the inverse function, and it brings me back to x. So I really have a chain of functions. The chain has a very special result. And our situation is if we know how to take the derivative of f, this ought to tell us-- the chain rule-- how to take the derivative of the inverse function, f inverse. Let me try it with this all-important example. So which-- and notice also, chain goes the other way. If I start with y, do the inverse function, then I've reached x. Then f of x is y.

Maybe before I take e to the x, let me take the function that always is the starting point. So practice is-- I'll call this example of-- just to remember how inverse functions work-- linear functions. y equals ax plus b. That's f of x there. Linear. What's the inverse of that? Now the point about inverses is I want to solve for x in terms of y. I want to get x by itself. So I move b to the opposite side. So in the end, I want to get x equals something.

And how do I do that? I move b to the opposite side, and then I still have ax, so I divide by a. Then I've got x by itself. This is the inverse. This is f inverse of y.

Notice something about inverse functions, that here we did-- this function f of x was created in two steps. it was sort of a chain in itself. The first step was multiply by a. We multiply and then we add. What's inverse function do?

The inverse function takes y. Subtracts b. So it does subtract first to get y minus b, and then it divides.

What's my point? My point is that if a function is built in two steps, multiply and then add in this nice case, the inverse function does the inverse steps. Instead of multiplying it divides. Instead of add, it subtracts. What it does though is the opposite order. Notice the multiply was done first, then the divide is last. The add was done second, the subtract was done first.

When you invert things, the order-- well, you know it. It has to be that way. It's that way in life, right? If we're standing on the beach and we walk to the water and then we swim to the dock, so that's our function f of x from where we were to the dock, then how do we get back? Well, wise to swim first, right? You don't want to walk first. From the dock, you swim back. So you swam one way at the end, and then in the inverse, you swim-- oh, you get it.

OK, so now I'm ready for the real thing. I'll take one of those chains and take its derivative. Let me take the first one. So the first one says that the log of e to the x is x. That's-- The logarithm was defined that way. That's what the log is.

Let's take the derivative of that equation, the derivative of both sides. We will be learning what's the derivative of the log. That's what we don't know. So if I take the derivative, well, on the right-hand side, I certainly get 1. On the left-hand side, this is where it's interesting. So it's the chain rule. The log of something, so I should take the derivative-- oh, I need a little more space here, over here.

The derivative of log y, y is e to the x. Everybody's got that, right? So this is log y, and I'm taking its derivative with respect to y. But then I have to, as the chain rule tells me, take the derivative of what's inside. What's inside is e to the x, so I have dy dx. So I put dy dx. And the derivative on the right-hand side, the neat point here is that the x-derivative of that is 1.

OK, now I'm going to learn what this is because I know what this is: the derivative of dy dx, the derivative of the exponential. Well, now comes the most important property that we use to construct this exponential. dy dx is e to the x. No problem, OK? And now, I'm going to divide by it to get what I want-- almost. Almost, I say. Well, I've got the derivative of log y here. Correct, but there's a step to take still.

I have to write-- I want a function of y. The log is a function of y. It's derivative is-- the answer is a function of y. So I have to go back from x to y. But that's simple. e to the x is y. Oh! Look at this fantastic answer. The derivative of the log, the thing we wanted, is 1 over y.

Why do I say fantastic? Because out of the blue almost, we've discovered the function log y, which has that derivative 1 over y. And the point is this is the minus 1 power. It's the only power that we didn't produce earlier as a derivative. I have to make that point.

You remember the very first derivatives we knew were the derivatives of x to the n-th, powers of x. Everybody knows that that's n times x to the n minus 1. The derivative of every power is one power below. With one exception. With one exception. If n is 0, so I have to put except n equals 0.

Well, it's true when n is 0, so I don't mean the formula doesn't fail. What fails is when n is 0, this right-hand side is 0, and I don't get the minus 1 power. No power of x produces the minus 1 power when I take the derivative. So that was like an open hole in the list of derivatives. Nobody was giving the derivative to be the minus 1 power when we were looking at the powers of x.

Well, here it showed up. Now, you'll say the letter y's there. OK, that's the 25th letter of the alphabet. I'm perfectly happy if you prefer the 26th letter. You can write d log z dz equals 1/z if you want to. You can write, as you might like to, d by dx. Use the 24th letter of log x is 1/x. I'm perfectly OK for you to do that, to write the x there, now after we got the formula.

Up to this point, I really had to keep x and y straight because I was beginning from y is e to the x. That was my starting point. OK, so that keeping them straight got me the derivative of log y as 1/y. End. Now, I'm totally happy if you use any other letter. Use t if you have things growing.

And remember about the logarithm now. We can see why it grows so slowly. Because its slope is 1/y. Or let's look at this one, because we're used to thinking of graphs with x along the axis. And this is telling us that the slope of the log curve-- the log curve is increasing, but the slope is decreasing, getting smaller and smaller. As x gets very small, it's just barely increasing. It does keep going on up to infinity, but very, very slowly. And why is that? That's because the exponential is going very, very quickly. And you remember that the one graph is just the flip of the other graph, so if one is climbing like mad, the other one is growing slowly.

OK, that's the main facts, the most important formula of today's lecture. I could-- do you feel like practice to take the chain in the opposite direction just to see what would happen? So what's the opposite direction?

I guess the opposite direction is to start with-- which did I start with? I started with log of e to the x is x. The opposite direction would be to start with e to the log y is y, right? That's the same chain. That's the f inverse coming before the f. What do I do? Take derivatives. Take the derivative of everything, OK?

