Calculus Revisited
Part 1
A Self-Study Course

Study Guide

Block V
Transcendental Functions

Block VI
More Integration Techniques

Block VII
Infinite Series

Center for Advanced Engineering Study

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CALCULUS REVISITED
PART 1

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STUDY GUIDE: Calculus of a Single Variable

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Solutions

BLOCK V: TRANSCENDENTAL FUNCTIONS

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PRETEST

1. Perform the indicated operations:

   a. $\int \frac{\ln^3 x}{x}\,dx$ (x>0)

   b. $d[\ln(\ln x)]/dx$ (x>1)

   c. $\lim_{n\to\infty} (1 + \frac{2}{n})^n$

2. Find y as a function of x if $y'' - 2y' - 15y = 0$ and $y(0) = 8$ while $y'(0) = 16$.

3. Find the volume generated when R is revolved about the y-axis if R is the region bounded above by $y = e^{-x^2}$, below by the x-axis, on the left by the y-axis, and on the right by the line $x = 3$.

4. Determine sinh x if $\cosh x = \frac{13}{12}$.

5. Simplify $\frac{1}{2}\ln\left[\frac{1 + \tanh x}{1 - \tanh x}\right]$

6. Find $\frac{dy}{dx}$ if $y = \sinh^{-1}(\tan x)$

7. Find the equation of the line which is tangent to the curve $x = \cosh y$ at the point $[2, \ln(2 - \sqrt{3})]$.

STUDY GUIDE: Calculus of a Single Variable - Block V: Transcendental Functions

UNIT 1: Logarithms Revisited

1. View: Lecture 5.010 "Logarithms Without Exponents".
2. Read: Supplementary Notes, Chapter IX.
3. Read: Thomas 7.4, 7.5, 7.6, 7.7 .
4. Exercises:

5.1.1(L)

a. Use the mean value theorem for ln x on [5,6] to prove that $\frac{1}{6} < \ln \frac{6}{5} < \frac{1}{5}$.

b. Use the inequality in (a) to conclude that:

$$\left(\frac{6}{5}\right)^5 < e < \left(\frac{6}{5}\right)^6$$

c. Using the definition that $\ln t = \int_1^t \frac{dx}{x}$, compute ln 1.2 approximately by trapezoidal approximation with n = 2. Discuss the bounds on the size of the error.

5.1.2(L)

a. Show that $\ln(x^n) = n \ln x$ for any <u>rational number</u>, n.

b. Use logarithmic differentiation to compute $\frac{dy}{dx}$ if $y = (x^2 + 1)^5 \sqrt{x^4 + 2}\ \sqrt[4]{x^4 + 2x^2 + 3}$.

c. Assuming that $\ln x^n = n \ln x$ is true for all <u>real</u> numbers n, use logarithmic differentiation to find $\frac{dy}{dx}$ if $y = x^x$ $(x>0)$

d. Under the same assumption as in (c), show that $\frac{d(x^n)}{dx} = nx^{n-1}$ for any real number, n.

5.1.3 Perform the indicated operations

a. $\dfrac{d(\ln\sqrt{x^2+1})}{dx}$ b. $\dfrac{d(\ln x^2\sqrt{x^2+1})}{dx}$

c. $\displaystyle\int\frac{(\ln x)^3\,dx}{x}$ $(x>0)$ d. $\displaystyle\int\frac{\cos x\,dx}{1+\sin x}$

e. $\dfrac{d[\ln(\ln x)]}{dx}$ f. $\displaystyle\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$

5.1.4

a. Find the area of region R if R is bounded above by $y = \dfrac{x}{x^2+1}$, below by the x-axis, on the left by the y-axis and on the right by the line $x = 1$.

b. Let R be the region which is bounded above by $y = \dfrac{x}{\sqrt{x^3+1}}$, below by the x-axis, on the left by the y-axis, and on the right by $x = 2$. Find the volume generated when R is rotated about the x-axis.

5.1.5(L)

a. Show that $\ln x = \displaystyle\int_0^{x-1}\frac{du}{1+u}$ (where $x>1$)

b. Use long division to show that

$$\frac{1}{1+u} = 1 - u + u^2 - \frac{u^3}{1+u}$$

c. Combine (a) and (b) to show:

$$\ln x = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \int_0^{x-1}\frac{u^3\,du}{1+u}$$

[5.1.5(L)(cont'd)]

d. Show that $\left|\int_0^{x-1} \frac{u^3 du}{1 + u}\right| < \frac{1}{4}(x - 1)^4$

e. Combine (c) and (d) to find a value for ln 1.2 that is accurate to three decimal places.

5.1.6 Mimic the procedure of the previous exercise to find a value for ln 1.2 which is accurate to the 6th decimal place.

5.1.7(L)

a. By comparing $\int_1^x \frac{dt}{t}$ with $\int_1^x \frac{dt}{\sqrt{t}}$,

prove that:

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0$$

b. From (a) deduce that $\lim_{x\to\infty} \frac{\ln(x^n)}{x} = 0$ for any constant, n.

c. Use (b) to show that $\lim_{u\to\infty} \frac{\ln u}{\sqrt[n]{u}} = 0$.

5.1.8

a. Sketch the curve $y = \frac{\ln x}{x}$ $(x>0)$

b. If $\frac{\ln x}{x} = \frac{\ln 2}{2}$ does it follow that $x = 2$? Explain.

c. If $\frac{\ln x}{x} = \frac{\ln \frac{1}{2}}{\frac{1}{2}} = -2 \ln 2$, does it follow that $x = \frac{1}{2}$? Explain.

UNIT 2: The Exponential Function

1. View: Lecture 5.020

2. Read: Thomas 7.8, 7.9, 7.10

3. Exercises:

5.2.1 Perform the indicated operations

a. $\dfrac{d\left(e^{\frac{1}{x}}\right)}{dx}$ b. $\dfrac{d}{dx}\left[\ln \dfrac{e^x}{1 + e^x}\right]$

c. $\displaystyle\int e^{\sin 2x} \cos 2x \, dx$

d. $\displaystyle\int \frac{e^x dx}{3 + 4e^x}$ e. $\displaystyle\int_{e^2}^{e^3} \frac{dx}{x \ln x}$

5.2.2 Find the volume generated when R is revolved about the y-axis where R is the region bounded above by $y = e^{-x^2}$, below by the x-axis, on the left by the y-axis, and on the right by $x = 3$.

5.2.3 (L) Find y as a function of x if

$$y'' - 2y' - 15y = 0$$

and $y(0) = 8$, $y'(0) = 16$

5.2.4 Find y as a function of x if $y(0) = 1$, $y'(0) = 2$ and

$$y'' - 7y' + 12y = 0$$

5.2.5 (L) Find a number c such that if $x > c$ then $e^x > x^{100}$. (To help standardize our approach, utilize the information that $\ln 2 = 0.69315$.)

5.2.6 Find a number c such that $x > c$ implies $e^x > x^{1000}$ and show that the minimum c which works is between 8,192 and 16,384.

5.2.7 Assume that (1) $u = e^{\ln u}$ and (2) $\ln u^r = r \ln u$ for any real r,

a. Simplify $e^{2 \ln x}$

b. Let b denote any positive real number. Develop the formula for $\frac{db^x}{dx}$.

c. Evaluate $\int_0^1 4^x dx$

5.2.8 Compute $\lim_{h\to 0} \frac{1}{h} \int_3^{3+h} e^{-x^2} dx$

STUDY GUIDE: Calculus of a Single Variable - Block V: Transcendental Functions

UNIT 3: The Hyperbolic Functions

1. View: Lecture 5.030
2. Read: Thomas 8.1, 8.2, 8.3 (skim 8.4 if desired; otherwise omit.)
3. Exercises:

5.3.1

a. Determine cosh x if sinh x = $\frac{5}{12}$

b. Determine sinh 2x if sinh x = $\frac{5}{12}$

c. Determine sinh x if cosh x = $\frac{13}{12}$

5.3.2

a. Simplify $\frac{1}{2} \ln \left[\frac{1 + \tanh x}{1 - \tanh x}\right]$

b. Use the basic definitions $\cosh u = \frac{e^u + e^{-u}}{2}$ and $\sinh u = \frac{e^u - e^{-u}}{2}$ to perform the following integrations.

(i) $\int \frac{\cosh\theta \, d\theta}{\sinh\theta + \cosh\theta}$ (ii) $\int e^x \sinh 2x \, dx$

(iii) $\int \frac{e^{2x} - 1}{e^{2x} + 1} \, dx$

5.3.3 Find the arclength of the segment of the curve y = cosh x between x = 0 and x = 1.

5.3.4 Use the fact that $\int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right)$

[5.3.4 (cont'd)]

to evaluate $\lim_{n\to\infty}\left[\dfrac{\cosh\frac{1}{n} + \cosh\frac{2}{n} + \cdots + \cosh\frac{n}{n}}{n}\right]$

5.3.5 (L)

a. At what point does the line tangent to $x^2 - y^2 = 1$, $x > 0$, at (x_1,y_1) intersect the x-axis?

b. Use (a) to describe a way of constructing sech u for any real number u.

5.3.6 Find a construction for csch u by finding where the line in Exercise 5.3.5(L), part (a), intersects the y-axis.

5.3.7 (L)

a. Show that $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$ for any constant n.

b. Show that $(\cosh x - \sinh x)^n = \cosh nx - \sinh nx$.

c. Use the result of (a) and (b) to find identities for sinh 2x and cosh 2x.

5.3.8 Use the technique of Exercise 5.3.7(L) to find identities for sinh 3x and cosh 3x.

STUDY GUIDE: Calculus of a Single Variable - Block V: Transcendental Functions

UNIT 4: The Inverse Hyperbolic Functions

1. View: Lecture 5.040
2. Read: Thomas 8.5, 8.6
3. Exercises:

5.4.1(L) Find the equation of the line which is tangent to the curve $x = \cosh y$ at the point $(2, \ln(2 - \sqrt{3}))$.

5.4.2(L) Show that if $x = \cosh y$ and $y < 0$ then $y = \ln(x - \sqrt{x^2 - 1})$.

5.4.3 Prove that $\operatorname{sech}^{-1} x = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right)$ where $0 < x \leq 1$.

5.4.4(L)

a. Evaluate $\int_1^b \left(\frac{1}{\sqrt{x^2 - 1}} - \frac{1}{x}\right) dx$

b. Compute $\lim_{b\to\infty} \int_1^b \left(\frac{1}{\sqrt{x^2 - 1}} - \frac{1}{x}\right) dx$

5.4.5 Compute $\lim_{b\to\infty} \int_1^b \left(\frac{1}{\sqrt{1 + x^2}} - \frac{1}{x}\right) dx$

5.4.6

a. Simplify $e^{\sinh^{-1} x} - \sqrt{x^2 + 1}$

b. Find $\frac{dy}{dx}$ if $y = \sinh^{-1}(\tan x)$

[5.4.6 (cont'd)]

c. Determine f(x) if $f'(x) = \dfrac{e^x}{\sqrt{1 + e^{2x}}}$

5.4.7

a. Sketch the curve $y = \dfrac{1}{1 - x^2}$

b. Let R denote the region bounded above by $y = \dfrac{1}{1 - x^2}$, below by the x-axis, on the left by the y-axis, and on the right by the line $x = \frac{1}{2}$. Find the area of R.

c. With R as in part (b), find the volume generated when R is revolved about the y-axis.

QUIZ

5.Q.1 Find $\frac{dy}{dx}$ if:

a. $y = \ln^2(x^3 + 1)$ b. $y = \tanh^{-1}(\sin x)$ c. $y = x^{\sin x}$, $x > 0$

5.Q.2 Integrate each of the following:

a. $\int \frac{\sec^2 x\, dx}{1 + \tan x}$ b. $\int \frac{\sinh \sqrt{x}\, dx}{\sqrt{x}}$

c. $\int_0^{\frac{1}{2}} \frac{dx}{1 - x^2}$

5.Q.3 Let R be the region which is bounded above by $y = \frac{4x^2}{x^4 + 1}$, below by the x-axis, on the left by the y-axis, and on the right by the line $x = 1$. Find the volume generated when R is rotated above the y-axis.

5.Q.4 Find y as a function of x if it is known that (i) $y'' - 8y' + 7y = 0$, and (ii) $y(0) = 0$ while $y'(0) = 12$.

5.Q.5 Use the fact that $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$ to prove that:

$$\cosh 4x = \cosh^4 x + 6 \cosh^2 x \sinh^2 x + \sinh^4 x$$

BLOCK VI: MORE INTEGRATION TECHNIQUES

Content

PRETEST

1. Determine f(x) if $f'(x) = \dfrac{\cos x}{\sqrt{2 + \sin x}}\,dx$ and $f(0) = 3$.

2. Determine f(x) if $f'(x) = \dfrac{x^4}{x + 1}$ and $f(0) = 2$.

3. Evaluate $\displaystyle\int \frac{dx}{(x - 1)(x - 2)(x - 3)}$.

4. Compute $\displaystyle\int x^2 \cos x\,dx$.

5. Evaluate $\displaystyle\int_0^3 (x - 2)^{-4}\,dx$.

STUDY GUIDE: Calculus of a Single Variable - Block VI: More Integration Techniques

UNIT 1: A Review of Some Basic Inverse Derivatives

1. View: Lecture 6.010

2. Read: Thomas 9.1, 9.2, 9.3, 9.4, 9.5

3. Exercises:

6.1.1 Determine f(x) if:

a. $f'(x) = \dfrac{\cos x}{\sqrt{2 + \sin x}}$ and $f(0) = 3$.

b. $f'(x) = \tan^4 5x \sec^2 5x$ and $f(0) = 1$.

6.1.2 Determine f(x) if $f'(x) = \dfrac{1}{(4 + x^2)^2}$ and $f(0) = 1$.

6.1.3 Determine f(x) if $f'(x) = \dfrac{1}{4x^2 + 4x + 17}$ and $f(\frac{3}{2}) = 0$.

6.1.4

a. Compute $\displaystyle\int \frac{\cos x\, dx}{1 + \sin x}$.

b. Find the area of region R if R is bounded above by $y = \dfrac{\cos x}{1 + \sin x}$, below by the x-axis, on the left by the y-axis, and on the right by $x = \frac{\pi}{2}$.

c. A particle moves along the x-axis according to the rule $v = \dfrac{\cos t}{1 + \sin t}$ $0 \leqslant t \leqslant \pi$ (where t is measured in seconds and v in feet per second). What is the displacement of the particle during this time interval? How far (total distance) does it travel?

6.1.5

a. Compute $\displaystyle\int \sin^4 x\, dx$.

b. The arch of the curve $y = \sin^2 x$ between $x = 0$ and $x = \pi$ is rotated about the x=axis. Find the volume generated.

6.1.6 Let R be the region which is bounded above by $y = \frac{1}{x^2 - 2x + 17}$, below by the x-axis, on the left by x = 1 and on the right by x = 5. Find the area of R.

6.1.7 (L)

a. Determine f(x) if $f'(x) = \sqrt{1 + x^2}$ and f(0) = 0.

b. Check a. by differentiating your answer to a.

c. The region R is that which is bounded above by y = cos x, below by the x-axis, on the left by the y-axis, and on the right by $x = \frac{\pi}{2}$. Find the surface area generated when R is revolved about the x-axis.

UNIT 2: Partial Fractions

1. View: Lecture 6.020
2. Read: Thomas 9.6, 9.8
3. Exercise:

6.2.1 (L) Determine f(x) if $f'(x) = \frac{x^4}{x + 1}$ (x > -1) and f(0) = 2.

6.2.2 Compute $\int \frac{x^4\,dx}{(x - 1)(x - 2)}$.

6.2.3 (L) Evaluate $\int_4^5 \frac{dx}{(x - 1)(x - 2)(x - 3)}$

6.2.4 Determine A, B, and C if $\frac{2x + 1}{x(x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{x + 2}$.

6.2.5 (L) Compute

a. $\int \frac{dx}{x(x^2 + 1)}$

b. $\int \frac{dx}{x(x + 1)^2}$.

6.2.6 Compute

a. $\int \frac{x^4\,dx}{(x^2 + 1)^2}$

b. $\int \frac{dx}{(x^2 + 1)^2}$.

6.2.7 Compute

a. $\int \frac{\cos\theta\,d\theta}{\sin^2\theta + 7\sin\theta + 12}$

b. $\int \frac{dx}{e^x - 1}$.

6.2.8 Use the substitution $Z = \tan\frac{x}{2}$ to compute $\int \frac{dx}{1 + \sin x}$.

6.2.9 Compute $\int \sec x\,dx$ by transforming the integral into one which lends itself to the technique of partial fractions.

STUDY GUIDE: Calculus of a Single Variable - Block VI: More Integration Techniques

UNIT 3: Integration by Parts

1. View: Lecture 6.030

2. Read: Thomas 9.7

3. Exercises:

6.3.1 (L) Perform the indicated integrations:

a. $\int x^2 \cos x \, dx$

b. $\int \sin \sqrt{x} \, dx$

c. $\int e^x \sin x \, dx$.

6.3.2 Compute $\int \sin(\ln x) \, dx$.

6.3.3 R is the region bounded above by $y = \cos^2 x$, below by the x-axis, on the left by the y-axis, and on the right by the line $x = \frac{\pi}{2}$. Find the volume generated when R is rotated about the y-axis.

6.3.4

a. Use integration by parts to show $\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$.

b. Use a. to compute $\int x^3 e^x \, dx$.

c. R is the region bounded above by $y = x^3 e^x$, below by the x-axis, on the left by the y-axis and on the right by the line $x = 1$. Find the area of R .

6.3.5 Let R be as in Exercise 6.3.4 c. Find the volume generated when R is rotated about the y-axis.

6.3.6

a. Use parts to show $\int \ln^n x \, dx = x \ln^n x - n \int \ln^{n-1} x \, dx$.

b. Let R be the region bounded above by $y = \ln x$, below by the x-axis, and on the sides by the lines $x = 1$, $x = 2$.

[6.3.6 cont'd]

(1) Find the area of R.

(2) Find the volume generated when R is rotated about the x-axis.

(3) Find the volume generated when R is rotated about the y-axis.

UNIT 4: Improper Integrals

1. View: Lecture 6.040
2. Read: Thomas 9.10
3. Exercises:

6.4.1 Discuss the convergence (or divergence) of each of the following:

a. $\int_0^3 (x - 2)^{-4}\, dx$

b. $\int_0^1 (x - 2)^{-4}\, dx$

c. $\int_2^3 (x - 2)^{-\frac{1}{4}}\, dx$.

6.4.2 Show that $\int_1^\infty x^r\, dx$ converges if and only if $r < -1$.

6.4.3 Show that $\int_0^1 x^r\, dx$ converges if and only if $r > -1$.

6.4.4 (L) Show that $\int_1^\infty t^{50} e^{-t}\, dt$ converges, <u>but</u> do not evaluate the integral.

6.4.5 Decide whether $\int_1^\infty \frac{\ln x}{x^3}\, dx$ converges .

6.4.6 (L) Discuss the convergence of $\int_0^1 t^n e^{-t}\, dt$.

6.4.7 (L)

a. Show that $\int_0^\infty t^{x-1} e^{-t}\, dt$ converges if and only if $x > 0$. (This integral occurs in advanced analysis in several different contexts such as in complex variables, differential equations and probability. It is known as the Gamma Function and it is denoted by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$. It is beyond our scope to motivate $\Gamma(x)$ here but is does serve as an interesting example of an improper integral.)

b. Use integration by parts to show that if $x > 0$ then $\Gamma(x + 1) = x\Gamma(x)$.

c. Use b. to show how the Gamma Functions is an extension of the concept of factorials. In particular, show that $\Gamma(n) = (n - 1)!$ for any positive whole number.

6.4.8 (L) Show that $\lim_{n\to\infty} (1 + \frac{1}{4} + \ldots + \frac{1}{n^2}) \leq 2$ by making appropriate use of the convergent improper integral

$$\int_1^\infty \frac{dx}{x^2}.$$

6.4.9 By making appropriate use of the divergent improper integral $\int_1^\infty \frac{dx}{x}$, show that $\lim_{n\to\infty} (1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}) = \infty$.

QUIZ

Find f(x) in each of the exercises 1 through 6 .

1. $f'(x) = \dfrac{\cos 3x}{(1 + \sin 3x)^3}$ and $f(0) = 2$.

2. $f'(x) = \cos^4 x$ and $f(0) = 1$.

3. $f'(x) = \dfrac{x^2 - 1}{x^3(x - 2)}$ and $f(1) = \dfrac{7}{2}$.

4. $f'(x) = \dfrac{\cos x}{\sin^2 x - 5 \sin x + 6}$ and $f(0) = \ln \dfrac{3}{2}$.

5. $f'(x) = x^3 \cos x$ and $f(0) = 3$.

6. $f'(x) = \cos \sqrt{x}$ and $f(0) = 3$.

7. Evaluate each of the following:

   a. $\displaystyle\int_0^5 (x - 3)^{-2} dx$

   b. $\displaystyle\int_0^2 (x - 3)^{-2} dx$

   c. $\displaystyle\int_3^4 (x - 3)^{-\frac{1}{2}} dx$.

BLOCK VII: INFINITE SERIES

Content

PRETEST

1. Which of the following series converge and which diverge? In each case, explain your choice.

   (a) $\sum_{n=2}^{\infty} \frac{2n+7}{5n-6}$ (b) $\sum_{n=1}^{\infty} \frac{(1000)^n}{n!}$ (c) $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$

2. For what values of c does the series $\sum_{n=0}^{\infty} (-2)^n(n+1)(c-1)^n$ converge absolutely.

3. Use the power series which represents e^x to compute $\frac{1}{e}$ correct to three decimal places.

4. For what values of x does $\sum_{n=0}^{\infty} (5x)^n$ converge uniformly?

5. Use series to compute $\int_0^{\frac{1}{2}} xe^{-x^3}dx$ correct to four decimal places.

STUDY GUIDE: Calculus of a Single Variable - Block VII: Infinite Series

UNIT 1: Sequences and Series

1. View: Lecture 7.010

2. Read: Supplementary Notes, Chapter X, Sections A, B, C, and D.

3. Read: Thomas 18.1

4. Exercises:

7.1.1(L)

a. Suppose $\{a_n\}$ converges (i.e. $\lim_{n\to\infty} a_n = L$). Show that there exist numbers m and M such that for every n, $m \leqslant a_n \leqslant M$.

b. Define the sequence $\{a_n\}$ by $a_n = (-1)^{n+1}$. Show that $\{a_n\}$ diverges.

c. Use (b) to show that the converse of (a) is false. That is, show that $\{a_n\}$ may diverge even though there exist numbers m and M such that $m \leqslant a_n \leqslant M$ for every n.

7.1.2(L) Prove that if $\lim_{n\to\infty} a_n = L_1$ and if $\lim_{n\to\infty} a_n = L_2$ then $L_1 = L_2$.

7.1.3(L)

a. Show that if $\{a_n\}$converges so also does $\{|a_n|\}$.

b. Does $\{a_n\}$ converge if $\{|a_n|\}$ does? Explain.

c. Suppose $\lim_{n\to\infty} |a_n| = 0$. What can we conclude about $\lim_{n\to\infty} a_n$?

7.1.4 Determine whether $\{a_n\}$converges, and, if it does, determine the limit.

[7.1.4 cont'd]

a. $a_n = \frac{2n+7}{5n-6}$ b. $a_n = \frac{2n+7}{5n^2-6}$ c. $a_n = \frac{2n^2+7}{5n-6}$

d. $a_n = 1 + (-1)^n$ e. $\frac{1 + (-1)^n}{n}$

7.1.5(L) Show that each of the following series diverges.

a. $\sum_{n=1}^{\infty} \frac{2n+7}{5n-6}$ b. $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

7.1.6 Tell which of the following series converge and which diverge. In the event that a series converges compute its sum.

a. $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ b. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ c. $\sum_{n=1}^{\infty} (-1)^{n+1}$

7.1.7(L) Find the common fraction whose decimal representation is given by $0.513513\overline{513}\ldots$ (where 513 repeats endlessly).

7.1.8 Find the common fraction whose decimal representation is given by $0.5131\overline{3}13\ldots$ (where 13 repeats endlessly).

UNIT 2: Positive Series

1. View: Lecture 7.020

2. Read: Thomas 18.2

3. Exercises:

7.2.1 (L) Determine which of the following series converge and which diverge.

a. $\sum_{n=1}^{\infty} \frac{(1000)^n}{n!}$ b. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ c. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

d. $\sum_{n=1}^{\infty} \frac{1}{n^{1.000001}}$ e. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$.

7.2.2 Test each of the following series for convergence:

a. $\sum_{n=1}^{\infty} \frac{2n+7}{3n+2}$ b. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

c. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ d. $\sum_{n=0}^{\infty} \frac{(n+2)!}{n!\ 3^n}$

7.2.3 (L)

a. Suppose that $\sum_{n=1}^{\infty} a_n$ is a positive convergent series.

Show that $\sum_{n=1}^{\infty} a_n^{\ 2}$ is also convergent.

b. On the other hand show that it is possible that $\sum_{n=1}^{\infty} a_n$ diverges even thought $\sum_{n=1}^{\infty} a_n^{\ 2}$ converges.

7.2.4 (L)

a. Show that for any positive integer n,

$$\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \geqslant \frac{1}{2} \quad .$$

b. Use a. to prove the following:

Suppose $\{a_n\}$ is a positive non-increasing sequence (that is, $a_1 \geqslant a_2 \geqslant a_3 \geqslant \ldots \geqslant a_{n-1} \geqslant a_n \geqslant a_{n+1} \geqslant \ldots$) and that $a_2 + a_4 + a_8 + a_{16} + \ldots + a_{2^n} + \ldots$ diverges. Then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ $(= a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \ldots)$ also diverges.

c. Use b. to show that $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

7.2.5 Suppose $\{a_n\}$ is a non-increasing positive sequence and that $a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots + 2^n a_{2^n} + \ldots$ diverges. Prove that $\sum_{n=1}^{\infty} a_n$ diverges.

7.2.6 (L)

a. Let $\sum_{n=1}^{\infty} a_n$ be a positive series, and let $L = \lim_{n\to\infty} \sqrt[n]{a_n}$. Use the comparison test to show that $\sum_{n=1}^{\infty} a_n$ converges if $L < 1$ and diverges if $L > 1$.

b. Use a. to test the convergence of $\sum_{n=1}^{\infty} \left(\frac{n+1}{3n}\right)^n$.

7.2.7 Test the convergence of $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$.

7.2.8 (L)

a. Prove that if $\sum_{n=1}^{\infty} a_n$ is any positive series and c is any non-zero constant, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} c\, a_n$ converges.

b. Use a. to show how we may test a negative series for convergence.

7.2.9 (L) Test the following negative series for convergence:

a. $\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n}\right)$ b. $\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$.

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UNIT 3: Absolute Convergence

1. View: Lecture 7.030
2. Read: Supplementary Notes, Chapter X, Section E
3. Read: Thomas 18.9, 18.10
4. Exercises:

7.3.1 Rearrange the terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ so that the resulting series converges to zero.

7.3.2 Let $a_n = \frac{(-1)^{n+1}}{\sqrt{n}}$. Show that $\sum_{n=1}^{\infty} a_n{}^2$ diverges even though $\sum_{n=1}^{\infty} a_n$ converges. (In particular, observe that the result stated in Ex. 7.2.3(L), part (a), depends on the fact that $\sum a_n$ is a positive series.)

7.3.3(L) For what values of c does the series $\sum_{n=0}^{\infty} \frac{(-2)^n}{(n+1)(c-1)^n}$ converge absolutely?

7.3.4 In each of the following find the values of c for which the series is absolutely convergent.

a. $\sum_{n=1}^{\infty} nc^n$ b. $\sum_{n=0}^{\infty} n!c^n$ c. $\sum_{n=0}^{\infty} \frac{(c+5)^n}{n+1}$ d. $\sum_{n=0}^{\infty} \frac{c^n}{n!}$

7.3.5(L) Given the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, we define

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) \text{ to equal } \sum_{n=0}^{\infty} c_n \text{ where}$$

[7.3.5(L) cont'd]

$c_n = \sum_{k=0}^{n} a_k b_{n-k}$ $(n = 0,1,2,3\ldots)$. Use this definition to write the series which is equal to $\left(\sum_{n=0}^{\infty} \frac{1}{n+1}\right)^2$.

7.3.6 Write the series which equals $\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}\right)^2$.

7.3.7(L)

a. Expand (n-k+1)(k+1) and complete the square to show that

$$(n-k+1)(k+1) = \left(\frac{n}{2} + 1\right)^2 - \left(\frac{n}{2} - k\right)^2 \leqslant \left(\frac{n}{2} + 1\right)^2$$

b. Combine (a) with the answer to Ex. 7.3.6 to show that $\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}\right)^2$ diverges.

c. From (b) deduce that the product of two convergent series need not be a convergent series.

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UNIT 4: Polynomial Approximations

1. View: Lecture 7.040
2. Read: Supplementary Notes, Chapter X, section F
3. Read: Thomas 18.3, 18.4, and 18.6
4. Exercises:

(P_n and P when used below are defined as follows: for a given function f, $P_n(x) = \sum_{k=0}^{n} \frac{f^{(n)}(0)}{n!} x^n$ and P is defined by $P(x) = \lim_{n\to\infty} P_n(x)$.)

7.4.1 (L)

a. Determine $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$, and $P_4(x)$ if $f(x) = x^3 - 6x^2 + 9x + 1$.

b. With f(x) as in a. determine P(x) and discuss the accuracy with which P(x) approximates f(x) .

7.4.2 (L)

Determine P(x) if $f(x) = \begin{cases} x^3 - 6x^2 + 9x + 1 \text{ if } 0 \leqslant x \leqslant 1 \\ x^2 + 4, \text{ if } x > 1 \end{cases}$

and discuss the accuracy with which P(x) approximates f(x).

7.2.3 (L) For each of the following choices of f(x), determine

a. $P_n(x)$

b. $P(x)$

c. the interval of absolute convergence of P(x)

(1) $f(x) = \sin x$
(2) $f(x) = \cosh x$
(3) $f(x) = \frac{1}{1 - x}$
(4) $f(x) = 6x^4 + 3x^2 + 7x - 5$.

7.4.4 (L) Recall that Taylor's Theorem with Remainder states that if f and its first (n + 1) derivatives are defined and continuous on the interval $I = \{x: |x - a| < R\}$, then for all $x \in I$, we have:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k + R_n(x,a)$$

where $R_n(x,a) = \frac{1}{n!} \int_a^x (x - t)^n f^{(n+1)}(t)\, dt$.

a. Show that if there exists a number M such that $|f^{(n+1)}(x)| \leq M$ for all $x \in I$, then

$$|R_n(x,a)| \leq \frac{M |x - a|^{n+1}}{(n + 1)!} .$$

b. Use a. to show that for every real number, x,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x .$$

c. Use b. to determine $\frac{1}{e}$ correct to three decimal places.

7.4.5 Show that for any real number x,

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)!}$. In particular, determine $\sin \frac{1}{2}$ correct to three decimal places.

7.4.6 (L)

a. Use division to show that

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n + \frac{(-1)^{n+1} x^{n+1}}{1 + x} .$$

[7.4.6 (L) cont'd]

b. Observing that $\ln(1 + x) = \int_0^x \frac{dt}{1 + t}$, use part a. to

show that $\ln(1 + x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n + 1}$ if $0 \leqslant x \leqslant 1$

(actually the result is true if $-1 < x \leqslant 1$).

c. Compute $\ln 1.2$ correct to three decimal place.

d. Use b. to determine $\sum_{n=0}^{\infty} \frac{(-1)^n}{n + 1}$.

7.4.7 Mimic the procedure of Exercise 7.4.6 (L) to find a method for computing π. Namely observe that

$$\frac{\pi}{4} = \tan^{-1} 1 = \int_0^1 \frac{dt}{1 + t^2}$$

then use division or an equivalent to express $\frac{1}{1 + t^2}$ as the sum of a polynomial and a remainder term.

STUDY GUIDE: Calculus of a Single Variable - Block VII: Infinite Series

UNIT 5: Uniform Convergence

1. View: Lecture 7.050
2. Read: Supplementary Notes, Chapter X, Section G
3. Read: Thomas 18.7 (after Exercise 7.5.6(L))
4. Exercises:

7.5.1(L)

a. Show that $\{x^n\}$ converges uniformly to 0 in the interval $(0,b)$ where $0 < b < 1$.

b. What happens in (a) if $b = 1$?

7.5.2 Let $b > 0$. Show that $\left\{\frac{1}{x+n}\right\}$ converges uniformly to 0 on $[0,b]$.

7.5.3 Let $f_n(x) = \frac{n}{x+n}$, where dom $f_n = [0,b]$, $b > 0$. Show that $\{f_n\}$ converges uniformly to 1 on $[0,b]$.

7.5.4 For any real x, define f_n by $f_n(x) = \frac{n}{x+n}$. Show that $\lim_{x\to\infty}\left\{\lim_{n\to\infty} f_n(x)\right\} \neq \lim_{n\to\infty}\left\{\lim_{x\to\infty} f_n(x)\right\}$.

7.5.5(L) Let $f_n(x) = \frac{nx}{1 + nx}$, dom $f_n = [0,1]$.

a. Show that f_n is continuous on $[0,1]$.

b. Let $f(x) = \lim_{n\to\infty} f_n(x)$. Show that f is <u>not</u> continuous on $[0,1]$.

c. Does $\{f_n\}$ converge uniformly to f on $[0,1]$? Explain in terms of (a) and (b).

d. Show analytically that $\left\{\frac{nx}{1 + nx}\right\}$ converges to 1 on $[b,1]$ if $0 < b < 1$ but that the convergence is not uniform in $(0,1]$.

7.5.6(L) Let $f_n(x) = \frac{n^2x}{1 + n^3x^2}$, dom $f_n = [0,1]$.

a. Show that $\int_0^1 \lim_{n\to\infty} f_n(x)dx = \lim_{n\to\infty} \int_0^1 f_n(x)dx$

b. Describe the curve $y = f_n(x)$ in general and $y = f_{100}(x)$ in particular. Based on the curve does it seem that $\{f_n\}$ converges uniformly to 0 on [0,1]? Explain.

c. Show analytically that $\{f_n\}$ does not converge uniformly to 0 on [0,1].

d. Combining (a) and (c) does it follow that $\{f_n\}$ converges uniformly to f on [a,b] if $\int_a^b \lim_{n\to\infty} f_n(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$? Explain.

7.5.7

a. Show that $\lim_{n\to\infty} 2nxe^{-nx^2} = 0$.

b. Letting $f_n(x) = 2nxe^{-nx^2}$ show that $\int_0^1 \lim_{n\to\infty} f_n(x)dx = 0$.

c. With $f_n(x)$ again equal to $2nxe^{-nx^2}$ show that

$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 1$.

d. Does $\left\{2nxe^{-nx^2}\right\}$ converge uniformly to 0 on [0,1]? Explain.

7.5.8(L) Let $f_n(x) = \sum_{k=0}^{n} \frac{x^2}{(1 + x^2)^k}$, dom $f_n = [0,b]$, $b > 0$, and let $f = \lim_{n\to\infty} f_n(x)$.

[7.5.8(L) cont'd]

a. Show that f is discontinuous at x = 0.

b. Does $\{f_n\}$ converge uniformly to f on [0,1]? Explain.

7.5.9(L) Let $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$ and let $\lim\limits_{n\to\infty} f_n(x) = f$.

a. Determine f and show that $\{f_n(x)\}$ converges uniformly to f(x) on [0,b] where b > 0.

b. Show that $f_n'(x)$ exists for every n and that f'(x) exists but that

$$\lim_{n\to\infty} f_n'(x) \neq f'(x) \left(=\left[\lim_{n\to\infty} f_n(x)\right]'\right)$$

STUDY GUIDE: Calculus of a Single Variable - Block VII: Infinite Series

UNIT 6: Uniform Convergence Applied to Power Series

1. View: Lecture 7.060

2. Read: Supplementary Notes, Chapter X, Section H

3. Exercises:

7.6.1 (L) Use the Weierstrass M-test to find the values of x for which $\sum_{n=0}^{\infty} (5x)^n$ converges uniformly.

7.6.2 Find the values of x for which $\sum_{n=0}^{\infty} \frac{1}{1 + x^{2n}}$ converges uniformly.

7.6.3 (L)

a. Show that if $\sum_{n=0}^{\infty} b_n x^n$ and $\sum_{n=0}^{\infty} c_n x^n$ converges uniformly to the same function then $b_n = c_n$ for every n = 1,3,2,4.. (That is, two power series are unequal unless the coefficients are equal term by term.)

b. Write the power series which represents x sin x .

c. Use your answer in b. to compute $\int_0^{1/2} x \sin dx$.

d. Use integration by parts to compute $\int_0^{1/2} x \sin x \, dx$, and compare your answer with the one obtained in c.

7.6.4 Use series to compute $\int_0^{1/2} x e^{-x^3} dx$ correct to four decimal places.

7.6.5

a. Observing that $\frac{1}{(x-2)(x-3)} = \frac{1}{x-3} - \frac{1}{x-2}$, find a power series which represents $\frac{1}{(x-2)(x-3)}$.

b. Use the series in a. to compute $\int_0^1 \frac{1}{(x-2)(x-3)}$.

c. Compute $\int_0^1 \frac{dx}{(x-2)(x-3)}$ by the use of partial fractions.

7.6.6 Define a function f by $f(x) = \int_0^x \cos t^2\, dt,\ x > 0$.

Express f(x) as a power series.

7.6.7 Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Show that if $f(x) = f(-x)$ (i.e., f is an even function) then $f(x) = a_0 + a_2 x^2 + a_4 x^4 + \ldots + a_{2n} x^{2n} + \ldots$

7.6.8 (L)

a. Determine the power series which represents $\frac{\sin x}{1 - x}$ for $|x| < 1$.

b. Use a. to determine $\int_0^{0.01} \frac{\sin x\, dx}{1 - x}$ to three significant figures.

QUIZ

1. Tell which of the following series converge and which diverge. In each case, give a reason for your choice.

(a) $\sum_{n=1}^{\infty} \frac{5n+7}{8n-3}$ (b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ (c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

(d) $\sum_{n=0}^{\infty} \frac{10^{6n}}{n!}$ (e) $\sum_{n=1}^{\infty} \frac{2n}{1+n^2}$

2. Find the interval of convergence for each of the following power series. (In (b) do not test the endpoints of the interval.)

(a) $\sum_{n=1}^{\infty} \frac{x^n}{n}$ (b) $\sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$ (c) $\sum_{n=1}^{\infty} \frac{n^2x^n}{3^n}$

3. Use the Weierstrass M-test to prove each of the following.

(a) $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ converges uniformly

(b) If $\sum |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n \sin nx$ converges uniformly

4. (a) How many terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ must we take if we want the sum to be within 0.01 of the exact answer?

[4. cont'd]

(b) How many terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ must we take if we want the sum to be correct to within 0.01?

5. Use power series to compute $\int_0^{\frac{1}{2}} \sin(t^2)dt$ correct to within 0.00001.

6. Use the first four non-vanishing terms of the power series expansion for $\frac{1}{e^x(1-x)}$ to estimate $\int_0^{\frac{1}{10}} \frac{dx}{e^x(1-x)}$ correct to four decimal places.

Calculus of a Single Variable

SOLUTIONS

PRETEST

1. a. $\frac{1}{4}\ln^4 x + c$ b. $\frac{1}{x\ln x}$ c. e^2

2. $y = 5e^{5x} + 3e^{-3x}$

3. $\pi[1 - \frac{1}{e^9}]$

4. $\pm\frac{5}{12}$

5. x

6. $|\sec x|$

7. $x + \sqrt{3}y = 2 + \sqrt{3}\ln(2 - \sqrt{3})$

UNIT 1: Logarithms Revisited

5.1.1(L)

a. Recall that the mean value theorem says that

$$\frac{f(b) - f(a)}{b - a} = f'(c) \quad \text{for some } c \varepsilon (a,b)$$

provided that f is continuous on [a,b] and differentiable in (a,b).

Now, observing that $\ln \frac{6}{5} = \ln 6 - \ln 5$, $f'(c) = \frac{1}{c}$, and that ln x is continuous and differentiable on [5,6], we may invoke the mean value theorem to obtain

$$\frac{\ln 6 - \ln 5}{6 - 5} = \frac{1}{c} \quad \text{for some } c \varepsilon (5,6) \tag{1}$$

Since $5 < c < 6$, it follows that $\frac{1}{5} > \frac{1}{c} > \frac{1}{6}$; hence, (1) becomes:

$$\frac{1}{6} < \frac{\ln 6 - \ln 5}{6 - 5} < \frac{1}{5}$$

$$\therefore \quad \frac{1}{6} < \ln 6 - \ln 5 < \frac{1}{5} \text{ , and since}$$

$$\ln 6 - \ln 5 = \ln \frac{6}{5} \text{ ,}$$

we have: $\frac{1}{6} < \ln \frac{6}{5} < \frac{1}{5}$ (2)

b. From (2) it follows that:

$$\left.\begin{array}{l} \frac{5}{6} < 5 \ln \frac{6}{5} < 1 \\ \text{and} \\ 1 < 6 \ln \frac{6}{5} < \frac{6}{5} \end{array}\right\} \tag{3}$$

[5.1.1(L) cont'd]

Now $5 \ln \frac{6}{5} = \ln(\frac{6}{5})^5$, while $6 \ln \frac{6}{5} = \ln(\frac{6}{5})^6$

Putting these results into (3), we obtain

$$\ln(\tfrac{6}{5})^5 < 1 < \ln(\tfrac{6}{5})^6 \qquad (4)$$

and since $\ln e = 1$, (4) may be written as:

$$\ln(\tfrac{6}{5})^5 < \ln e < \ln(\tfrac{6}{5})^6 \qquad (5)$$

Then since $\ln x$ is an increasing function, (5) implies that:

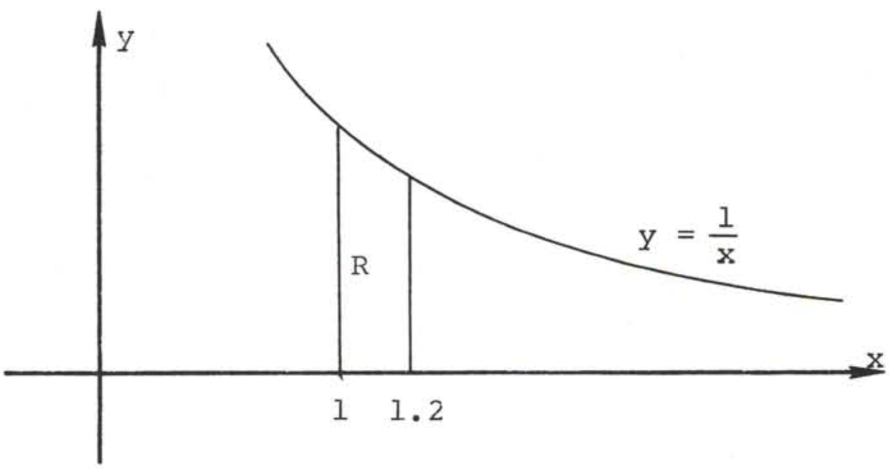
$$(\tfrac{6}{5})^5 < e < (\tfrac{6}{5})^6 \qquad (6)$$

A menial but straight-forward computation shows that $(\frac{6}{5})^5 = 2.48732$ and $(\frac{6}{5})^6 = 2.984784$. Putting this into (6) yields:

$$2.48 < e < 2.98 \qquad (7)$$

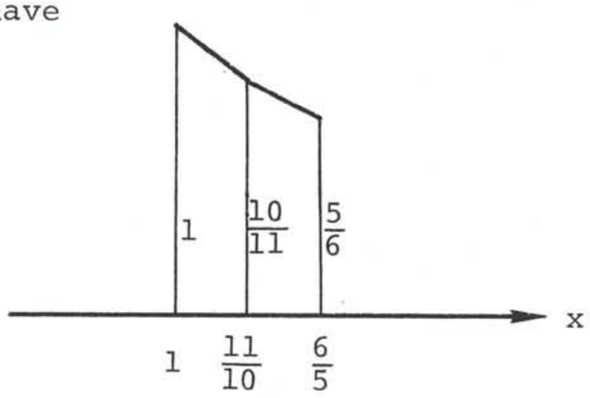
c. $\ln 1.2 = \int_1^{1.2} \frac{dx}{x}$

Thus $\ln 1.2$ is the area of R where



[5.1.1(L) cont'd]

Using trapezoidal approximations for R with n = 2, we have



$$T_2 = \frac{1}{2}\left(1 + \frac{10}{11}\right)\left(\frac{1}{10}\right) + \frac{1}{2}\left(\frac{10}{11} + \frac{5}{6}\right)\frac{1}{10}$$

$$= \frac{1}{20}\left(\frac{21}{11} + \frac{115}{66}\right) = \frac{1}{1320}(126 + 115) = \frac{241}{1320}$$

$$= 0.1826 \qquad (8)$$

Notice that (8) is compatible with the results of (2). The interesting point, however, is that (8) is a much better estimate for ln 1.2 than is the inequality in (2) - and it was much easier to obtain (8) (only 2 trapezoids) than it was to obtain (2).