So take the derivative, the y-derivative. I get the nice thing. I mean, that's the fun part, taking the derivative on the right-hand side. On the left side, a little more work, but I know how to take the derivative of e to the something. It's the chain rule. Of course it's the chain rule. We got a chain here.

So the derivative of e to the something, now you remember with the chain rule, is e to that same something times the derivative of what's inside. The derivative and what's inside is this guy: the derivative of log y dy. This is what we want to know, the one we know, and what is e to the log y? It's sitting up there on the line before. e to the log y is y. So this parenthesis is just containing y. Bring it down. Set it under there, and you have it again. The derivative of log y dy is 1 over e to the log y, which is y.

OK, we sort of have done more about inverse functions than typical lectures might, but I did it really because they're kind of not so simple. And yet, they're crucially important in this situation of connecting exponential with log. And by the way, I prefer to start with exponential. The logic goes also just fine. In fact, some steps are a little smoother if you start with a logarithm function, define that somehow, and then take its inverse, which will be the exponential. But for me, the exponential is so all important. The logarithm is important, but it's not in the league of e to the x. So I prefer to do it this way to know e to the x.

Now if you bear with me, I'll do the other derivative for today. The other derivative is this one. Can we do that? OK, so I want the derivative of this arc sine function, all right?

So I'm going to-- let me bring that. This side of the board is now going to be x is the inverse sine of y, or it's often called the arc sine of y. OK, good. All right. So again, I have a chain. I start with x. I create y. So y is sine x. So y is the sine of x, but x is the arc sine of y. That's the chain. Start with a y. Do f inverse. Do f, and you got y again, all right?

Now, I'm interested in the derivative, the derivative of this arc sine of y. I want the y-derivative. I'm just going to copy this plan, but instead of e, I've got sines here. So take the y-derivative of both sides, the y-derivative of both sides. Well, I always like that one. The y-derivative of this is the chain rule. So I have the sine of some inside function. So the derivative is the cosine of that inside function times the derivative of the inside function, which is the guy we want. OK, so I have to figure out that thing. In other words, I guess I've got to think a little bit about these inverse trig functions.

OK, so what's the story with the inverse trig functions? The point will be this is an angle. Ha! That's an angle. Let me draw the triangle. Here is my angle theta. Here is my sine theta. Here is my cos theta, and everybody knows that now the hypotenuse is 1. So here is theta. OK, whoa! Wait a minute. I would love theta to be the angle whose-- oh, maybe it is. This is the angle whose sine-- theta should be the angle whose sine is y, right? OK, theta is the angle whose sine is y. OK, let me make that happen.

And now, tell me the other side because I got to get a cosine in here somewhere. What is this side? Back to Pythagoras, the most important fact about a right triangle. This side will be the square root of-- this squared plus this squared is 1, so this is the square root of 1 minus y squared. And that's the cosine. The cosine of this angle theta is this guy divided by 1. We're there, and all I've used pretty quickly was I popped up a triangle there. I named an angle theta. I took its sine to be y, and I figured out what its cosine had to be. OK, so there's the theta. Its cosine has to be this, and now I'm ready to write out the answer. I'm ready to write down the answer there.

That has a 1 equals-- the cosine of theta, that's this times the derivative of the inverse sine. You see, I had to get this expression into something-- I had to solve it for y. I had to figure out what that quantity is as a function of y. And now I just put this down below. So if I cross this out and put it down here, I've got the answer. There is the derivative of the arc sine function: 1 over the square root of 1 minus y squared.

OK, it's not as beautiful as 1/y, but it shows up in a lot of problems. As we said earlier, sines and cosines are involved with repeated motion, going around a circle, going up and down, going across and back, in and out. And it will turn out that this quantity, which is really coming from the Pythagoras, is going to turn up, and we'll need to know that it's the derivative of the arc sine.

And may I just write down what's the derivative of the arc cosine as long as we're at it? And then I'm done. The derivative of the arc cosine, well, you remember what-- what's the difference between sines and cosines when we take derivatives? The cosine has a minus. So there'll be a minus 1 over the square root of 1 minus y squared.

That's sort of unexpected. This function has this derivative. This function has the same derivative but with a minus sign. That suggests that somehow if I add those, yeah, let's just think about that for the last minute here. That says that if I add sine inverse y to cosine inverse y, their derivatives will cancel. So the derivative of that sum of this one-- can I do a giant plus sign there?-- is 0. The derivative of that plus the derivative of that is a plus thing and a minus thing, giving 0.

So how could that be? Have you ever thought about what functions have derivative 0? Well, actually, you have. You know what functions have no slope. Constant functions. So I'm saying that it must happen that the arc sine function plus the arc cosine function is a constant. Then its derivative is 0, and we are happy with our formulas.

And actually, that's true. The arc sine function gives me this angle. The arc cosine function would give me-- shall I give that angle another name like alpha? This one would be the theta. That one would be the alpha. And do you believe that in that triangle theta plus alpha is a constant and therefore has derivative 0? In fact, yes, you know what it is. Theta plus alpha in a right triangle, if I add that angle and that angle, I get 90 degrees. A constant. Well, 90 degrees, but I shouldn't allow myself to write that. I must write it in radians. A constant.

OK, don't forget the great result from today. We filled in the one power that was missing, and we're ready to go. Thank you.

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