This stems from the fact that for most curves trapezoids fill in the region rather well, rather quickly. More objectively, recall the estimate:

$$\int_a^b f(x)dx = T - \frac{b-a}{12} f''(c)(\Delta x)^2 \quad \text{for some } c \in (a,b)^* \qquad (9)$$

*See, for example, Thomas: pg 180

[5.1.1(L) cont'd]

In our case $a = 1$, $b = \frac{6}{5}$, $f(x) = \frac{1}{x}$, $T = 0.1826$, $\Delta x = \frac{1}{10}$. Then: $f(x) = \frac{1}{x} \rightarrow f'(x) = -\frac{1}{x^2} \rightarrow$ $f''(x) = \frac{2}{x^3}$; on $[1,\frac{6}{5}]$, $\frac{2}{x^3}$ is maximum when $x = 1$ since $\frac{2}{x^3}$ is maximum when x is minimum. Hence, an upper bound on our error is obtained from (9) by letting $c = 1$. We obtain:

$$\text{Maximum error} = \left[\frac{\frac{1}{5}}{12}\right](2)\left[\frac{1}{10}\right]^2$$

$$= \frac{1}{3000} = 0.0003^{+} \qquad (10)$$

The minimum error is obtianed from (9) by letting $c = \frac{6}{5}$ and we obtain:

$$\text{Minimum error} = \left[\frac{\frac{1}{5}}{12}\right]\left[\frac{2}{(\frac{6}{5})^3}\right]\left(\frac{1}{10}\right)^2$$

$$= \frac{1}{5184} = 0.0001^{+} \qquad (11)$$

Notice that (10) and (11) tell us that our answer in (8) is correct to three decimal places and that the fourth place is off by less than 3 and more than 1. In other words, our answer lies between 0.1823 and 0.1824^{+}. In any event, correct to three decimal places.

$$\ln 1.2 = 0.182 \qquad (12)$$

[5.1.1(L) cont'd]

Aside from affording us a review of several major principles of calculus in general, our hope is that this exercise helps to establish the "realness" of the natural logarithm in general, and the number e in particular.

5.1.2(L)

a. We have $\frac{d}{dx}\ln(x^n) = \frac{d\ln(x^n)}{d(x^n)}\frac{d(x^n)}{dx}$

$$= \frac{1}{x^n}[nx^{n-1}]$$

$$= \frac{n}{x}$$

While $\frac{d}{dx}[n\ln x] = n\frac{d(\ln x)}{dx} = \frac{n}{x}$

Since $\ln(x^n)$ and $n \ln x$ have the same derivative, we may conclude that:

$$\ln(x^n) = n\ln x + c \quad (1)$$

If we let $x = 1$ in (1), we find:

$$\ln 1 = n\ln 1 + c$$

or

$$0 = 0 + c$$

$\therefore$ $c = 0$, and (1) becomes

$$\ln(x^n) = n\ln x \quad (2)$$

[5.1.2(L) cont'd]

At first glance, (2) might seem to be redundant, since one of our properties of any logarithmic function was $f(x^n) = n\,f(x)$. You may recall, however, that this general property, at least as proven in our supplementary notes, was restricted to n being a positive integer (otherwise we couldn't have used induction). In our present proof, the result is valid for any rational number n, since the recipe $\frac{d(x^n)}{dx} = nx^{n-1}$ is valid for any rational number, n.

b.

$$y = (x^2 + 1)^5 \sqrt{x^4 + 2}\ \sqrt[4]{(x^4 + 2x^2 + 3)}$$

$$y = (x^2 + 1)^5 (x^4 + 2)^{\frac{1}{2}} (x^4 + 2x^2 + 3)^{\frac{1}{4}}$$

$$\ln y = \ln\left[(x^2 + 1)^5 (x^4 + 2)^{\frac{1}{2}} (x^4 + 2x^2 + 3)^{\frac{1}{4}}\right]$$

$$\ln y = \ln(x^2 + 1)^5 + \ln(x^4 + 2)^{\frac{1}{2}} + \ln(x^4 + 2x^2 + 3)^{\frac{1}{4}}$$

$$\ln y = 5\ln(x^2 + 1) + \frac{1}{2}\ln(x^4 + 2) + \frac{1}{4}\ln(x^4 + 2x^2 + 3) \quad (1)$$

Differentiating (1) implicitly with respect to x, yields

$$\frac{1}{y}\frac{dy}{dx} = 5\left(\frac{2x}{x^2 + 1}\right) + \frac{1}{2}\left(\frac{4x^3}{x^4 + 2}\right) + \frac{1}{4}\left(\frac{4x^3 + 4x}{x^4 + 2x^2 + 3}\right)$$

[5.1.2(L) cont'd]

$$\frac{dy}{dx} = y\left[\frac{10x}{x^2+1} + \frac{2x^3}{x^4+2} + \frac{x^3+x}{x^4+2x^2+3}\right]$$

$$= (x^2+1)^5(x^4+2)^{\frac{1}{2}}(x^4+2x^2+3)^{\frac{1}{4}}\left[\frac{10x}{x^2+1} + \frac{2x^3}{x^4+2} + \frac{x^3+x}{x^4+2x^2+3}\right]$$

c. First of all, do not confuse x^x and x^n. That is, in x^n the exponent is fixed while in x^x the exponent varies with x. The point is that as of this moment we have a recipe for differentiating x^n not x^x. At any rate if we assume that $\ln x^m = m \ln x$ for any real number m, (we already know its true if m is a rational number) we have

$$y = x^x$$

$$\ln y = \ln(x^x) = x \ln x \qquad (2)$$

Now $\frac{d}{dx}\ln y = \frac{1}{y}\frac{dy}{dx}$

While $\frac{d}{dx}(x \ln x) = x\frac{d}{dx}(\ln x) + \frac{dx}{dx}\ln x$

$= \frac{x}{x} + \ln x = 1 + \ln x$

Thus, from (2) we may conclude that

[5.1.2(L) cont'd]

$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$$

$$\therefore \quad \frac{dy}{dx} = y(1 + \ln x), \quad \text{and since } y = x^x,$$

$$\frac{dy}{dx} = x^x(1 + \ln x)$$

In particular, $\frac{dx^x}{dx} \neq x(x)^{x-1} \quad (=x^x)$

d. $$y = x^n$$

$$\ln y = \ln x^n = n \ln x$$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \frac{n}{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{ny}{x} = \frac{nx^n}{x} = nx^{n-1}$$

Thus, if we assume that $\ln x^n = n \ln x$ for all real numbers n, the recipe $\frac{d(x^n)}{dx} = nx^{n-1}$ is now valid for all real numbers n.

5.1.3

a. $$\frac{d}{dx}\ln\sqrt{x^2 + 1} = \frac{d}{dx}\ln(x^2 + 1)^{\frac{1}{2}}$$

$$= \frac{d}{dx}[\tfrac{1}{2}\ln(x^2 + 1)]$$

[5.1.3 cont'd]

$$= \frac{1}{2}\left[\frac{2x}{x^2 + 1}\right]$$

$$= \frac{x}{x^2 + 1}$$

b. $\frac{d}{dx}\ln[x^2\sqrt{x^2 + 1}] = \frac{d}{dx}\ln[x^2(x^2 + 1)^{\frac{1}{2}}]$

$$= \frac{d}{dx}[\ln x^2 + \ln(x^2 + 1)^{\frac{1}{2}}]$$

$$= \frac{d}{dx}[2\ln x + \ln(x^2 + 1)^{\frac{1}{2}}]$$

$$= \frac{2}{x} + \frac{x}{x^2 + 1} \qquad \text{[from (a)]}$$

$$= \frac{3x^2 + 2}{x(x^2 + 1)}$$

c. Let $u = \ln x$. Then $du = \frac{dx}{x}$

$$\therefore \int (\ln x)^3 \frac{dx}{x} = \int u^3 du = \frac{1}{4}u^4 + c = \frac{1}{4}\ln^4 x + c$$

d. Let $u = 1 + \sin x$, then $du = \cos x\, dx$

$$\therefore \int \frac{\cos x\, dx}{1 + \sin x} = \int \frac{du}{u} = \ln|u| + c$$

[5.1.3 cont'd]

$$= \ln|1 + \sin x| + c, \text{ and since } 1 + \sin x \geqslant 0$$

$$= \ln(1 + \sin x) + c$$

e. $y = \ln(\ln x) \longrightarrow \dfrac{dy}{dx} = \dfrac{\frac{d}{dx}(\ln x)}{\ln x} = \dfrac{1}{x \ln x}$

$\left(\text{That is, let } u = \ln x \text{ ; then } \dfrac{d \ln u}{dx} = \dfrac{1}{u}\dfrac{du}{dx} = \left(\dfrac{1}{\ln x}\right)\dfrac{1}{x}\right)$

f. $\lim\limits_{n\to\infty} (1 + \frac{1}{n})^n = e$ (from the supplementary notes)

Now given $1 + \frac{2}{n}$, let $m = \frac{n}{2}$ then

$$(1 + \tfrac{2}{n})^n = (1 + \tfrac{1}{m})^{2m} = [(1 + \tfrac{1}{m})^m]^2$$

$$\therefore \lim_{n\to\infty} (1 + \tfrac{2}{n})^n = \lim_{m\to\infty} [(1 + \tfrac{1}{m})^m (1 + \tfrac{1}{m})^m]$$

$$= (e)(e) = e^2$$

5.1.4

a. The area of R is given by $\displaystyle\int_0^1 \frac{x\,dx}{x^2 + 1}\,dx$

If we let $u = x^2 + 1$, then $du = 2xdx$; $u = 1$ when $x = 0$, and $u = 2$ when $x = 1$.

Hence $\displaystyle\int_0^1 \frac{x\,dx}{x^2 + 1} = \int_1^2 \frac{\frac{1}{2}\,du}{u} = \frac{1}{2}\int_1^2 \frac{du}{u}$

[5.1.4 cont'd]

$$= \frac{1}{2}\left[\ln|u|\;\right|_1^2 = \frac{1}{2}[\ln 2 - \ln 1]$$

$$= \frac{1}{2}\ln 2$$

b. We have that:

$$V_x = \pi\int_a^b f^2(x)\,dx$$

Thus, in this case;

$$V_x = \pi\int_0^2\left\{\frac{x}{\sqrt{x^3+1}}\right\}^2 dx$$

$$= \pi\int_0^2 \frac{x^2}{x^3+1}\,dx$$

Letting $u = x^3 + 1$, we see that $du = 3x^2dx$ or $x^2dx = \frac{1}{3}du$. Moreover, when $x = 0$, $u = 1$ and when $x = 2$, $u = 9$. Hence:

$$V_x = \pi\int_1^9 \frac{du}{3u} = \frac{1}{3}\pi\ln|u|\;\Big|_1^9$$

$$= \frac{1}{3}\pi\ln 9$$

[5.1.4 cont'd]

(Notice that there is no new theory in this exercise. All that's new is that our integrals involve $\int \frac{du}{u}$ which we couldn't handle before this exercise.)

5.1.5 (L) Aside from the fact that an exercise such as this one provides us with an excellent review of several major topics as well as drill with our new function, ln x, there is yet another very important aspect to this exercise. In its own way, this exercise is a forerunner of Block VII in which we shall discuss the most important concept of power series. Since this will be discussed in Block VII, we will not take the time here to delve too deeply into the notion of power series, but we would like to focus some attention on the result that for values of x in excess of 1, we can approximate ln x quite nicely by the cubic polynomial equation

$$(x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3}$$

provided only that x doesn't exceed 1 by too much.

There is nothing sacred about our choice of using a cubic equation nor in our choice of picking powers of (x - 1). For example, had we been interested in ln x for values of x near 2, we would have proceeded in a similar way, using powers of (x - 2), and had we desired more accuracy we could have proceeded beyond a third degree polynomial and taken as a high a degree polynomial as we desired to produce the

[5.1.5 (L) cont'd]

accuracy we needed. At any rate, let us now proceed with the exercise.

a. $$\int_0^{x-1} \frac{du}{1+u} = \ln(1+u)\Big|_0^{x-1} = \ln(1+x-1) - \ln(1+0)$$

$$= \ln x - \ln 1$$

$$= \ln x$$

From the integral calculus point of view, we have:

$$\ln x = \int_1^x \frac{dt}{t}$$

If we now let $u = t - 1$ we obtain:

$$\ln x = \int_0^{x-1} \frac{du}{1+u}$$

b. By long division, we have:

$$\begin{array}{rl} 1 & \\ \underline{1 + u} & \\ -u & \\ \underline{-u - u^2} & \\ u^2 & \end{array} \qquad \begin{array}{|l} 1 + u \\ \hline 1 - u + u^2 \cdots \end{array}$$

At each stage, our remainder is the next power of u and the signs keep alternating.

[5.1.5 (L) cont'd]

Thus:

$$\frac{1}{1+u} = 1 - \frac{u}{1+u}$$

$$= 1 - u + \frac{u^2}{1+u}$$

$$= 1 - u + u^2 - \frac{u^3}{1+u}$$

etc.

c. $$\ln x = \int_0^{x-1} \frac{du}{1+u}$$

$$= \int_0^{x-1} \left[1 - u + u^2 - \frac{u^3}{u+1}\right] du$$

$$= \int_0^{x-1} (1 - u + u^2)\,du - \int_0^{x-1} \frac{u^3 du}{u+1}$$

$$= u - \frac{1}{2}u^2 + \frac{1}{3}u^3 \Big|_0^{x-1} - \int_0^{x-1} \frac{u^3 du}{u+1}$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \int_0^{x-1} \frac{u^3 du}{u+1} \quad (1)$$

d. From (1), the difference between ln x and (x - 1) $-\frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$ is $\int_0^{x-1} \frac{u^3 du}{u+1}$. Since

[5.1.5 (L) cont'd]

$0 \leqslant u \leqslant x^{-1}$ and $x > 1$, it follows that u is positive. Hence:

$$\frac{u^3}{u+1} < u^3$$

$$\therefore \int_0^{x-1} \frac{u^3}{u+1}\,du < \int_0^{x-1} u^3 du = \frac{1}{4}u^4 \Big|_0^{x-1} = \frac{(x-1)^4}{4}$$

Thus an upper bound for the error is $\frac{(x-1)^4}{4}$. Hence:

$$\frac{(x-1)^4}{4} < \frac{1}{1024} \longleftrightarrow (x-1)^4 < \frac{4}{1024} = \frac{1}{256} = \frac{1}{4^4}$$

$$\longleftrightarrow |x-1| < \frac{1}{4}$$

$$\longleftrightarrow -\frac{1}{4} < x - 1 < \frac{1}{4}$$

$$\longleftrightarrow \frac{3}{4} < x < \frac{5}{4}$$

$$\longleftrightarrow x < \frac{5}{4}$$, since our assumption is $x > 1$.

e. Since $x = 1.2$ is in the range $1 < x < \frac{5}{4}$, it follows that ln 1.2 can be expressed with an error of less than 0.001 by $(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$

$$\therefore \ln 1.2 \simeq (1.2-1) - \frac{(1.2-1)^2}{2} + \frac{(1.2-1)^3}{3}$$

[5.1.5 (L) cont'd]

$$\approx (.2) - \frac{(.2)^2}{2} + \frac{(.2)^3}{3}$$

$$\approx 0.2 - 0.02 + 0.0027$$

$$\approx 0.1827$$

$$\approx 0.183$$

5.1.6 We have:

$$\ln x = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4}$$

$$+ \frac{(x - 1)^5}{5} - \ldots + (-1)^n \frac{(x - 1)^{n-1}}{n - 1} + R$$

where the error R is no greater than $\left|\frac{(x - 1)^n}{n}\right|$

if with $x = 1.2$, $\frac{(x - 1)^n}{n} = \frac{(.2)^n}{n}$

we want $\frac{(.2)^n}{n} < 10^{-6}$

To solve $\frac{(.2)^n}{n} < 10^{-6}$ we can use many techniques, one of which is simply to look at $\frac{(0.2)^n}{n}$ for various n.

For example, with $n = 6$, $\frac{(.2)^n}{n} = \frac{1}{6}(2 \times 10^{-1})^6$

[5.1.6 cont'd]

$$= 10.6667 \times 10^{-6}$$

$$= 0.0000106667\ldots$$

$$n = 7; \quad \frac{.2^n}{n} = \frac{1}{7}(128) \times 10^{-7} = 18^+ \times 10^{-7} = 1.8^+ \times 10^{-6} > 10^{-6}$$

$$n = 8; \quad \frac{.2^n}{n} = \frac{1}{8}(256) \times 10^{-8} = 32 \times 10^{-8} = .32 \times 10^{-6}$$
$$(= .00000032) < 10^{-6}$$

∴ We must carry out our computations through $\frac{1}{7}(x - 1)^7$ to get the required accuracy. In other words, to 6 place accuracy.

$$\ln 1.2 = .2 - \frac{(.2)^2}{2} + \frac{(.2)^3}{3} - \frac{(.2)^4}{4} + \frac{(.2)^5}{5} - \frac{(.2)^6}{6} + \frac{(.2)^7}{7}$$

$$= .2 - .02 + .0026667 - .0004 + .000064 - 0000107 + .0000018$$

$$= .2027325 - .0204107$$

$$= 0.182321$$

5.1.7(L)

a. $$\ln x = \int_1^x \frac{dt}{t}$$

[5.1.7 (L) cont'd]

Now for $t > 1$, $t > \sqrt{t}$

$$\therefore \int_1^x \frac{dt}{t} < \int_1^x \frac{dt}{\sqrt{t}} = \int_1^x t^{-\frac{1}{2}}\,dt = 2\sqrt{t}\,\Big|_1^x = 2\sqrt{x} - 2$$

$$\therefore \ln x < 2\sqrt{x} - 2$$

$$\therefore \frac{\ln x}{x} < \frac{2\sqrt{x} - 2}{x} = \frac{2}{\sqrt{x}} - \frac{2}{x}$$

$$\therefore \lim_{x\to\infty} \frac{\ln x}{x} = 0 \qquad (1)$$

b.

$$\ln x^n = n \ln x$$

$$\lim_{x\to\infty} \frac{\ln x^n}{x} = \lim_{x\to\infty} \frac{n \ln x}{x} = n \lim_{x\to\infty} \frac{\ln x}{x} = n\cdot 0 \text{ [by (1)]}$$

$$= 0 \qquad (2)$$

Equation (2) shows that x "gets large" faster than $\ln x^n$ no matter how great n is. Part (c) says the same thing from a different point of view. Namely,

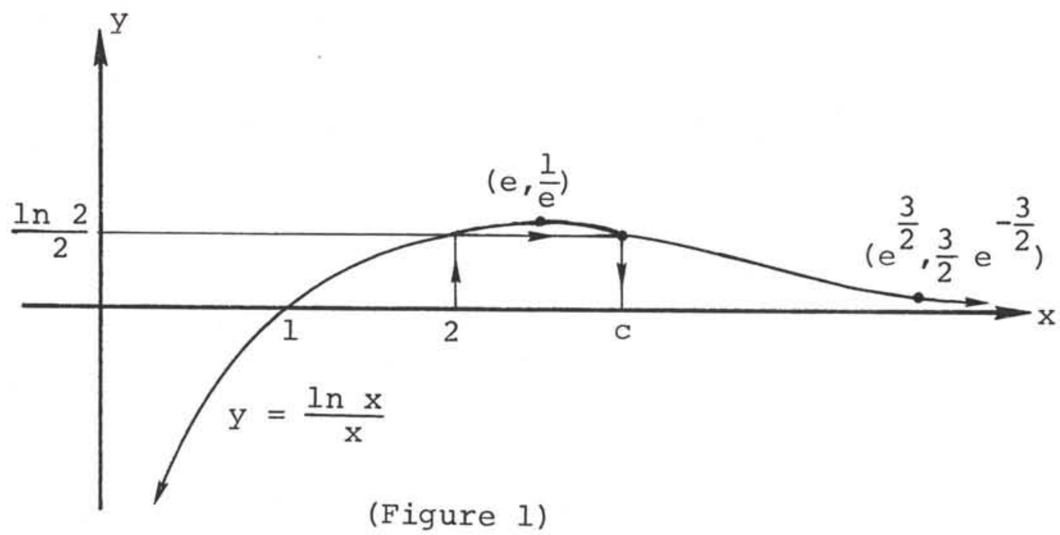
c. In (2), let $u = x^n$. Then $x = \sqrt[n]{u}$ and we obtain:

$$\lim_{u\to\infty} \frac{\ln u}{\sqrt[n]{u}} = 0 \qquad (3)$$

Thus $\ln u$ goes to zero more rapidly than any positive power of u, no matter how small.

5.1.8

a. The curve $y = \frac{\ln x}{x}$ $(x > 0)$ is given by:



(Figure 1)

The key steps in obtaining this result are:

$$y = \frac{\ln x}{x} \quad (x > 0) \tag{1}$$

$$\therefore \quad y' = \frac{x(\frac{1}{x}) - \ln x}{x^2} = \frac{1 - \ln x}{x^2} \tag{2}$$

$$y'' = \frac{x^2(-\frac{1}{x}) - (1 - \ln x)2x}{x^4}$$

$$= \frac{-3x + 2x \ln x}{x^4}$$

$$= \frac{2 \ln x - 3}{x^3} \quad (\text{since } x \neq 0) \tag{3}$$

[5.1.8 cont'd]

From (1) we see that the curve passes through (1,0).

From (2) we see that $y' > 0$ if $x < e$, $y' = 0$ if $x = e$, and $y' < 0$ if $x > e$ $\therefore$ $x = e$ yields a high point; and since $y = \frac{\ln x}{x}$, the high point is $(e,\frac{1}{e})$

From (3): $\ln x = \frac{3}{2}$ (or $x = e^{3/2}$) yields a point of inflection with the curve "spilling water" for $x < e^{3/2}$ and "holding water" if $x > e^{3/2}$. Moreover, when $x = e^{3/2}$, $y = \frac{\ln x}{x} = \frac{3}{2}\, e^{-3/2}$.

Finally, from the previous exercise $\frac{\ln x}{x} \to 0$ as $x \to \infty$.

Figure 1 is the compilation of these results.

b. When we look at Figure 1, it is easy to see that $y = \frac{\ln x}{x}$ is not 1 - 1 if $x > 1$. In particular, (see Figure 1) $\frac{\ln c}{c} = \frac{\ln 2}{2}$

(In general solving for c is at best "messy". However, in this special case:

$$\frac{\ln 2}{2} = \frac{\ln c}{c} \to c \ln 2 = 2 \ln c \to \ln 2^c$$

$$= \ln c^2 \to 2^c = c^2$$

It happens that $2^x = x^2$ is solved by $x = 2$ and $x = 4$.

In any event $\frac{\ln 4}{4} = \frac{\ln 2}{2}$)

c. $y = \frac{\ln x}{x}$ is 1 - 1 when y is negative (see Figure 1).

Hence, since $x = \frac{1}{2} \to \frac{\ln x}{x} = -2 \ln 2 < 0$ we see that $x = \frac{1}{2}$ is the only such number.

UNIT 2: The Exponential Function

5.2.1

a. $\frac{de^u}{dx} = \frac{de^u}{du}\frac{du}{dx} = e^u \frac{du}{dx}$

with $u = \frac{1}{x}$, $\frac{du}{dx} = -\frac{1}{x^2}$

Hence $\frac{d(e^{\frac{1}{x}})}{dx} = \frac{-e^{\frac{1}{x}}}{x^2}$

b. $\ln\left[\frac{e^x}{1 + e^x}\right] = \ln e^x - \ln(1 + e^x)$

$\frac{d}{dx} \ln e^x = \frac{e^x}{e^x} = 1$

$\frac{d}{dx} \ln(e^x + 1) = \frac{e^x}{e^x + 1}$

$\therefore \frac{d}{dx} \ln\left[\frac{e^x}{1 + e^x}\right] = 1 - \frac{e^x}{e^x + 1} = \frac{1}{e^x + 1}$

c. let $u = \sin 2x$

Then $du = 2\cos 2x\, dx$ or $\cos 2x\, dx = \frac{du}{2}$

$\therefore \int e^{\sin 2x} \cos 2x\, dx = \int e^u \frac{du}{2} = \frac{1}{2} e^u + c$

$= \frac{1}{2} e^{\sin 2x} + c$

[5.2.1 cont'd]

d. Let $u = 3 + 4e^x$

Then $du = 4e^x dx$ or, $e^x dx = \frac{1}{4}\,du$

$$\therefore \int \frac{e^x dx}{3 + 4e^x} = \int \frac{\frac{1}{4}\,du}{u} = \frac{1}{4}\ln|u| + c$$

$$= \frac{1}{4}\ln|3 + 4e^x| + c = \frac{1}{4}\ln(3 + 4e^x) + c \quad (\text{since } 3 + 4e^x > 0)$$

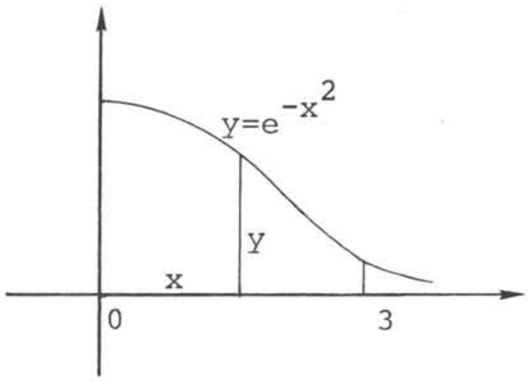
e. Let $u = \ln x$. Then $du = \frac{dx}{x}$

$$\therefore \int_{x=e^2}^{e^3} \frac{dx}{x \ln x} = \int_{u = \ln e^2}^{\ln e^3} \frac{du}{u} = \ln|u| \Big|_{u = \ln e^2}^{\ln e^3}$$

and since $\ln e^u = u$

$$\ln|u| \Big|_{u=2}^{u=3} = \ln 3 - \ln 2 = \ln \frac{3}{2}$$

5.2.2



[5.2.2 cont'd]

$$V_y = 2\pi \int_0^3 xy\, dx$$

$$= 2\pi \int_0^3 xe^{-x^2}\, dx$$

$$= -\pi e^{-x^2} \Big|_0^3$$

$$= -\pi[e^{-9} - e^0]$$

$$= -\pi[\frac{1}{e^9} - 1]$$

$$= \pi[1 - \frac{1}{e^9}]$$

<u>5.2.3 (L)</u> We try $y = e^{rx}$

Then $y' = re^{rx}$

$y'' = r^2e^{rx}$

$\therefore\ y'' - 2y' - 15y = 0 \longrightarrow$

$r^2e^{rx} - 2re^{rx} - 15e^{rx} = 0 \longrightarrow$

$e^{rx}(r^2 - 2r - 15) = 0$, and since $e^{rx} \neq 0$

[5.2.3 (L) cont'd]

$$r^2 - 2r - 15 = 0 \quad \text{or} \quad (r-5)(r+3) = 0$$

$$\therefore r = 5 \quad \text{or} \quad r = -3$$

Therefore, e^{5x} and e^{-3x} satisfy $y'' - 2y' - 15y = 0$. It can now be verified by direct computation that

$$c_1e^{5x} + c_2e^{-3x}$$

is a solution of $y'' - 2y' - 15y = 0$ for any choice of constants c_1, c_2.

Letting $y = c_1e^{5x} + c_2e^{-3x}$ (1)

It follows that

$$y' = 5c_1e^{5x} - 3c_2e^{-3x} \qquad (2)$$

When $x = 0$, (1) and (2) yield:

$$\left.\begin{aligned} y(0) &= c_1 + c_2 \\ y'(0) &= 5c_1 - 3c_2 \end{aligned}\right\} \qquad (3)$$

Recalling that $y(0) = 8$ and $y'(0) = 16$, (3) yields:

$$\left.\begin{aligned} 8 &= c_1 + c_2 \\ 16 &= 5c_1 - 3c_2 \end{aligned}\right\} \quad \begin{aligned} 24 &= 3c_1 + 3c_2 \\ 16 &= 5c_1 - 3c_2 \\ \hline 40 &= 8c_1 \end{aligned}$$

$$\therefore c_1 = 5$$

$$c_2 = 3$$

$$\therefore \quad y = 5e^{5x} + 3e^{-3x}$$

[5.2.3 (L) cont'd]

Check: $y(0) = 8$

$$y'(0) = 25e^{5x} - 9e^{-3x}\Big|_{x=0} = 16$$

$$y'' = 125e^{5x} + 27e^{-3x}$$

$$\therefore\ y'' - 2y' - 15y =$$

$$(125e^{5x} + 27e^{-3x}) - 2(25e^{5x} - 9e^{-3x}) - 15(5e^{5x} + 3e^{-3x}) =$$

$$125e^{5x} - 50e^{5x} - 75e^{5x} + 27e^{-3x} + 18e^{-3x} - 45e^{-3x} \equiv 0$$

This is the main idea behind a general technique for solving differential equations which have constant coefficients.

5.2.4 Letting $y = e^{rx}$, $y' = re^{rx}$, $y'' = r^2e^{rx}$, we have

$y'' - 7y' + 12y = 0 \longrightarrow r^2e^{rx} - 7re^{rx} + 12e^{rx} = 0 \longrightarrow$

$r^2 - 7r + 12 = 0 \longrightarrow (r - 4)(r - 3) = 0 \longrightarrow r = 3$

or $r = 4$

Hence, $y = c_1e^{3x} + c_2e^{4x}$ is a solution of $y'' - 7y' + 12y = 0$.

Now, $y' = 3c_1e^{3x} + 4c_2e^{4x}$.

$$\therefore \left.\begin{aligned} y(0) &= 1 = c_1 + c_2 \\ y'(0) &= 2 = 3c_1 + 4c_2 \end{aligned}\right\} \left.\begin{aligned} 3 &= 3c_1 + 3c_2 \\ 2 &= 3c_1 + 4c_2 \end{aligned}\right\} \quad c_2 = -1$$

[5.2.4 cont'd]

$$\therefore c_1 = 1 - c_2 = 1 - (-1) = 2$$

$$\therefore \underline{y} = 2e^{3x} - e^{4x}$$

5.2.5 (L) In the last section, we showed that $\lim_{x\to\infty} \frac{\ln x}{x^n} = 0$ if $n > 0$. In essence, this meant that for sufficiently large values of x, x^n "dwarfed" $\ln x$ no matter how small n was chosen, as long as it was positive.

This exercise is meant to establish the "inverse" of the result, namely:

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0 \qquad (1)$$

no matter how large n is!

Rather than prove (1) rigorously, we pick $n = 100$ and ask to show that when x is "sufficiently large" then e^x exceeds x^{100}. (There is no important reason for choosing e in this exercise other than the fact that we are studying e in this section. In general, if $b > 1$ then $\lim_{x\to\infty} \frac{x^n}{b^x} = 0$.)

There are several ways of comparing e^x and x^{100}. The method we shall choose will afford us the opportunity to gain more experience with the use of logarithms. The fact that ln is an increasing function implies that

$$e^x > x^{100} \leftrightarrow \ln e^x > \ln x^{100}$$

$$\leftrightarrow x \ln e > 100 \ln x$$

$$\leftrightarrow x > 100 \ln x$$

[5.2.5 (L) cont'd]

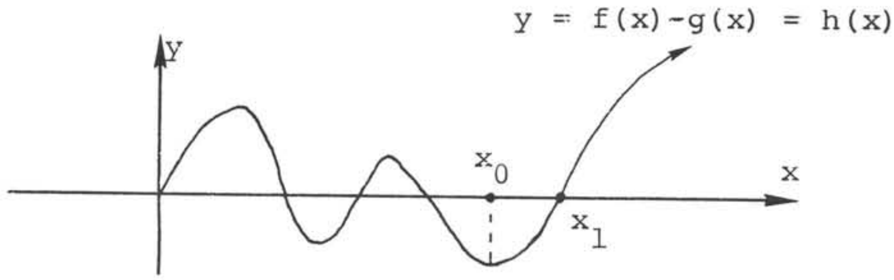
In other words, to find x such that $e^x > x^{100}$ it is <u>necessary</u> <u>and</u> <u>sufficient</u> that we find x such that

$$x > 100 \ln x \qquad (2)$$

To solve (2), we have a particular application of a more general technique which we shall now describe.

If f(x) and g(x) are differentiable functions and we wish to prove that $f(x) > g(x)$, we study $y = f(x) - g(x)$ If, for example, we can show that the minimum value of $f(x) - g(x)$ is positive the result will follow since if the minimum value of $f(x) - g(x)$ is positive, $f(x) - g(x)$ is always positive, and this is precisely the definition that $f(x) > g(x)$.

Of course f(x) might not always exceed g(x). What might happen is that $\frac{d[f(x) - g(x)]}{dx} > 0$ for $x \geq x_0$ where x_0 is some constant. If we can then find $x_1 \geq x_0$ such that $f(x_1) > g(x_1)$ then $f(x) > g(x)$ for all $x \geq x_1$. Pictorially:



$y = f(x)-g(x)$ always rises when $x>x_0$ and it crosses the x-axis when $x=x_1>x_0$

Hence it is above the x-axis (i.e. $f(x)-g(x) > 0$ or $f(x) > g(x)$) whenever $x > x_1$

(Figure 1)

[5.2.5 (L) cont'd]

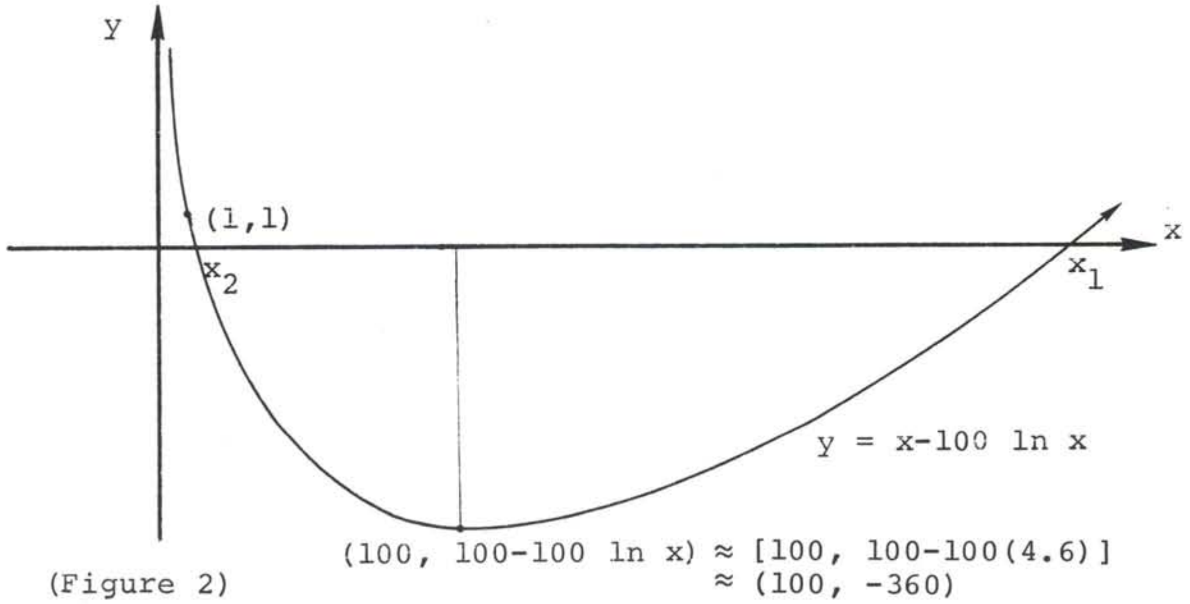
In any event, in our exercise we consider

$$y = x - 100 \ln x \qquad (x > 0) \tag{3}$$

Then $$y' = 1 - \frac{100}{x} \tag{4}$$

and $$y'' = \frac{100}{x^2} \tag{5}$$

From (3), (4), and (5) we see that our curve passes through (1,1), always "holds water" and rises when x > 100 but falls when x < 100. Again, pictorially;



(Figure 2)

Notice that our curve falls quite rapidly at first and reaches its minimum when y is a rather large negative number. This indicates that until x gets "sufficiently large", x is considerably less than 100 ln x. For example when x = <u>10</u> , 100 ln 10 ≈ 100(2.3) ≈ <u>230</u>. Correspondingly, when x = 10:

$$e^x = e^{10} \approx 22{,}026 < 10^5$$

$$x^{100} = 10^{100}$$

[5.2.5 (L) cont'd]

That is, 10^{100} is very much greater than e^{10}.

At any rate, the smallest x beyond which e^x always exceeds x^{100} is x_1 as shown in Figure 2. To locate x_1 analytically we must solve the equation

$$x - 100 \ln x = 0$$

or

$$x = 100 \ln x \tag{6}$$

Again, a glance at Figure 2 indicates that (6) has two solutions, one of which is "near" x = 1.

Equation (6) is known as a transcendental equation and one way of tackling it is through tables. For example when

$$\begin{cases} x = 1.01, \ln x = .01 \therefore 100 \ln x = 1 \therefore x - \ln x = 1.01 - 1 = .01 > 0 \\ x = 1.02, \ln x = .0198 \therefore 100 \ln x = 1.98 \therefore x - \ln x = 1.02 - 1.98 < 0 \end{cases}$$

$\therefore$ x - 100 ln x = 0 (i.e., y = x - 100 ln x crosses the x-axis) between x = 1.01 and 1.02 etc.

For our needs, it is necessary to find the solution of (6) for which x > 100. Again proceeding by tables:

$$x = 1000 \rightarrow \ln x = 6.9078 \rightarrow 100 \ln x = 690.78 \rightarrow x - 100 \ln x > 0$$

$\therefore$ y = x - 100 ln x crosses the x-axis somewhere between x = 100 and x = 1000.

While this estimate is quite crude, we were not required in this exercise to find the smallest x which worked. In other words, while we might like

[5.2.5 (L) cont'd]

a better estimate, it is nonetheless correct to state:

$$x \geqslant 1000 \longrightarrow e^x > x^{100} \qquad (7)$$

There are ways of sharpening the result in (7).
For example, we can find from the table that ln 2 = 0.6931.
Hence, $\ln 2^k = k \ln 2 = 0.6931k$. Letting $x = 2^k$,
(6) assumes the form:

$$\underline{2^k} = 100 \ln 2^k = 100k \ln 2 = \underline{69.31k} \qquad (8)$$

Equation (8) is easy to compute for k = 1, 2, 3,..etc.
For example:

k	2^k		$69.31k$ (= $100 \ln 2^k$)	
1	2	<	69.31	
2	4	<	138.62	
3	8	<	207.93	
4	16	<	277.24	
5	32	<	346.55	
6	64	<	415.86	
7	128	<	485.17	512 < 100 ln 572
8	256	<	554.48	∴ 512 is a lower
9	512	<	623.79	bound on c in
10	1024	>	693.10	this problem.

[5.2.5 (L) cont'd]

If we wanted a better estimate we could now replace (8) by:

$$3^k = 100 \ln 3^k = 100k \ln 3 = 109.86k \qquad (9)$$

Now we know that c > 512, and the first power of 3 that exceeds 512 is $3^6 = 729$. Letting k = 6 in (9), we obtain:

$$3^k = 729$$

$$100 \ln 3^k = 100 \ln 728 = (109.86)6 = 659.16$$

$$\therefore 729 - 100 \ln 729 > 0$$

Hence our minimum c is now given by

$$512 < c < 729$$

We could continue making refinements by appropriate use of ln tables or we could use such analytic devices as Newton's Method, etc. However, we feel that we need not carry this discussion further here in the sense that our main aim was to establish the existence of c as well as to review some fundamental ideas about exponents and logarithms.

5.2.6

$$e^x > x^{1000} \longleftrightarrow$$

$$x > 1000 \ln x$$

Pick $x = 2^k$

[5.2.6 cont'd]

$$2^k > 1000 \ln 2^k = 1000 k \ln 2 = 693.15k$$

k	2^k		693.15k
10	1024	<	6931.5
11	2048	<	7624.65
12	4096	<	8317.80
13	8192	<	9010.95
14	16,384	>	9704.10

$$\therefore e^{16,384} > 16,384^{1000}$$

Check:

$$\left.\begin{array}{l} e^{16,384} > 2^{16,384} \\ (16,384)^{1000} = (2^{14})^{1000} = 2^{14,000} \end{array}\right\} \therefore e^{16,384} > 16,384^{1000}$$

$e^x - x^{1000}$ is an increasing function when $x > 16,384$ (In fact $x - 1000 \ln x = g(x) \longrightarrow g'(x) = 1 - \frac{1000}{x} \longrightarrow$ $g'(x) > 0$ when $x > 1000$). Hence we may choose c = 16,384.

5.2.7 Note: we have not proved that $\ln x^r = r \ln x$ for all real numbers. We have proved the result for any rational number r. The point is that once we know $\ln x^r = r \ln x$ when r is rational, we define $\ln x^r = r \ln x$ for all real numbers.

[5.2.7 cont'd]

a. $e^{2\ln x} = e^{\ln x^2} = x^2$

b. $b^x = e^{\ln b^x} = e^{x\ln b}$

$\therefore \dfrac{d(b^x)}{dx} = \dfrac{d(e^{x\ln b})}{dx} = (e^{x\ln b})\dfrac{d(x\ln b)}{dx}$

$= (e^{x\ln b})\ln b = (e^{\ln b^x})\ln b = (b^x)\ln b$

c. $\int 4^x dx = \int e^{\ln 4^x} dx = \int e^{x\ln 4}\, dx$

$= \dfrac{1}{\ln 4} e^{x\ln 4} + c$

$= \dfrac{1}{\ln 4} e^{\ln 4^x} + c$

$= \dfrac{4^x}{\ln 4} + c$

$$\therefore \int_0^1 4^x dx = \frac{4^x}{\ln 4}\Bigg|_{x=0}^{x=1} = \frac{4^1}{\ln 4} - \frac{4^0}{\ln 4} = \frac{4-1}{\ln 4} = \frac{3}{\ln 4}$$

5.2.8

Recall that if $G(t) = \int_0^t e^{-x^2}dx$ then $G'(t) = e^{-t^2}$.

Hence,

$$\frac{1}{h}\left[\int_3^{3+h} e^{-x^2}dx\right] = \frac{1}{h}\left[\int_0^{3+h} e^{-x^2}dx - \int_0^3 e^{-x^2}dx\right]$$

$$= \frac{1}{h}\left[G(3+h) - G(3)\right]$$

$$= \frac{G(3+h) - G(3)}{h}$$

[5.2.8 cont'd]

$$\therefore \lim_{h\to 0}\left[\frac{1}{h}\int_3^{3+h} e^{-x^2}\,dx\right] = \lim_{h\to 0}\left[\frac{G(3+h)-G(3)}{h}\right]$$

$$= G'(3)$$

But $G'(t) = e^{-t^2} \longrightarrow G'(3) = e^{-3^2} = e^{-9}$

$$\therefore \lim_{h\to 0}\left[\frac{1}{h}\int_3^{3+h} e^{-x^2}\,dx\right] = e^{-9}$$

UNIT 3: The Hyperbolic Functions

5.3.1

a. Our basic identity is $\cosh^2 x - \sinh^2 x = 1$. Since $\sinh x = \frac{5}{12}$ we have:

$$\cosh^2 x - \frac{25}{144} = 1 \quad \text{or} \quad \cosh^2 x = 1 + \frac{25}{144} = \frac{169}{144}$$

$$\therefore \cosh x = \sqrt{\frac{169}{144}} = \frac{13}{12}$$

(Notice that $\cosh x = -\frac{13}{12}$ is excluded since $\cosh x \geqslant 1$ for all x.)

b. Here we use the identity (which we shall derive as an exercise in Exercise 5.3.7(L), but which is stated in the text)

$$\sinh 2x = 2\sinh x \cosh x$$

$$\sinh x = \frac{5}{12} \rightarrow \cosh x = \frac{13}{12} \quad \text{[by part (a)]}$$

$$\therefore \sinh 2x = 2\left(\frac{5}{12}\right)\left(\frac{13}{12}\right) = \frac{65}{72}$$

c. We have $\cosh^2 x - \sinh^2 x \equiv 1$.

Hence, $\sinh^2 x = \cosh^2 x - 1$

Since $\cosh x = \frac{13}{12}$, we obtain

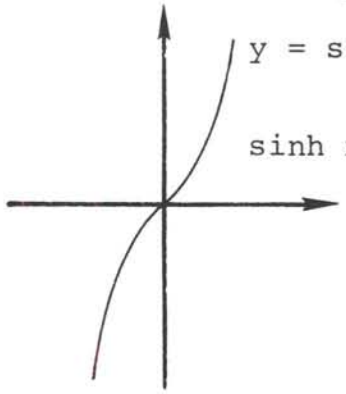
$$\sinh^2 x = \left(\frac{13}{12}\right)^2 - 1 = \frac{169}{144} - 1 = \frac{25}{144}$$

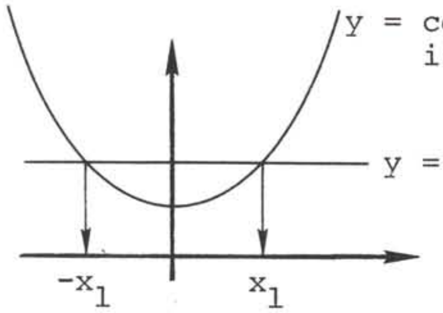
$$\therefore \quad \sinh x = \sqrt{\frac{25}{144}}$$

$$= \pm\frac{5}{12}$$ (but now we cannot disregard the minus sign.)

[5.3.1 cont'd]

Pictorially,





$\sinh x_1 = \frac{5}{12}$

$\sinh(-x_1) = -\frac{5}{12}$

5.3.2

a. $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{(e^x - e^{-x})/2}{(e^x + e^{-x})/2} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$$\therefore \quad 1 + \tanh x = 1 + \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^x + e^{-x} + (e^x - e^{-x})}{e^x + e^{-x}} = \frac{2e^x}{e^x + e^{-x}}$$

$$1 - \tanh x = 1 - \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^x + e^{-x} - (e^x - e^{-x})}{e^x + e^{-x}} \quad \frac{2e^{-x}}{e^x + e^{-x}}$$

$$\therefore \quad \frac{1 + \tanh x}{1 - \tanh x} = \left(\frac{2e^x}{e^x + e^{-x}}\right) \div \left(\frac{2e^{-x}}{e^x + e^{-x}}\right)$$

$$= \left(\frac{2e^x}{e^x + e^{-x}}\right) \cdot \left(\frac{e^x + e^{-x}}{2e^{-x}}\right) = \frac{e^x}{e^{-x}} = e^{2x}$$

$$\therefore \quad \ln\left[\frac{1 + \tanh x}{1 - \tanh x}\right] = \ln e^{2x} = 2x$$

Finally,

$$\frac{1}{2} \ln\left[\frac{1 + \tanh x}{1 - \tanh x}\right] = \frac{1}{2}(2x) = x$$

[5.3.2 cont'd]

b. $\cosh u = \dfrac{e^u + e^{-u}}{2}$ and $\sinh u = \dfrac{e^u - e^{-u}}{2}$ $\rightarrow$ $\sinh u + \cosh u = e^u$ etc. $\cosh u - \sinh u = e^{-u}$

$\therefore$ (i) $$\int \frac{\cosh\theta\, d\theta}{\sinh\theta + \cosh\theta} = \int \frac{\frac{1}{2}[e^\theta + e^{-\theta}]\, d\theta}{e^\theta}$$

$$= \frac{1}{2}\int \left[\frac{e^\theta + e^{-\theta}}{e^\theta}\right] d\theta = \frac{1}{2}\int \left[\frac{e^\theta}{e^\theta} + \frac{e^{-\theta}}{e^\theta}\right] d\theta$$

$$= \frac{1}{2}\int \left[1 + e^{-2\theta}\right] d\theta \qquad = \frac{1}{2}\left[\theta - \frac{1}{2}e^{-2\theta}\right] + c$$

$$= \frac{\theta}{2} - \frac{1}{4}e^{-2\theta} + c$$

or, since $e^{-2\theta} = \cosh 2\theta - \sinh 2\theta$,

$$\int \frac{\cosh\theta\, d\theta}{\sinh\theta + \cosh\theta} = \frac{\theta}{2} + \frac{1}{4}(\cosh 2\theta - \sinh 2\theta) + c$$

(ii) $$\int e^x \sinh 2x\, dx = \int e^x \left[\frac{e^x - e^{-x}}{2}\right] dx$$

$$= \frac{1}{2}\int (e^{2x} - 1)\, dx$$

$$= \frac{1}{2}\left[\frac{1}{2}e^{2x} - x\right] + c$$

$$= \frac{1}{4}e^{2x} - \frac{x}{2} + c$$

(iii) We observe that $\dfrac{e^x + e^{-x}}{2} = \dfrac{e^x + \frac{1}{e^x}}{2} = \dfrac{e^{2x} + 1}{2e^x}$

while $\dfrac{e^x - e^{-x}}{2} = \dfrac{e^x - \frac{1}{e^x}}{2} = \dfrac{e^{2x} - 1}{2e^x}$

[5.3.2 cont'd]

Hence:

$$\int \frac{e^{2x} - 1}{e^{2x} + 1}\,dx = \int \frac{(e^{2x} - 1)/2e^x}{(e^{2x} + 1)/2e^x}\,dx$$

$$= \int \frac{\sinh x}{\cosh x}\,dx$$

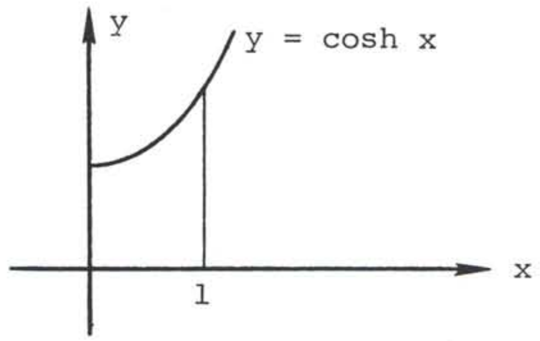
Letting $v = \cosh x$, $dv = \sinh x\,dx$, we obtain:

$$\int \frac{e^{2x} - 1}{e^{2x} + 1}\,dx = \int \frac{\sinh x\,dx}{\cosh x} = \int \frac{dv}{v} = \ln|v| + c$$

$$= \ln|\cosh x| + c$$

$$= \ln(\cosh x) + c \quad \text{(since } \cosh x > 0 \text{ for all } x)$$

5.3.3



$$s = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Now $y = \cosh x \rightarrow \frac{dy}{dx} = \sinh x \rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2 x$

But $\cosh^2 x - \sinh^2 x \equiv 1 \rightarrow 1 + \sinh^2 x = \cosh^2 x$

$\therefore \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \sinh^2 x} = \sqrt{\cosh^2 x} = \cosh x$

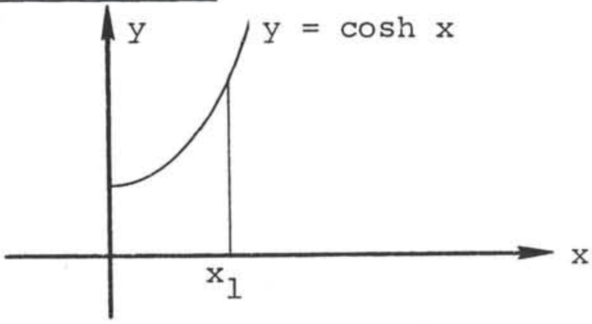
[5.3.3 cont'd]

$$\therefore \quad s = \int_0^1 \cosh x \, dx = \sinh 1 - \sinh 0 = \sinh 1$$

$$= \frac{e^1 - e^{-1}}{2} = \frac{e - \frac{1}{e}}{2}$$

$$= \frac{e^2 - 1}{2e}$$

Generalization



$$s = \int_0^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_0^{x_1} \cosh x \, dx = \sinh x_1$$

$$= \frac{e^{x_1} - \frac{1}{e^{x_1}}}{2} = \frac{e^{2x_1} - 1}{2e^{x_1}}$$

In other words, the arc length of $y = \cosh x$ between 0 and x_1 is simply $\sinh x_1$.

5.3.4 Here we are merely applying the first fundamental theorem of integral calculus to a "new" function. Namely, if we let $f(x) = \cosh x$, we have:

[5.3.4 cont'd]

$$\int_0^1 \cosh x\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\cosh\left(\frac{k}{n}\right)$$

$$= \lim_{n\to\infty}\left[\frac{\cosh\frac{1}{n} + \cosh\frac{2}{n} + \cdots + \cosh\frac{n}{n}}{n}\right]$$

But $\displaystyle\int_0^1 \cosh x\,dx = \sinh x\Big|_0^1 = \sinh 1 - \sinh 0 = \sinh 1$

$$= \frac{e^2 - 1}{2e}$$

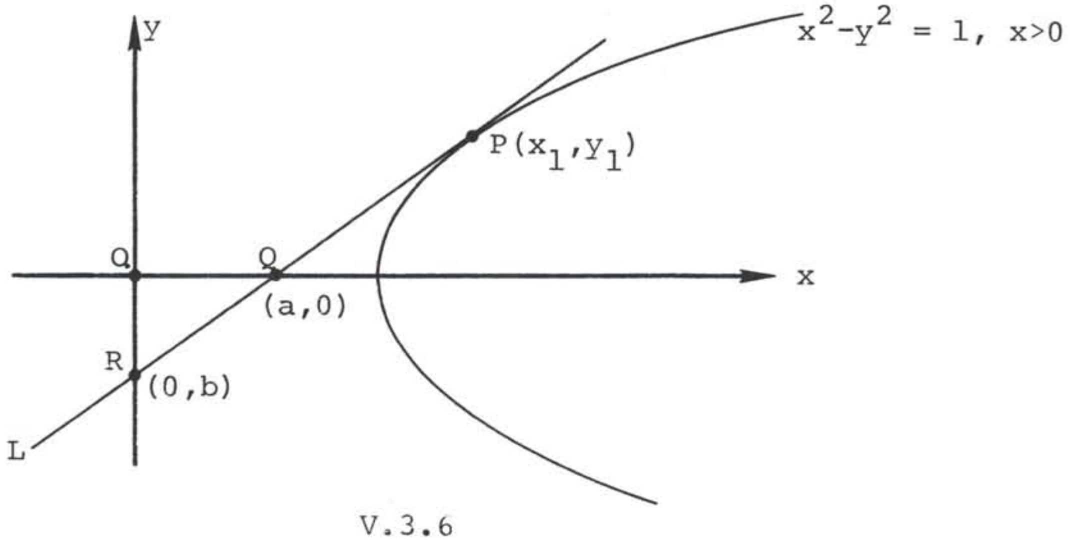
Hence:

$$\lim_{n\to\infty}\left[\frac{\cosh\frac{1}{n} + \cosh\frac{2}{n} + \cdots + \cosh\frac{n}{n}}{n}\right]$$

$$= \frac{e^2 - 1}{2e} = \sinh 1$$

5.3.5 (L) The main (perhaps, only) reason for calling this a learning exercise is so that we can emphasize the fact that we can construct the hyperbolic functions geometrically, just as we can the circular functions. To be sure, circles "behave more nicely" than hyperbolas, but this is hardly the important part.

a.



[5.3.5 (L) cont'd]

Our line L passes through (x_1, y_1) on $x^2 - y^2 = 1$; (hence, $x_1^2 - y_1^2 = 1$). Moreover, $x^2 - y^2 = 1 \longrightarrow$

$$2x - 2y\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = \frac{x}{y}$$

$$\therefore \quad m_L = \left.\frac{dy}{dx}\right|_{\substack{x=x_1 \\ y=y_1}} = \frac{x_1}{y_1}$$

Thus, the equation of L is given by

$$\frac{y - y_1}{x - x_1} = m_L = \frac{x_1}{y_1}$$

$$\therefore \quad (y - y_1)y_1 = (x - x_1)x_1 \tag{1}$$

To find where the line L intersects the x-axis, we merely set $y = 0$ in (1) and solve for x. (Notice how the theory is exactly the same as it was in Block I, the only difference being that we can now handle more classes of functions.)

$$\therefore \quad (0 - y_1)y_1 = (x - x_1)x_1$$

$$-y_1^2 = x\,x_1 - x_1^2$$

$$\text{or} \qquad x = \frac{x_1^2 - y_1^2}{x_1}$$

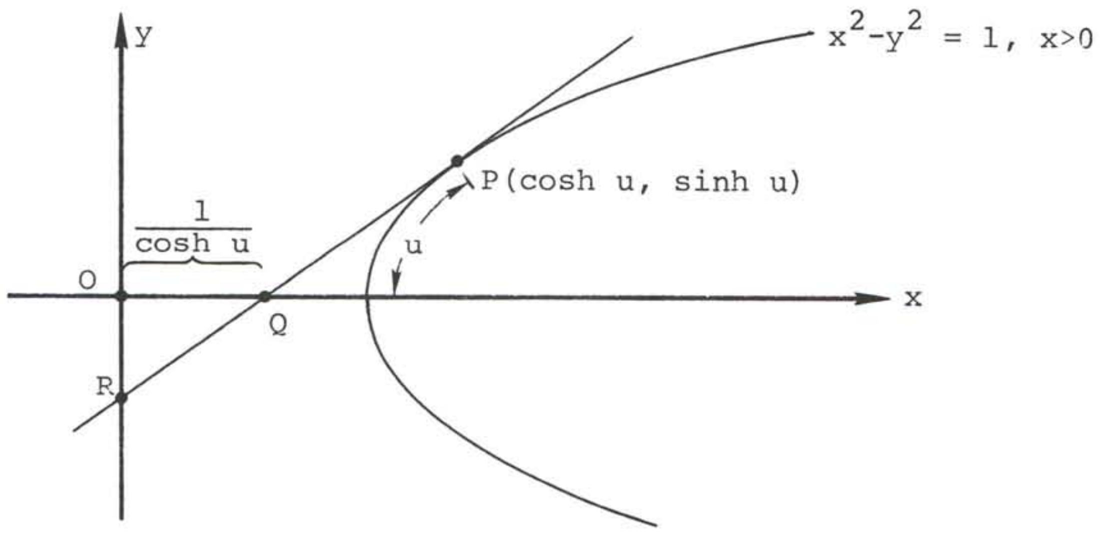
and since $x_1^2 - y_1^2 = 1$, we obtain:

[5.3.5 (L) cont'd]

$$x = \frac{1}{x_1} \qquad (2)$$

(An interesting aside is that since $x_1 \geqslant 1$, (2) shows us that the x-intercept of L must lie between 0 and 1. In other words, the line tangent to the curve $x^2 - y^2 = 1$, $x > 0$, at any point intersects the x-axis to the right of the y-axis.)

b. If we recall that $\begin{cases} x = \cosh u \\ y = \sinh u \end{cases}$ is the parametric form of $x^2 - y^2 = 1$, we see that $\frac{1}{x} = \frac{1}{\cosh u} = \text{sech } u$. In other words:



Hence, sech u is precisely the length of OQ.

5.3.6 Here we repeat the procedure of Exercise 5.3.5(L), through equation (1) of our solution. That is,

$$(y - y_1)y_1 = (x - x_1)x_1 \tag{1}$$

We now find the y-intercept of L by letting $x = 0$ in (1) and solving for y. Thus,

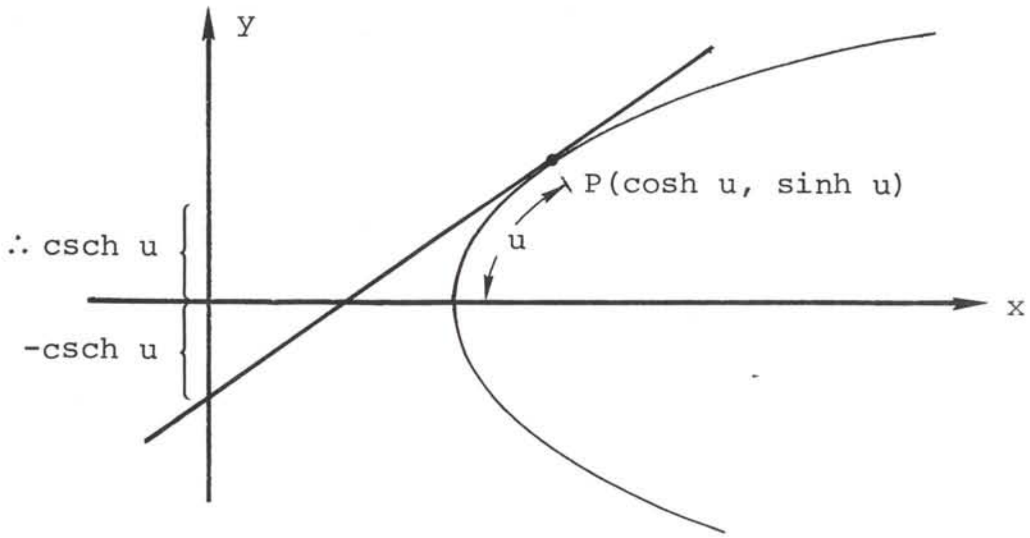
$$(y - y_1)y_1 = (-x_1)x_1$$

$$y\,y_1 - {y_1}^2 = -{x_1}^2$$

$$\therefore \quad -y\,y_1 = {x_1}^2 - {y_1}^2 = 1$$

$$\therefore \quad y = -\frac{1}{y_1},$$ (With respect to the diagrams in Exercise 5.3.5 (L), $(0,-\frac{1}{y_1})$ is the point we labeled R)

Now, since y_1 is sinh u, $-\frac{1}{y_1}$ is -csch u. Thus the y-intercept of the tangent line is the negative of csch u. Pictorially,



[5.3.6 cont'd]

(If u is negative the result still holds. Notice that the minus sign simply tells us that the y-intercept is on the opposite side of the x-axis from P.)

5.3.7 (L) [In our lecture, we pointed out the resemblance of $x^2 + y^2 = 1$ and $x^2 - y^2 = 1$ in terms of complex numbers. The result mentioned in this exercise has as its analog, in the complex numbers, DeMoivre's Theorem. Namely:

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx$$]

In general,

$$\cosh u + \sinh u = \frac{e^u + e^{-u}}{2} + \frac{e^u - e^{-u}}{2} = e^u \quad (1)$$

This is the fundamental building block in this exercise. For example,

a. From (1) $\cosh x + \sinh x = e^x$

$$\therefore \quad (\cosh x + \sinh x)^n = e^{nx} \quad (2)$$

But letting u = nx in (1), we have:

$$e^{nx} = \cosh nx + \sinh nx \quad (3)$$

[5.3.7 (L) cont'd]

Substituting (3) for (2) yields:

$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$$

b. We repeat the procedure in (a) as follows: Let $u = -x$ in (1). Then:

$$\cosh(-x) + \sinh(-x) = e^{-x} \tag{4}$$

Since $\cosh x = \cosh(-x)$ (i.e., $\cosh x$ is an even function)
and $\sinh(-x) = -\sinh x$ (i.e., $\sinh x$ is an odd function), (4) becomes

$$\cosh x - \sinh x = e^{-x}$$

Whereupon:

$$(\cosh x - \sinh x)^n = (e^{-x})^n = e^{-nx} \tag{5}$$

If we now let $u = -nx$ in (1), we obtain $\cosh(-nx) + \sinh(-nx) = e^{-nx}$ or:

$$\cosh nx - \sinh nx = e^{-nx} \tag{6}$$

Substituting (6) into (5),

$$(\cosh x - \sinh x)^n = \cosh nx - \sinh nx$$

[5.3.7 (L) cont'd]

c. With n = 2, (a) yields

$$(\cosh x + \sinh x)^2 = \cosh 2x + \sinh 2x \qquad (7)$$

while (b) yields

$$(\cosh x - \sinh x)^2 = \cosh 2x - \sinh 2x \qquad (8)$$

Adding (7) and (8) yields:

$$2 \cosh 2x = (\cosh x + \sinh x)^2 + (\cosh x - \sinh x)^2$$

$$= 2 \cosh^2 x + 2 \sinh^2 x$$

$\therefore \quad \cosh 2x = \cosh^2 x + \sinh^2 x$ (Notice that this isn't quite the same as the corresponding circular function identity.)

Similarly, subtracting (8) from (7) yields:

$$2 \sinh 2x = (\cosh x + \sinh x)^2 - (\cosh x - \sinh x)^2$$

$$= 4 \sinh x \cosh x$$

$$\therefore \sinh 2x = 2 \sinh x \cosh x$$

<u>5.3.8</u> From the previous exercise we have:

$$\cosh 3x + \sinh 3x = (\cosh x + \sinh x)^3$$

$$= \cosh^3 x + 3 \cosh^2 x \sinh x + 3 \cosh x \sinh^2 x + \sinh^3 x \qquad (1)$$

[5.3.8 cont'd]

and:

$$\cosh 3x - \sinh 3x = (\cosh x - \sinh x)^3$$

$$= \cosh^3 x - 3 \cosh^2 x \sinh x + 3 \cosh x \sinh^2 x - \sinh^3 x \quad (2)$$

Adding (1) and (2):

$$2 \cosh 3x = 2 \cosh^3 x + 6 \cosh x \sinh^2 x$$

or

$$\cosh 3x = \cosh^3 x + 3 \cosh x \sinh^2 x$$

Similarly, subtracting (2) from (1):

$$2 \sinh 3x = 6 \cosh^2 x \sinh x + 2 \sinh^3 x$$

or:

$$\sinh 3x = 3 \cosh^2 x \sinh x + \sinh^3 x$$

UNIT 4: The Inverse Hyperbolic Functions

5.4.1(L) A major aim of this exercise is to emphasize the difference between $y = \cosh^{-1} x$ and $x = \cosh y$ (see diagrams at the end of this solution).

For example $(2, \ln[2-\sqrt{3}])$ doesn't belong to $y = \cosh^{-1} x$ even though it does belong to $x = \cosh y$. The reason is that $y = \cosh^{-1} x$ was defined to be $x = \cosh y$ where $y > 0$. However, $\ln(2 - \sqrt{3}) < \ln 1 \therefore \ln(2 - \sqrt{3}) < 0$.

To find $\frac{dy}{dx}$ in this case, we mimic the procedure used in the lecture to find $\frac{dy}{dx}$ when $y = \cosh^{-1} x$. By way of review, we have

$$x = \cosh y \qquad y < 0$$

$$\therefore \quad \frac{dx}{dy} = \sinh y$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sinh y}$$

Now, $\cosh^2 y - \sinh^2 y = 1$ implies that

$$\sinh y = \pm \sqrt{\cosh^2 - 1}$$

and since $x = \cosh y$, $\sinh y = \pm\sqrt{x^2 - 1}$
However, $\sinh y$ and y have the same sign
$\therefore$ Since $y < 0$, $\sinh y < 0$ and hence the plus sign doesn't apply (in the lecture, since $y > 0$, the minus sign was discarded) $\therefore \sinh y = -\sqrt{x^2 - 1}$

[5.4.1 (L) cont'd]

and:

$$\frac{dy}{dx} = \frac{-1}{\sqrt{x^2 - 1}}$$

$$\left.\frac{dy}{dx}\right|_{x=2} = \frac{-1}{\sqrt{3}}$$

$\therefore \dfrac{y - \ln(2 - \sqrt{3})}{x - 2} = -\dfrac{1}{\sqrt{3}}$ is the equation of the desired line.

Simplified:

$$\sqrt{3}\left(y - \ln(2 - \sqrt{3})\right) = -(x - 2)$$

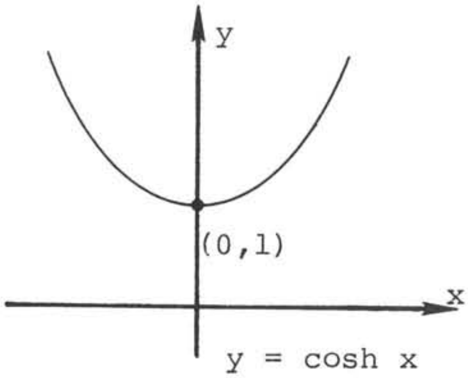
or

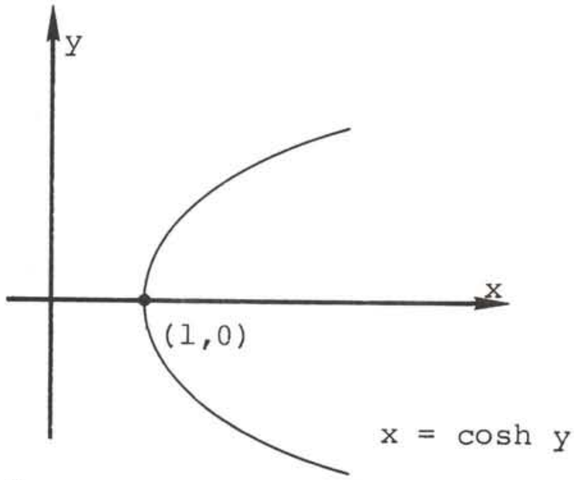
$$x + \sqrt{3}y = 2 + \sqrt{3}\ln(2 - \sqrt{3})$$

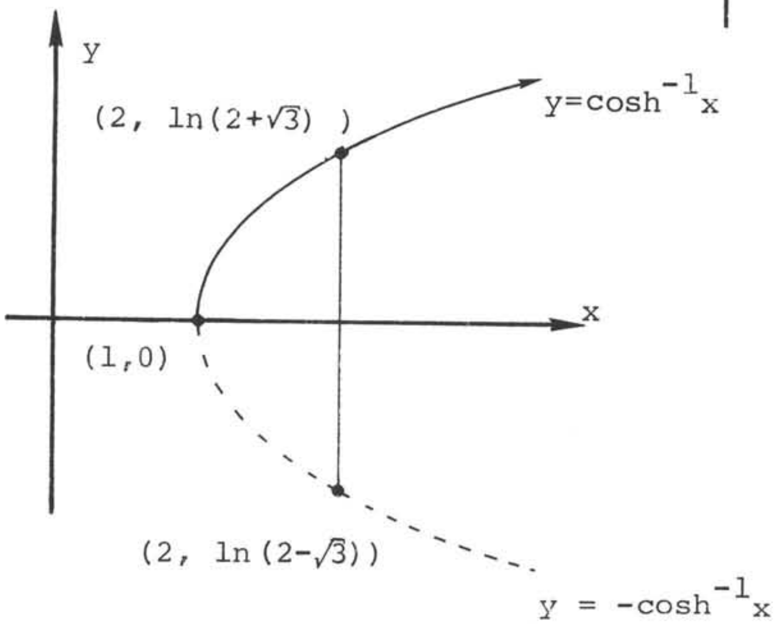
It is most important to see that we would obtain the incorrect answer had we used $y = \cosh^{-1} x$. For not only is the given point not on $y = \cosh^{-1} x$, but the "recipe" $\dfrac{dy}{dx} = +\dfrac{1}{\sqrt{x^2 - 1}}$ also doesn't apply.

Another way of seeing this is to observe that $x = \cosh y$ means $y = \pm\cosh^{-1} x$ and the minus sign is used when y is negative while the plus sign is used when y is positive.

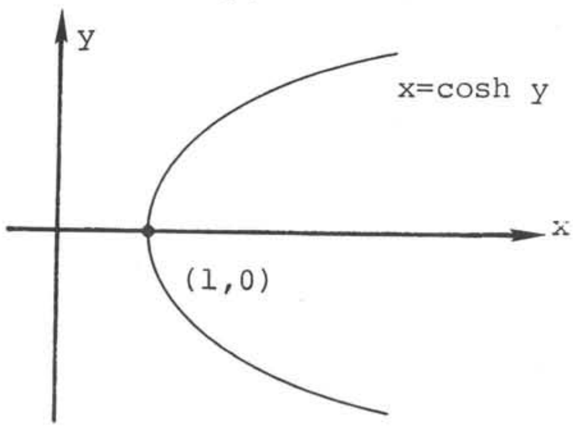
[5.4.1 (L) cont'd]







<u>5.4.2 (L)</u> If we look at $x = \cosh y$, we find



(Figure 1)

[5.4.2 (L) cont'd]

and this shows that for $x > 1$ we have a double valued function $y = \pm \cosh^{-1} x$.

Now the formula

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$$

was derived on the basis that $\cosh^{-1} x > 0$. The point is that if $x = \cosh y$, then

$$x = \frac{e^y + e^{-y}}{2} \quad \text{or} \quad 2x = e^y + \frac{1}{e^y}$$

$$\therefore\ (e^y)^2 - 2xe^y + 1 = 0$$

or:

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

$$\therefore \quad y = \ln(x \pm \sqrt{x^2 - 1} \qquad (1)$$

Equation (1) represents two curves. Namely,

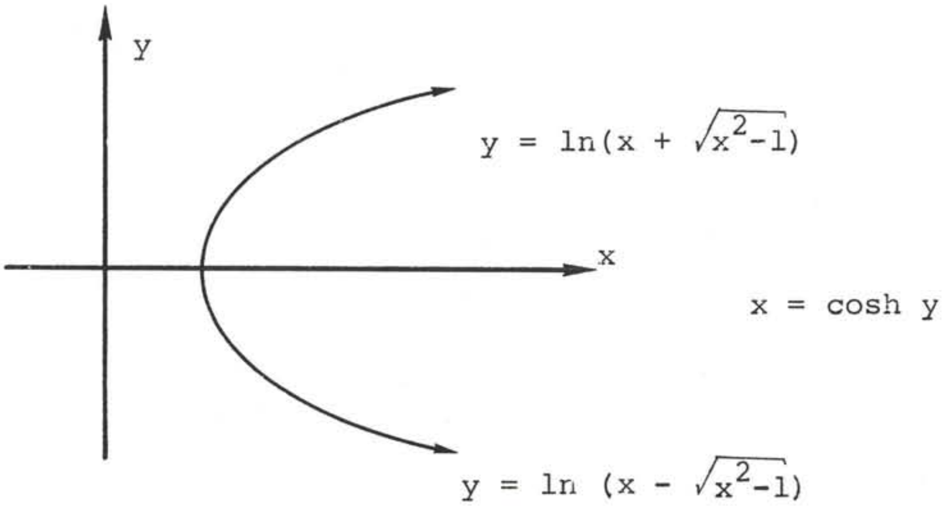
$$y = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad (2)$$

and

$$y = \ln\left(x - \sqrt{x^2 - 1}\right) \qquad (3)$$

Correlating (2) and (3) with Figure 1 (since $x + \sqrt{x^2 - 1} > x - \sqrt{x^2 - 1}$) it appears that:

[5.4.2 (L) cont'd]



As a final check, the lower branch of x = cosh y, by symmetry, should also be given by:

$$y = -\ln (x + \sqrt{x^2 - 1})$$

Observe that:

$$-\ln(x + \sqrt{x^2 - 1}) = \ln(x + \sqrt{x^2 - 1})^{-1}$$

$$= \ln \left[\frac{1}{x + \sqrt{x^2 - 1}} \right]$$

$$= \ln \left[\frac{x - \sqrt{x^2 - 1}}{(x + \sqrt{x^2 - 1})(x - \sqrt{x^2 - 1})} \right]$$

[5.4.2 (L) cont'd]

$$= \ln\left[\frac{x - \sqrt{x^2 - 1}}{x^2 - (x^2 - 1)}\right]$$

$$= \ln(x - \sqrt{x^2 - 1})$$

5.4.3

$$y = \operatorname{sech}^{-1} x$$

$$x = \operatorname{sech} y = \frac{1}{\cosh y} \qquad y > 0$$

$$\left(1 \leqslant \cosh y < \infty \rightarrow \frac{1}{1} \geqslant \frac{1}{\cosh y} > \text{"}\frac{1}{\infty}\text{" or } 0 < x \leqslant 1\right)$$

$$\therefore\ x = \left[\frac{e^y + e^{-y}}{2}\right]^{-1} = \frac{2}{e^y + \frac{1}{e^y}} = \frac{2e^y}{e^{2y} + 1}$$

$$\therefore\ xe^{2y} + x = 2e^y$$

$$\therefore\ x(e^y)^2 - 2e^y + x = 0$$

$$\therefore\ e^y = \frac{2 \pm \sqrt{4 - 4x^2}}{2x}$$

$$= \frac{1 \pm \sqrt{1 - x^2}}{x}$$

$$y = \ln\left(\frac{1 \pm \sqrt{1 - x^2}}{x}\right) \qquad \text{and since } y > 0:$$

[5.4.3 cont'd]

$$y = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right)^{*}$$

Alternate Method

We already know that $\cosh^{-1} u = \ln(u + \sqrt{u^2 - 1})$, $u \geq 1$. Now

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x} \quad {}^{**}$$

$$\therefore \operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x} = \ln\left[\frac{1}{x} + \sqrt{\left(\frac{1}{x}\right)^2 - 1}\right] \qquad \frac{1}{x} \geq 1$$

$$= \ln\left[\frac{1}{x} + \frac{\sqrt{1 - x^2}}{x}\right] \qquad 0 < x \leq 1$$

$$= \ln\left[\frac{1 + \sqrt{1 - x^2}}{x}\right]$$

---

*Note that, just as in the previous exercise,

$$-\ln\left[\frac{1 + \sqrt{1 - x^2}}{x}\right] = \ln\left[\frac{x}{1 + \sqrt{1 - x^2}}\right]$$

$$= \ln\left[\frac{x(1 - \sqrt{1 - x^2})}{(1 + \sqrt{1 - x^2})(1 - \sqrt{1 - x^2})}\right]$$

$$= \ln\left[\frac{x(1 - \sqrt{1 - x^2})}{1 - (1 - x^2)}\right]$$

$$= \ln\left[\frac{1 - \sqrt{1 - x^2}}{x}\right]$$

**Let $y = \operatorname{sech}^{-1} x$. Then $\operatorname{sech} y = x = \dfrac{1}{\cosh y}$

$\therefore \cosh y = \dfrac{1}{x} \qquad \therefore y = \cosh^{-1} \dfrac{1}{x}$

5.4.4 (L)

a. $\displaystyle\int_1^b \left(\frac{1}{\sqrt{x^2 - 1}} - \frac{1}{x}\right) dx =$

$$\int_1^b \frac{dx}{\sqrt{x^2 - 1}} - \int_1^b \frac{dx}{x} =$$

$$\cosh^{-1} x \Big|_1^b - \ln x \Big|_1^b = \cosh^{-1} b - \cosh^{-1} 1 - \ln b + \ln 1$$

$$\cosh^{-1} b - \ln b \qquad (\text{since } \cosh^{-1} 1 = \ln 1 = 0)$$

b. The major problem here is that $\lim_{b\to\infty}(\cosh^{-1} b - \ln b)$ has the indeterminate form $\infty - \infty$. What we do here is rewrite $\cosh^{-1} b$ as a natural logarithm.
In particular,

$$\cosh^{-1} b = \ln(b + \sqrt{b^2 - 1})$$

$$\therefore \cosh^{-1} b - \ln b = \ln(b + \sqrt{b^2 - 1}) - \ln b$$

$$= \ln\left[\frac{b + \sqrt{b^2 - 1}}{b}\right]$$

$$= \ln\left[1 + \sqrt{1 - \frac{1}{b^2}}\right]$$

$$\therefore \lim_{b\to\infty}\left[\cosh^{-1} b - \ln b\right] = \lim_{b\to\infty}\left[\ln\left(1 + \sqrt{1 - \frac{1}{b^2}}\right)\right]$$

$$= \ln(1 + 1) = \ln 2$$

[5.4.4 (L) cont'd]

$$\therefore \lim_{b\to\infty} \int_1^b \left(\frac{1}{\sqrt{x^2 - 1}} - \frac{1}{x}\right) dx = \ln 2$$

5.4.5

$$\int_1^b \left(\frac{1}{\sqrt{1 + x^2}} - \frac{1}{x}\right) dx =$$

$$\sinh^{-1} x - \ln x \Big|_1^b =$$

$$\sinh^{-1} b - \ln b - \sinh^{-1} 1$$

Now $\sinh^{-1} u = \ln(u + \sqrt{u^2 + 1})$

$$\therefore \begin{cases} \sinh^{-1} b = \ln(b + \sqrt{b^2 + 1}) \\ \sinh^{-1} 1 = \ln(1 + \sqrt{2}) \end{cases}$$

$$\therefore \int_1^b \left(\frac{1}{\sqrt{1 + x^2}} - \frac{1}{x}\right) dx = \ln(b + \sqrt{b^2 + 1}) - \ln b - \ln(1 + \sqrt{2})$$

$$= \ln\left[\frac{b + \sqrt{b^2 + 1}}{b}\right] - \ln(1 + \sqrt{2})$$

$$= \ln\left[1 + \sqrt{1 + \frac{1}{b^2}}\right] - \ln(1 + \sqrt{2})$$

[5.4.5 cont'd]

$$\therefore \lim_{b\to\infty}\int_1^b \left(\frac{1}{\sqrt{1+x^2}} - \frac{1}{x}\right) dx = \lim_{b\to\infty}\left[\ln\left(1+\sqrt{1+\frac{1}{b^2}}\right)\right] - \ln(1+\sqrt{2})$$

$$= \ln 2 - \ln(1+\sqrt{2})$$

$$= \ln\left(\frac{2}{1+\sqrt{2}}\right)$$

5.4.6

a. $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$

$$\therefore e^{\sinh^{-1} x} = e^{\ln(x+\sqrt{x^2+1})}$$

$$= x + \sqrt{x^2+1}$$

$$\therefore e^{\sinh^{-1} x} - \sqrt{x^2+1} = x$$

b. $y = \sinh^{-1} (\tan x)$

Now $y = \sinh^{-1} u \rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$

Letting $u = \tan x$, we have:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+\tan^2 x}}\sec^2 x \quad (1)$$

But $\sec^2 x - \tan^2 x \equiv 1 \longrightarrow 1 + \tan^2 x = \sec^2 x$

$$\therefore \frac{dy}{dx} = \frac{1}{\sec x}\sec^2 x = \sec x \quad (2)$$

[5.4.6 cont'd]

However, in (1) $\frac{dy}{dx}$ must be non-negative (since $\sec^2 x$ and $\sqrt{1 + \tan^2 x}$ are)

$$\therefore \frac{dy}{dx} = |\sec x|$$

c. $$f(x) = \int \frac{e^x\, dx}{\sqrt{1 + e^{2x}}}$$

Let $u = e^x$ $\therefore du = e^x\, dx$, $1 + e^{2x} = 1 + (e^x)^2 = 1 + u^2$

$$\therefore \int \frac{e^x\, dx}{\sqrt{1 + e^{2x}}} = \int \frac{du}{\sqrt{1 + u^2}} = \sinh^{-1} u + c$$

$$= \sinh^{-1} (e^x) + c \qquad (\text{since } u = e^x)$$

5.4.7

a. $y = \frac{1}{1 - x^2}$ $\longrightarrow$ curve is symmetric with respect to y-axis since $f(x) = f(-x)$

$\therefore$ We need only concentrate on $x > 0$
Also denominator equals 0 when $x = 1$ $\therefore$ $x = 1$ is forbidden.

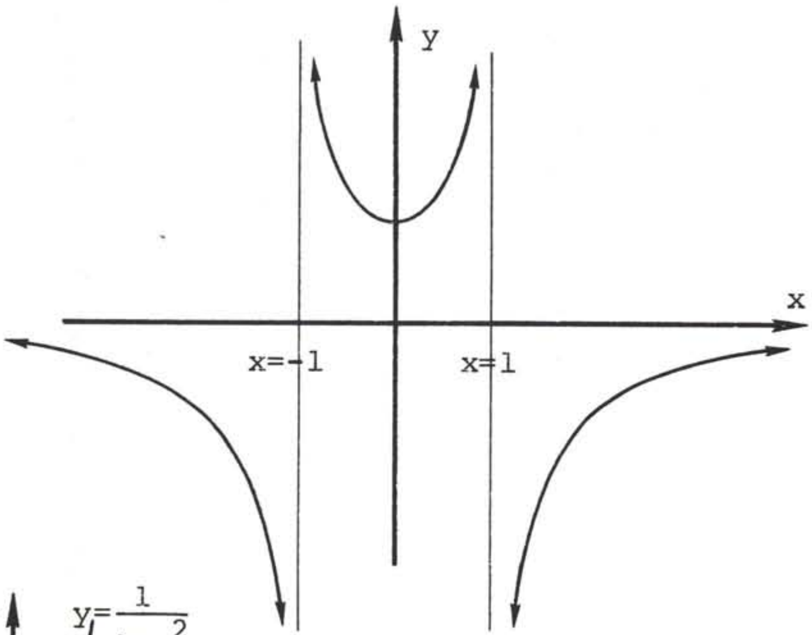
$$y' = \frac{(1 - x^2)\frac{d(1)}{dx} - 1(-2x)}{(1 - x^2)^2} = \frac{2x}{(1 - x^2)^2} \quad (> 0 \text{ if } x > 0)$$

[5.4.7 cont'd]

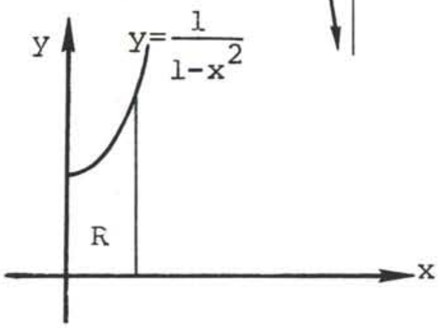
$$y'' = \frac{(1 - x^2)^2\, 2 - 2x[2(1 - x^2)(-2x)]}{(1 - x^2)^4}$$

$$= \frac{2(1 - x^2)[(1 - x^2) + 4x^2]}{(1 - x^2)^4}$$

$$= \frac{2(1 + 3x^2)}{(1 - x^2)^3} \quad \begin{cases} > 0 \text{ if } |x| < 1 \\ < 0 \text{ if } |x| > 1 \end{cases}$$



b.



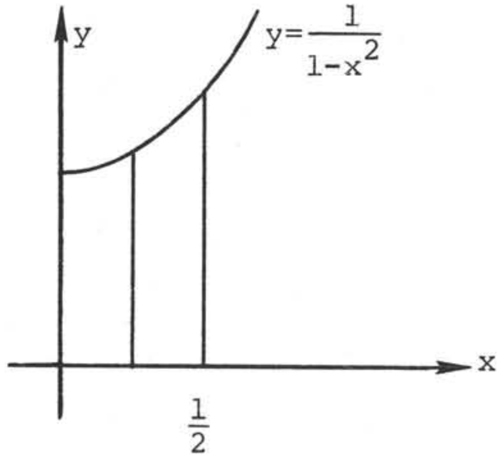
$$A_R = \int_0^{\frac{1}{2}} \frac{dx}{1 - x^2} = \tanh^{-1} x \Big|_0^{\frac{1}{2}}$$

[5.4.7 cont'd]

$$= \tanh^{-1}\frac{1}{2} - \tanh^{-1} 0 = \frac{1}{2} \ln\left[\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}}\right] - \frac{1}{2} \ln\left[\frac{1 + 0}{1 - 0}\right]$$

$$= \frac{1}{2} \ln \left[\frac{\frac{3}{2}}{\frac{1}{2}}\right] = \frac{1}{2} \ln 3 \approx 0.5493$$

c.



$$V_y = \int_0^{\frac{1}{2}} 2\pi\ xy\ dx$$

$$= 2\pi\int_0^{\frac{1}{2}} \frac{x}{1 - x^2}\ dx$$

$$= \pi \left[-\ln(1 - x^2)\right|_0^{\frac{1}{2}}$$

$$= \pi\ [-\ln(1 - \tfrac{1}{4})]$$

$$= \pi\ [-\ln \tfrac{3}{4}] = \pi \ln \tfrac{4}{3} \approx 0.9022$$

QUIZ

5.Q.1

a. $y = \ln^2(x^3 + 1)$ means:

$$y = \left[\ln(x^3 + 1)\right]^2$$

$$\therefore \frac{dy}{dx} = 2\ln(x^3 + 1)\frac{d}{dx}[\ln(x^3 + 1)]$$

$$= 2\ln(x^3 + 1)\left[\frac{3x^2}{x^3 + 1}\right]$$

$$= \frac{6x^2}{x^3 + 1}\ \ln(x^3 + 1)$$

b. If $y = \tanh^{-1} u$ then $\frac{dy}{du} = \frac{1}{1 - u^2}$

or:

$$\frac{dy}{dx} = \left(\frac{1}{1 - u^2}\right)\frac{du}{dx}$$

In (b), $u = \sin x$; hence:

$$\frac{dy}{dx} = \left[\frac{1}{1 - \sin^2 x}\right] \cos x$$

$$= \left[\frac{1}{\cos^2 x}\right] \cos x$$

$$= \sec x$$

[5.Q.1 cont'd]

c. $$y = x^{\sin x} \longrightarrow \ln y = \sin x \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \sin x \frac{d(\ln x)}{dx} + \ln x \frac{d(\sin x)}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\sin x}{x} + \ln x \cos x$$

$$\frac{dy}{dx} = y\left[\frac{\sin x}{x} + \ln x \cos x\right]$$

$$= x^{\sin x}\left[\frac{\sin x}{x} + \ln x \cos x\right]$$

5.Q.2

a. Let $u = 1 + \tan x$

Then $du = \sec^2 x\, dx$

Therefore:

$$\int \frac{\sec^2 x\, dx}{1 + \tan x} = \int \frac{du}{u}$$

$$= \ln |u| + c$$

$$= \ln |1 + \tan x| + c$$

b. Let $u = \sqrt{x}$ $(= x^{\frac{1}{2}})$

Then $du = \frac{1}{2} x^{-\frac{1}{2}} dx$ or $2\, du = \frac{dx}{\sqrt{x}}$

Therefore:

[5.Q.2 cont'd]

$$\int \frac{\sinh \sqrt{x}\, dx}{\sqrt{x}} = \int (\sinh \sqrt{x}) \left(\frac{dx}{\sqrt{x}}\right)$$

$$= \int (\sinh u)(2\, du)$$

$$= 2 \int \sinh u\, du$$

$$= 2 \cosh u + c$$

$$= 2 \cosh \sqrt{x} + c$$

c.

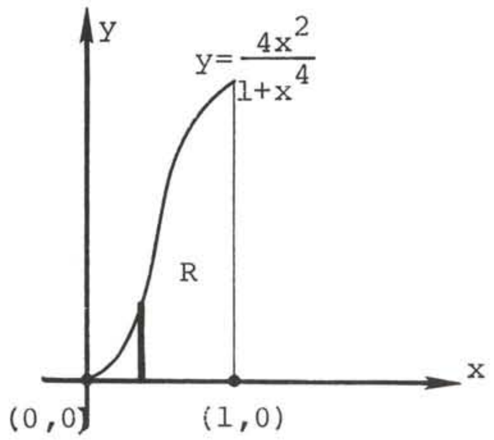
$$\int \frac{dx}{1 - u^2} = \tanh^{-1} u + c \quad \text{if } |u| < 1$$

$$\therefore \int_0^{\frac{1}{2}} \frac{dx}{1 - x^2} = \tanh^{-1} x \Big|_0^{\frac{1}{2}}$$

$$= \tanh^{-1} \tfrac{1}{2} - \tanh^{-1} 0$$

$$= \tanh^{-1} \tfrac{1}{2} \quad (= 0.5493)$$

5.Q.3



[5.Q.3 cont'd]

$$V = 2\pi \int_0^1 xy\, dx$$

$$= 2\pi \int_0^1 x \frac{4x^2}{x^4 + 1} dx$$

$$= 2\pi \int_{x=0}^1 \frac{4x^3}{x^4 + 1} dx$$

Letting $u = x^4 + 1$, we have: $du = 4x^3$

$$\therefore V = 2\pi \int_{u=1}^2 \frac{du}{u} = 2\pi \ln |u| \Big|_1^2 = 2\pi \ln 2$$

<u>5.Q.4</u> We let $y = e^{rx}$

Then: $y' = re^{rx}$, $y'' = r^2 e^{rx}$

Therefore:

$$y'' - 8y' + 7y = 0$$

$$r^2 e^{rx} - 8re^{rx} + 7e^{rx} = 0$$

$$e^{rx}(r^2 - 8r + 7) = 0$$

$$r^2 - 8r + 7 = 0$$

$$(r - 7)(r - 1) = 0$$

$$r = 1 \text{ or } r = 7$$

Hence:

[5.Q.4 cont'd]

$y = e^x$ and $y = e^{7x}$ are solutions of $y'' - 8y' + 7y = 0$

Therefore $y = C_1e^x + C_2e^{7x}$ is also a solution (1)

From $y = C_1e^x + C_2e^{7x}$ it follows that

$$y' = C_1e^x + 7C_2e^{7x} \quad (2)$$

From (1), the fact that $y(0) = 0$ implies that

$$0 = C_1 + C_2 \quad (3)$$

While from (2), we have

$$y'(0) = C_1 + 7C_2$$

Hence $y'(0) = 12$ implies

$$12 = C_1 + 7C_2 \quad (4)$$

Solving (3) and (4) simultaneously yields

$$C_2 = 2 \quad , \quad C_1 = -2$$

and putting this into (1) yields

$$y = -2e^x + 2e^{7x} = 2(e^{7x} - e^x)$$

5.Q.5 We have:

$$(\cosh x + \sinh x)^4 = \cosh 4x + \sinh 4x$$

[5.Q.5 cont'd]

$$(\cosh x - \sinh x)^4 = \cosh 4x - \sinh 4x \quad [\text{since } \sinh(-u) = -\sinh u]$$

$$\therefore (\cosh x + \sinh x)^4 + (\cosh x - \sinh x)^4 = 2\cosh 4x$$

$$\therefore \cosh 4x = \frac{(\cosh x + \sinh x)^4 + (\cosh x - \sinh x)^4}{2}$$

and using the binomial theorem* we obtain:

$$\cosh 4x = \frac{2\cosh^4 x + 12\cosh^2 x \sinh^2 x + 2\sinh^4 x}{2}$$

$$\cosh 4x = \cosh^4 x + 6\cosh^2 x \sinh^2 x + \sinh^4 x$$

---

*$(a \pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4$

$$(\cosh x + \sinh x)^4 = \cosh^4 x + 4\cosh^3 x \sinh x + 6\cosh^2 x \sinh^2 x + 4\cosh x \sinh^3 x + \sinh^4 x$$

$$(\cosh x - \sinh x)^4 = \cosh^4 x - 4\cosh^3 x \sinh x + 6\cosh^2 x \sinh^2 x - 4\cosh x \sinh^3 x + \sinh^4 x$$

SOLUTIONS: Calculus of a Single Variable - Block VI: More Integration Techniques

PRETEST

1. $2\sqrt{2 + \sin x} + 3 - 2\sqrt{2}$.

2. $\frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 - x + \ell n\,|x + 1| + 2$.

3. $\frac{1}{2}\ell n\,|x - 1| - \ell n\,|x - 2| + \frac{1}{2}\ell n\,|x - 3| + c$.

4. $x^2 \sin x + 2x \cos x - 2 \sin x + c$.

5. ∞; $\int_0^3 (x - 2)^{-4}\,dx$ is a divergent improper integral.

SOLUTIONS: Calculus of a Single Variable - Block VI: More Integration Techniques

UNIT 1: A Review of Some Basic Inverse Derivatives

6.1.1

(a) Essentially, we want the family $\int \frac{\cos x \, dx}{\sqrt{2 + \sin x}}$.

Observing that $d(2 + \sin x) = \cos x \, dx$, we make the substitution $u = 2 + \sin x \therefore du = \cos x \, dx$. Hence,

$$\int \frac{\cos x \, dx}{\sqrt{2 + \sin x}} = \int \frac{du}{\sqrt{u}} = \int u^{-1/2} \, du \quad .$$

But $\int u^{-1/2} \, du$ has the form $\int u^n \, du = \frac{u^{n+1}}{n+1} + c \quad (n \neq -1)$

$\therefore \int u^{-1/2} \, du = 2u^{1/2} + c$, and since $u = 2 + \sin x$

$$\int \frac{\cos x \, dx}{\sqrt{2 + \sin x}} = 2\sqrt{2 + \sin x} + c$$

$\therefore f(x) = 2\sqrt{2 + \sin x} + c$. Now, since $f(0) = 3$, it follows that $3 = f(0) = 2\sqrt{2} + c \quad \therefore c = 3 - 2\sqrt{2}$, and since c is a constant:

$$f(x) = 2\sqrt{2 + \sin x} + 3 - 2\sqrt{2} \quad .$$

(b) Since $d(\tan 5x) = 5 \sec^2 5x \, dx$ we make the substitution $u = \tan 5x$. Then $du = 5 \sec^2 5x \, dx$, or $\sec^2 5x \, dx = \frac{1}{5} du$

$$\therefore \int \tan^4 5x \sec^2 5x \, dx = \int u^4 \frac{1}{5} \, du = \frac{1}{5} \int u^4 \, du = \frac{1}{25} u^5 + c$$

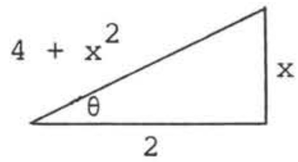
$$\therefore f(x) = \frac{1}{25} \tan^5 5x + c$$

$\therefore f(0) = c$, and since $f(0) = 1$ it follows that $c = 1$

$$\therefore f(x) = \frac{1}{25} \tan^5 5x + 1 \quad .$$

<u>6.1.2</u>

We have $f(x) = \int \frac{dx}{(4 + x^2)^2}$, and this should suggest (circular) trigonometric substitution, based on the reference triangle



$\therefore \tan\theta = \frac{x}{2}$

$\therefore \sec^2\theta \; d\theta = \frac{1}{2}\,dx;\; dx = 2\sec^2\theta \; d\theta$.

Moreover $\sec\theta = \frac{\sqrt{4 + x^2}}{2}$, or $\sqrt{4 + x^2} = 2\sec\theta$

$\therefore (4 + x^2)^2 = (\sqrt{4 + x^2})^4 = 16\sec^4\theta$

$$\therefore \int \frac{dx}{(4 + x^2)^2} = \int \frac{2\sec^2\theta \; d\theta}{16\sec^4\theta} = \frac{1}{8}\int \frac{d\theta}{\sec^2\theta} = \frac{1}{8}\int \cos^2\theta \; d\theta$$

$$= \frac{1}{8}\int \left(\frac{1 + \cos 2\theta}{2}\right) d\theta = \frac{1}{16}\int (1 + \cos 2\theta)\; d\theta$$

$$= \frac{1}{16}\left(\theta + \frac{1}{2}\sin 2\theta\right) + c \quad .$$

From our reference triangle, it follows that

$$\theta = \tan^{-1}\frac{x}{2}$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{x}{\sqrt{4 + x^2}}\right)\left(\frac{2}{\sqrt{4 + x^2}}\right) = \frac{4x}{4 + x^2}$$

$$\therefore \int \frac{dx}{(4 + x^2)^2} = \frac{1}{16}\left[\tan^{-1}\frac{x}{2} + \frac{2x}{(4 + x^2)}\right] + c$$

$$\therefore f(x) = \frac{1}{16}\left[\tan^{-1}\frac{x}{2} + \frac{2x}{(4 + x^2)}\right] + c$$

[6.1.2 cont'd]

$$\therefore f(0) = \frac{1}{16}[0 + 0] + c = c \quad \therefore c = 1 \text{ since } f(0) = 1$$

$$\therefore f(x) = \frac{1}{16}\left(\tan^{-1}\frac{x}{2} + \frac{2x}{(4 + x^2)}\right) + 1 \quad .$$

6.1.3

Here we want $\int \frac{dx}{4x^2 + 4x + 17}$.

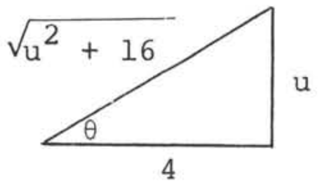
Since $4x^2 + 4x + 1 = (2x + 1)^2$, this should suggest the idea of completing the square: namely,

$$\int \frac{dx}{4x^2 + 4x + 17} = \int \frac{dx}{(4x^2 + 4x + 1) + 16} = \int \frac{dx}{(2x + 1)^2 + 16} \quad .$$

If we now let $u = 2x + 1$, $du = 2dx$ or $dx = \frac{du}{2}$

$$\therefore \int \frac{dx}{(2x + 1)^2 + 16} = \frac{1}{2}\int \frac{du}{u^2 + 16} \quad .$$

This, in turn, suggests the reference triangle



and, hence, the substitution

$\tan\theta = \frac{u}{4}$

$4\sec^2\theta\, d\theta = du$

$\sec\theta = \frac{\sqrt{u^2 + 16}}{4} \quad \therefore u^2 + 16 = 16\sec^2\theta$

[6.1.3 cont'd]

$$\therefore \int \frac{du}{u^2 + 16} = \int \frac{4\ \sec^2\theta\ d\theta}{16\ \sec^2\theta} = \frac{1}{4}\int d\theta = \frac{1}{4}\theta + c = \frac{1}{4}\tan^{-1}\frac{u}{4} + c$$

$$\therefore \frac{1}{2}\int \frac{du}{u^2 + 16} = \frac{1}{8}\tan^{-1}\frac{u}{4} + c$$

$$= \frac{1}{8}\tan^{-1}\frac{(2x + 1)}{4} + c \quad \text{(since } u = 2x + 1\text{)}$$

$$\therefore f(x) = \int \frac{dx}{4x^2 + 4x + 17} = \frac{1}{8}\tan^{-1}\left(\frac{2x + 1}{4}\right) + c$$

$$\therefore f\left(\tfrac{3}{2}\right) = \frac{1}{8}\tan^{-1} 1 + c = \frac{1}{8}\left(\frac{\pi}{4}\right) + c \qquad \therefore c = f\left(\tfrac{3}{2}\right) - \frac{\pi}{32} \qquad \therefore c = \frac{-\pi}{32}$$

since $f\left(\frac{3}{2}\right) = 0$

$$\therefore f(x) = \frac{1}{8}\tan^{-1}\frac{(2x + 1)}{4} - \frac{\pi}{32} \quad .$$

6.1.4

(a) Since $d(1 + \sin x) = \cos x\ dx$, we have:

$$\int \frac{\cos x\ dx}{1 + \sin x} \quad \int \frac{d(1 + \sin x)}{1 + \sin x} = \ln(1 + \sin x) + c$$

(actually, $\ln|1 + \sin x| + c$, but since $1 + \sin x \geqslant 0$ for all x we may omit the absolute value signs).

(b) $$A_R = \int_0^{\frac{\pi}{2}} \frac{\cos x\ dx}{1 + \sin x} = \ln(1 + \sin x)\Big|_0^{\frac{\pi}{2}} = \ln\left(1 + \sin\frac{\pi}{2}\right) - \ln(1 + \sin 0)$$

$$= \ln 2 - \ln 1$$

$$= \ln 2 \quad .$$

[6.1.4 cont'd]

(c) The displacement is given by

$$\Delta x = \int_0^{\pi} \frac{\cos t\, dt}{1 + \sin t} = \ln(1 + \sin t)\Big|_0^{\pi} = \ln 1 - \ln 1$$

$$= 0 \quad .$$

That is, the particle is at the same point when $t = \pi$ seconds as it was when $t = 0$. (The reason, at least in part, is that $1 + \sin t$ is positive, in fact $1 + \sin t \geqslant 1$ for $t \in [0,\pi]$ but $\cos t$ is negative when $\frac{\pi}{2} < t \leqslant \pi$.)

Now, since v changes sign (the particle reverses its direction) when $t = \frac{\pi}{2}$, we find that the total distance is given by:

$$\left|\int_0^{\frac{\pi}{2}} \frac{\cos x\, dx}{1 + \sin x}\right| + \left|\int_{\frac{\pi}{2}}^{\pi} \frac{\cos x\, dx}{1 + \sin x}\right|$$

$$= \left|\ln(1 + \sin t)\Big|_0^{\frac{\pi}{2}}\right| + \left|\ln(1 + \sin t)\Big|_{\frac{\pi}{2}}^{\pi}\right|$$

$$= |\ln 2 - \ln 1| + |\ln 1 - \ln 2|$$

$$= |\ln 2 - 0| + |0 - \ln 2| = \ln 2 + \ln 2$$

$$= 2 \ln 2 = \ln 2^2 = \ln 4 \quad .$$

6.1.5

(a) We want $\int \sin^4 x\, dx$. Observing that $\sin^2 x = \frac{1 - \cos 2x}{2}$ we have:

[6.1.5 cont'd]

$$\int \sin^4 x\, dx = \int (\sin^2 x)^2\, dx = \int \left(\frac{1 - \cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\, dx \quad .$$

We next recall that $\cos^2 2x = \dfrac{1 + \cos 4x}{2}$

$$\therefore \int \sin^4 x\, dx = \frac{1}{4}\int \left(1 - 2\cos 2x + \left[\frac{1 + \cos 4x}{2}\right]\right) dx$$

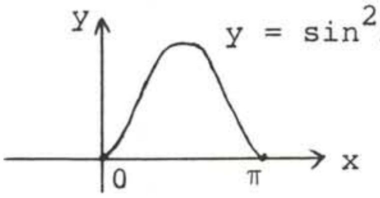
$$= \frac{1}{4}\int \left(\frac{2 - 4\cos 2x + 1 + \cos 4x}{2}\right) dx$$

$$= \frac{1}{8}\int (3 - 4\cos 2x + \cos 4x)\, dx$$

$$= \frac{1}{8}\left[3x - 2\sin 2x + \frac{1}{4}\sin 4x\right] + c$$

$$= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c \quad .$$

(b)



$$v_x = \pi\int_0^{\pi} (\sin^2 x)^2\, dx = \pi\int_0^{\pi} \sin^4 x\, dx$$

$$\therefore \text{from part (a), } v_x = \pi\left(\frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x\right)\Big|_{x=0}^{\pi} = \pi\left(\frac{3\pi}{8}\right)$$

$$= \frac{3\pi^2}{8} \quad .$$

6.1.6

$$A_R = \int_1^5 \frac{dx}{x^2 - 2x + 17}$$

$$= \int_1^5 \frac{dx}{x^2 - 2x + 1 + 16}$$

$$= \int_1^5 \frac{dx}{(x - 1)^2 + 16}$$

(Notice that our denomimator is never non-positive. In fact, $(x - 1)^2 + 16 \geqslant 16$ for all x.)

Let $u = x - 1$.

Then $du = dx$ and $\begin{cases} u = 0 \text{ when } x = 1 \\ u = 4 \text{ when } x = 5 \end{cases}$.

$$\therefore A_R = \int_0^4 \frac{du}{u^2 + 16} .$$

As shown in Exercise 6.1.3,

$$\int \frac{du}{u^2 + 16} = \frac{1}{4} \tan^{-1} \frac{u}{4} + c$$

$$\therefore A_R = \frac{1}{4} \tan^{-1} \frac{u}{4} \Big|_{u=0}^{4}$$

$$= \frac{1}{4}(\tan^{-1}1 - \tan^{-1}0)$$

$$= \frac{1}{4}(\frac{\pi}{4} - 0)$$

$$= \frac{\pi}{16} .$$

6.1.7 (L)

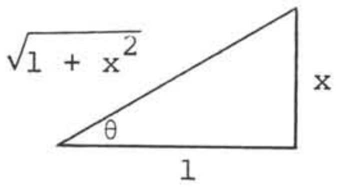
We have refrained from "learning exercises" in this unit if only because what we are doing is a review. However, the subtlety

[6.1.7 (L) cont'd]

of hyperbolic substitution may still be great enough to warrant one more "slow motion" approach.

The main idea is as follows:

When we see $\int \sqrt{1 + x^2}\, dx$ we are rather easily tempted to form the reference triangle



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In fact we have used this triangle in other exercises in this unit. At any rate the triangle leads to:

(1) $\tan\theta = x$, from which $\sec^2\theta\, d\theta = dx$

(2) $\sec\theta = \sqrt{1 + x^2}$

$$\therefore \int \sqrt{1 + x^2}\, dx = \int \sec\theta\, (\sec^2\theta\, d\theta) = \int \sec^3\theta\, d\theta \quad .$$

Now for the moment (this will change in the next units) we cannot handle $\int \sec^3\theta\, d\theta$ *.

Now with the preceeding as background, we come to the main idea of part (a) of this exercise.

(a) In our (circular) trigonometric substitution we made the substitution $x = \tan\theta$. Structurally, $\tan\theta$ is governed by the identity

$$\sec^2\theta - \tan^2\theta = 1 \quad .$$

---

* $\int \sec^3\theta\, d\theta = \int \sec^2\theta \sec\theta\, d\theta = \int (\tan^2\theta + 1) \sec\theta\, d\theta = \int \tan^2\theta \sec\theta\, d\theta + \int \sec\theta\, d\theta$ etc. The point is that the text handles $\int \sec\ d\theta$ by a "trick" which is far from self-evident. In the next unit we shall develop a general technique for solving $\int \sec\theta\, d\theta$ and other integrals.

[6.1.7 (L) cont'd]

This is structurally equivalent (that is, in the form of the difference of two squares) to the hyperbolic identity

$$\cosh^2\theta - \sinh^2\theta = 1 \quad .$$

In other words tan θ is to the circular identity what sinh θ is to the hyperbolic identity.

With this motivation, we now let $\sinh\theta = x$. Then:

$$\cosh\theta = dx \text{ and } \sqrt{1 + x^2} = \sqrt{1 + \sinh^2\theta} = \cosh\theta$$

$$\therefore \int\sqrt{1 + x^2}\,dx = \int(\cosh\theta)(\cosh\theta)\,d\theta = \int\cosh^2\theta\,d\theta$$

$$= \int\frac{1 + \cosh 2\theta}{2}\,d\theta$$

$$= \frac{\theta}{2} + \frac{1}{4}\sinh 2\theta + c \text{ where } \theta = \sinh^{-1}x \quad .$$

More explicitly, we have $\sinh\theta = x$, $\cosh\theta = \sqrt{1 + x^2}$

$$\sinh 2\theta = 2\sinh\theta\cosh\theta = 2x\sqrt{1 + x^2}$$

$$\therefore \int\sqrt{1 + x^2}\,dx = \frac{1}{2}\sinh^{-1}x + \frac{x\sqrt{1 + x^2}}{2} + c = f(x) \quad .$$

Since $f(0) = \frac{1}{2}\sinh^{-1}0 + 0 + c$, $c = f(0) = 0$ since $f(0) = 0$

$$\therefore f(x) = \frac{1}{2}\sinh^{-1}x + \frac{x\sqrt{1 + x^2}}{2} \quad .$$

(b) $\frac{d}{dx}[\frac{1}{2}\sinh^{-1}x + \frac{1}{2}x\sqrt{1 + x^2}]$

[6.1.7 (L) cont'd]

$$= \frac{1}{2}\frac{d}{dx}[\sinh^{-1}x + x(1 + x^2)^{1/2}]$$

$$= \frac{1}{2}\frac{d}{dx}(\sinh^{-1}x) + \frac{1}{2}\frac{d}{dx}[x(1 + x^2)^{1/2}]$$

$$= \frac{1}{2}[\frac{1}{\sqrt{1 + x^2}} + \frac{1}{2}[x\left\{\frac{1}{2}(1 + x^2)^{-\frac{1}{2}}\right\}2x + (1 + x^2)^{1/2}]$$

$$= \frac{1}{2\sqrt{1 + x^2}} + \frac{x^2}{2\sqrt{1 + x^2}} + \frac{\sqrt{1 + x^2}}{2}$$

$$= \frac{1 + x^2 + (\sqrt{1 + x^2})^2}{2\sqrt{1 + x^2}}$$

$$= \frac{2(1 + x^2)}{2\sqrt{1 + x^2}}$$

$$= \sqrt{1 + x^2} \quad .$$

(While this is simply an exercise in differentiation, it is important to notice that, while our techniques for integration may be sophisticated, the fact remains that once we obtain an answer we can always check its correctness by differentiating it. In this regard, notice how "complicated" a function we must construct just to get one whose derivative is $\sqrt{1 + x^2}$.)

(c) As usual, the surface area is given by

$$S = \int_a^b 2\pi y \, dx = \int_a^b 2\pi y \sqrt{1 + (\frac{dy}{dx})^2} \, dx \quad .$$

In our case $y = \cos x$, $a = 0$, $b = \frac{\pi}{2}$; $\frac{dy}{dx} = -\sin x$; $(\frac{dy}{dx})^2 = \sin^2 x$

$$\therefore S = 2\pi \int_0^{\pi} \cos x \sqrt{1 + \sin^2 x} \, dx \quad .$$

[6.1.7 (L) cont'd]

In theory our problem is solved. All we need is a technique for evaluating the definite integral. An obvious substitution is to let $u = \sin x$; then, $du = \cos x\,dx$ and $\begin{cases} u = 0 \text{ when } x = 0 \\ u = 1 \text{ when } x = \frac{\pi}{2} \end{cases}$.

$$\therefore S = 2\pi \int_0^1 \sqrt{1 + u^2}\,du \quad .$$

From part (a) $\int \sqrt{1 + u^2}\,du = \frac{1}{2}(\sinh^{-1}u + u\sqrt{1 + u^2})$

$$\therefore S = 2\pi\left[\tfrac{1}{2}(\sinh^{-1}u + u\sqrt{1 + u^2})\right]\Big|_0^1$$

$$= \pi[\sinh^{-1}1 + 1\sqrt{1 + 1^2} - 0]$$

$$= \pi[\sinh^{-1}1 + \sqrt{2}] \quad .$$

(If we wish, we may recall that $\sinh^{-1}x = \ln(x + \sqrt{x^2 + 1})$.)

$$\therefore S = \pi[\ln(1 + \sqrt{2}) + \sqrt{2}] \qquad (\approx 7.2)$$

UNIT 2: Partial Fractions

6.2.1 (L)

The main aim of this exercise is to give us experience in reducing quotients of polynomials by long division until the remainder has a degree less than that of the divisor. In this case:

$$\begin{array}{r|l} x^4 & \underline{x+1} \\ \underline{x^4 + x^3} & x^3 - x^2 + x - 1 \\ -x^3 & \\ \underline{-x^3 - x^2} & \\ x^2 & \\ \underline{x^2 + x} & \\ -x & \\ \underline{-x - 1} & \\ 1 & \end{array}$$

$$\therefore \frac{x^4}{x+1} = x^3 - x^2 + x - 1 + \frac{1}{x+1}$$

$$\therefore \int \frac{x^4 dx}{x+1} = \int \left(x^3 - x^2 + x - 1 + \frac{1}{x+1}\right) dx$$

$$= \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x - x + \ln|x+1| + c$$

$$\therefore f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x - x + \ln|x+1| + c$$

$f(0) = c \quad \therefore c = 2$ since $f(0) = 2$

$$\therefore f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x - x + \ln|x+1| + 2 \quad .$$

6.2.2

$$\frac{x^4}{(x-1)(x-2)} = \frac{x^4}{x^2 - 3x + 2}$$

[6.2.2 cont'd]

Since the degree of the numerator exceeds that of the denominator, we proceed by long division to obtain:

$$\begin{array}{r|l}
x^4 & x^2 - 3x + 2 \\ \cline{2-2}
\underline{x^4 - 3x^3 + 2x^2} & x^2 + 3x + 7 \\
3x^3 - 2x^2 & \\
\underline{3x^3 - 9x^2 + 6x} & \\
7x^2 - 6x & \\
\underline{7x^2 - 21x + 14} & \\
15x - 14 &
\end{array}$$

$$\therefore \frac{x^4}{(x-1)(x-2)} = x^2 + 3x + 7 + \frac{15x - 14}{(x-1)(x-2)}$$

$$\therefore \int \frac{x^4\,dx}{(x-1)(x-2)} = \int \left[x^2 + 3x + 7 + \frac{15x-14}{(x-1)(x-2)}\right] dx$$

$$= \frac{1}{3}x^3 + \frac{3}{2}x^2 + 7x + \int \frac{15x - 14}{(x-1)(x-2)}\,dx \quad . \qquad (1)$$

All that is now left is to handle $\int \frac{15x - 14}{(x-1)(x-2)}\,dx$. To this end,

$$\frac{15x - 14}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}\,^{*} \quad . \qquad (2 \qquad (2)$$

---

*This is why it is crucial that our numerator has a lower degree than our denominator. Otherwise, in setting up our partial fractions we would have to allow endless possibilities such as

$\frac{Ax^2 + Bx + C}{x - 1}$ etc. In other words, we do not have to worry about terms like $\frac{Ax^2 + Bx + C}{x - 1}$ occurring since such a term would lead to a sum in which the numerator had a greater degree than the denominator.

[6.2.2 cont'd]

Equation (2) leads to:

$$\frac{15x - 14}{(x - 1)(x - 2)} \equiv \frac{A(x - 2) + B(x - 1)}{(x - 1)(x - 2)} \equiv \frac{(A + B)x + (-2A - B)}{(x - 1)(x - 2)} \quad . \tag{3}$$

Equating numerators in (3), we obtain:

$$15x - 14 \equiv (A + B)x + (-2A - B), \tag{4}$$

and equating coefficients in (4) yields:

$$\left.\begin{array}{l} A + B = 15 \\ -2A - B = -14 \end{array}\right\} \quad . \tag{5}$$

Adding equals to equals, (5) yields:

$$A + (-2A) + B + (-B) = 15 + (-14); \text{ or}$$

$$A = -1$$

$$\therefore B = 16 \quad .$$

Putting these results in (2):

$$\frac{15x - 14}{(x - 1)(x - 2)} = \frac{-1}{x - 1} + \frac{16}{x - 2}$$

$$\therefore \int \frac{15x - 14}{(x - 1)(x - 2)} = -\ln|x - 1| + 16 \ln|x - 2| + C \quad . \tag{6}$$

Finally putting (6) into (1) yields:

$$\int \frac{x^4 \, dx}{(x - 1)(x - 2)} = \frac{1}{3} x^3 + \frac{3}{2} x^2 + 7x - \ln|x - 1| + 16 \ln|x - 2| + C \quad .$$

6.2.3 (L)

The reason for calling this a "learning exercise" is so that we may emphasize a computational device that simplifies our search for the undetermined coefficients.

Let us first observe that we can solve this problem, without "tricks," in the usual way. Namely,

$$\frac{1}{(x-1)(x-2)(x-3)} \equiv \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} \qquad (1)$$

$$\therefore \frac{1}{(x-1)(x-2)(x-3)} \equiv \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$\equiv \frac{Ax^2 - 5Ax + 6A + Bx^2 - 4Bx + 3B + Cx^2 - 3Cx + 2C}{(x-1)(x-2)(x-3)}$$

$$\equiv \frac{(A + B + C)x^2 + (-5A - 4B - 3C)x + (6A + 3B + 2C)}{(x-1)(x-2)(x-3)}$$

$\therefore 1 = 0x^2 + 0x + 1 \equiv (A + B + C)x^2 + (-5A - 4B - 3C)x + (6A + 3B + 2C)$ and equating coefficients yields the system of equations

$$\left.\begin{aligned} A+B+C&=0 \\ -5A-4B-3C&=0 \\ 6A+3B+2C&=1 \end{aligned}\right\} \qquad (2)$$

The system (2) can be solved, for example, by:

$$\left.\begin{aligned} A+B+C&=0 \\ -5A-4B-3C&=0 \\ 6A+3B+2C&=1 \end{aligned}\right\} \sim \left.\begin{aligned} A+B+C&=0 \\ B+2C&=0 \\ -3B-4C&=1 \end{aligned}\right\} \sim \left.\begin{aligned} A+B+C&=0 \\ B+2C&=0 \\ -3(-2C)-4C&=1 \end{aligned}\right\} \quad \therefore \begin{aligned} C &= \tfrac{1}{2}, \\ B &= -1, \\ A &= \tfrac{1}{2} \end{aligned}$$

$$\therefore \frac{1}{(x-1)(x-2)(x-3)} = \frac{1}{2}\left(\frac{1}{x-1}\right) - 1\left(\frac{1}{x-2}\right) + \frac{1}{2}\left(\frac{1}{x-3}\right)$$

[6.2.3 (L) cont'd]

$$\therefore \int \frac{dx}{(x-1)(x-2)(x-3)} = \frac{1}{2}\int \frac{dx}{x-1} - \int \frac{dx}{x-2} + \frac{1}{2}\int \frac{dx}{x-3} = \frac{1}{2}\ell n\ |x-1| - \ell n\ |x-2| + \frac{1}{2}\ell n\ |x-3| + C \quad .$$

$$\therefore \int_4^5 \frac{dx}{(x-1)(x-2)(x-3)} = \left(\frac{1}{2}\ \ell n\ |x-1| - \ell n\ |x-2| + \frac{1}{2}\ \ell n\ |x-3|\ \right|_4^5$$

$$= \left(\frac{1}{2}\ \ell n\ 4 - \ell n\ 3 + \frac{1}{2}\ \ell n\ 2\right) - \left(\frac{1}{2}\ \ell n\ 3 - \ell n\ 2 + \frac{1}{2}\ \ell n\ 1\right)$$

$$= \frac{1}{2}\ \ell n\ 4 - \frac{3}{2}\ \ell n\ 3 + \frac{3}{2}\ \ell n\ 2$$

or, since $\frac{1}{2}\ \ell n\ 4 = \ell n\ 4^{1/2} = \ell n\ 2$,

$$\int_4^5 \frac{dx}{(x-1)(x-2)(x-3)} = \ell n\ 2 - \frac{3}{2}\ \ell n\ 3 + \frac{3}{2}\ \ell n\ 2$$

$$= \frac{5}{2}\ \ell n\ 2 - \frac{3}{2}\ \ell n\ 3.$$

Admittedly, the procedure is cumbersome. An alternative method for determining A, B, and C in (1) is by a judicious use of limits. For example, since

$$\frac{1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} \qquad (1)$$

[6.2.3 (L) cont'd]

we can make A "stand alone" by multiplying both sides of (1) by (x-1) to obtain:

$$\frac{(x-1)}{(x-1)(x-2)(x-3)} = A + \frac{B(x-1)}{x-2} + \frac{C(x-1)}{x-2} \quad . \qquad (2)$$

To guard against x - 1 being 0, we simply let $x\to 1$ in (2). That is,

$$\lim_{x\to 1} \left[\frac{(x-1)}{(x-1)(x-2)(x-3)}\right] = A + B \lim_{x\to 1} \left(\frac{x-1}{x-2}\right) + C \lim_{x\to 1} \left(\frac{x-1}{x-2}\right)$$

or $\lim_{x\to 1} \frac{1}{(x-2)(x-3)} = A + 0 + 0$

or $\frac{1}{(x-2)(x-3)} = A$

$\therefore A = \frac{1}{(-1)(-2)} = \frac{1}{2}$.

Similarly, $\frac{1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$

$$\frac{(x-2)}{(x-1)(x-2)(x-3)} = \frac{A(x-2)}{(x-1)} + B + \frac{C(x-2)}{x-3}$$

and letting $x \to 2$, yields,

$$B = \lim_{x\to 2} \frac{1}{(x-1)(x-3)} = \frac{1}{(2-1)(2-3)} = \frac{1}{1(-1)} = -1$$

etc.

6.2.4

$$\frac{2x + 1}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2} \qquad (1)$$

To solve for A, multiply both sides of (1) by x ($x \neq 0$) to obtain:

$$\frac{2x + 1}{(x-1)(x+2)} = A + B\left(\frac{x}{x-1}\right) + C\left(\frac{x}{x+2}\right) \quad . \qquad (2)$$

In (2) take the limit as $x \to 0$

$$\therefore \frac{1}{(-1)(2)} = A + B(0) + C(0)$$

$$\therefore A = -\frac{1}{2} \quad .$$

To find B, multiply both sides of (1) by x-1 ($x \neq 1$) to obtain

$$\frac{2x + 1}{x(x + 2)} = A\left(\frac{x-1}{x}\right) + B + C\left(\frac{x-1}{x+2}\right) \quad . \qquad (3)$$

Letting $x \to 1$ in (3), yields

$$\frac{2(1) + 1}{1(1 + 2)} = A(0) + B + C(0)$$

$$\therefore B = \frac{3}{1(3)} = 1 \quad .$$

Finally, to find C multiply both sides of (1) by x+2 ($x \neq -2$) to obtain

$$\frac{2x + 1}{x(x-1)} = A\left(\frac{x+2}{x}\right) + B\left(\frac{x+2}{x-1}\right) + C \quad . \qquad (4)$$

Letting $x \to -2$ in (4), yields $\frac{2(-2) + 1}{(-2)(-2-1)} = A(0) + B(0) + C$

$$\therefore C = \frac{-4 + 1}{(-2)(-3)} = -\frac{3}{6} = -\frac{1}{2} \quad .$$

6.2.5 (L)

(a) We want to use partial fractions to "decompose" $\dfrac{1}{x(x^2+1)}$. To this end, we write

$$\frac{1}{x(x^2+1)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+1} \quad . \tag{1}$$

(The key point here is that since $x^2 + 1$ has degree 2, our numerator can be as much as degree 1, and the general first degree polynomial has the form $Bx + C$.)

At any rate, (1) yields:

$$\frac{1}{x(x^2+1)} \equiv \frac{A(x^2+1) + x(Bx+C)}{x(x^2+1)} = \frac{(A+B)x^2 + Cx + A}{x(x^2+1)} \tag{2}$$

And comparing coefficients of the numerators in (2), we obtain

$$\left.\begin{aligned} A + B &= 0 \\ C &= 0 \\ A &= 1 \end{aligned}\right\} \quad \therefore \left\{\begin{aligned} A &= 1 \\ B &= -1 \\ C &= 0 \end{aligned}\right. \quad .$$

Putting these results into (1) yields

$$\frac{1}{x(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1}$$

$$\begin{aligned} \therefore \int \frac{dx}{x(x^2+1)} &= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} \\ &= \ln|x| - \frac{1}{2}\ln(x^2+1) + C \quad . \end{aligned}$$

(b) Now we want to "decompose" $\dfrac{1}{x(x+1)^2}$. We might be tempted to write

$$\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x+1)^2} \tag{3}$$

just as we did in part (a).

[6.2.5 (L) cont'd]

The key point here is that a denominator like $(x + 1)^2$ comes from denominators like $(x + 1)^2$ <u>and</u> <u>also</u> $(x + 1)$. Thus, if we wish we can simplify (3) by writing

$$\frac{1}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2} \quad . \tag{4}$$

The major simplification in (4) is that when we have a power of a polynomial in the denominator we need only concern ourselves with the degree of the polynomial (not the power). (In this case the power is 2, the degree of $x + 1$ is 1.) The reason is that then when we put everything over a common denominator the proper degree comes into the numerator.

These remarks are concerned only with options for computation. Both (3) and (4) yield the same result. For example, from (3) it follows that

$$1 \equiv A(x + 1)^2 + x(Bx + C)$$

or:

$$\begin{aligned} 1 &\equiv Ax^2 + 2Ax + A + Bx^2 + Cx \\ &\equiv (A + B)x^2 + (2A + C)x + A \end{aligned} \tag{5}$$

$$\left.\begin{aligned} \therefore A + B &= 0 \\ 2A + C &= 0 \\ A &= 1 \end{aligned}\right\} \therefore \begin{aligned} A &= 1 \\ B &= -1 \\ C &= -2 \end{aligned} \quad .$$

Putting these results into (3) yields:

$$\frac{1}{x(x + 1)^2} = \frac{1}{x} + \left[\frac{-x - 2}{(x + 1)^2}\right] = \frac{1}{x} - \frac{x}{(x + 1)^2} - \frac{2}{(x + 1)^2} \quad . \tag{6}$$

From (6), it follows that

$$\int \frac{dx}{x(x + 1)^2} = \int \frac{dx}{x} - \int \frac{x\,dx}{(x + 1)^2} - 2\int \frac{dx}{(x + 1)^2}$$

[6.2.5 (L) cont'd]

$$= \int \frac{dx}{x} - \int \frac{(u - 1)}{u^2}\, du - 2\int \frac{dx}{(x + 1)^2}$$

$$= \ell n\ |x| - \int \frac{du}{u} + \int u^{-2} du + \frac{2}{x + 1} + C$$

$$= \ell n\ |x| - \ell n\ |x + 1| - \frac{1}{x + 1} + \frac{2}{x + 1} + C = \ell n\ x - \ell n\ |x + 1| + \frac{1}{x + 1} + C \quad . \qquad (7)$$

Had we elected to use (4), we could have solved for A and C by the technique in Exercises 6.2.3 and 6.2.4 . Namely,

$$\frac{1}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2} \longrightarrow \text{if } x \neq 0$$

$$\frac{1}{(x + 1)^2} = A + \left[\frac{B}{x + 1} + \frac{C}{(x + 1)^2}\right] x \quad .$$

Letting $x \longrightarrow 0$, yields,

$$\frac{1}{(0 + 1)^2} = A + (B + C)\ 0$$

A = 1 which agrees with our previous result.

To find C we multiply both sides of (4) by $(x + 1)^2$ to obtain:

$$\frac{1}{x} = \frac{A}{x}(x + 1)^2 + B(x + 1) + C$$

and letting $x \longrightarrow -1$, yields,

$$-1 = C \quad .$$

[6.2.5 (L) cont'd]

(Here, C doesn't seem to check with our previous value of C, but we must note that the C here is not the same as the C in (3). In particular, C in (3) is the constant term in our numerator when the denominator is $(x + 1)^2$ and no term has denominator $x + 1$. To correlate this with (4), write $\frac{B}{x+1} + \frac{C}{(x+1)^2} = \frac{B(x+1) + C}{(x+1)^2} = \frac{Bx + (B + C)}{(x+1)^2}$.

Thus B + C is in (4) what C is in (3).)

In any event,

$$\frac{1}{x(x+1)^2} = \frac{1}{x} + \frac{B}{x+1} \quad \frac{-1}{(x+1)^2} \quad . \qquad (8)$$

(It is interesting to note that we cannot find B in (8) by multiplying by x + 1 . For if we do get

$$\frac{1}{x(x+1)} = \frac{x+1}{x} + B - \frac{1}{x+1}$$

and if we now let $x \longrightarrow 1$ we get an indeterminate "$\infty - \infty$" form.)

To find B in (8) we can pick any value for x, except 0 or -1, and solve for B. For example, it is convenient to let x = -2 in (8) to obtain:

$$\frac{1}{-2(-2+1)^2} = \frac{1}{-2} + \frac{B}{-2+1} - \frac{1}{(-2+1)^2}$$

or

$$\frac{1}{(-2)(-1)^2} = -\frac{1}{2} - B - \frac{1}{(-1)^2}$$

$$\therefore -\frac{1}{2} = -\frac{1}{2} - B - 1$$

$$\therefore B = -1 \quad .$$

[6.2.5 (L) cont'd]

∴ Equation (4) becomes

$$\frac{1}{x(x+1)^2} = \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}$$

whereupon

$$\int \frac{dx}{x(x+1)^2} = \int \frac{dx}{x} - \int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2}$$

$$= \ln|x| - \ln|x+1| + \frac{1}{x+1} + C$$

which checks with (7) .

6.2.6

(a) We first want to decompose $\frac{x^4}{(x^2+1)^2}$.

This requires long-division, since the numerator and denominator have the same degree. An alternative to long-division is:

$$\frac{x^4}{(x^2+1)^2} = \frac{x^4}{x^4+2x^2+1} = \frac{[x^4+(2x^2+1)] - (2x^2+1)}{x^4+2x^2+1}$$

$$= \frac{x^4+2x^2+1}{x^4+2x^2+1} - \left[\frac{2x^2+1}{x^4+2x^2+1}\right]$$

$$= 1 - \left[\frac{2x^2+1}{x^4+2x^2+1}\right]$$

$$\therefore \int \frac{x^4\,dx}{(x^2+1)^2} = \int dx - \int \frac{2x^2+1}{(x^2+1)^2}\,dx \quad . \qquad (1)$$

Thus to complete this exercise we must decompose $\frac{2x^2+1}{(x^2+1)^2}$. To this end:

[6.2.6 cont'd]

$$\frac{2x^2 + 1}{(x^2 + 1)^2} = \frac{Ax + B}{(x^2 + 1)^2} + \frac{Cx + D}{x^2 + 1} \qquad (2)$$

$$= \frac{(Ax + B) + (Cx + D)(x^2 + 1)}{(x^2 + 1)^2}$$

$\therefore 2x^2 + 1 \equiv Ax + B + Cx^3 + Dx^2 + Cx + D \equiv Cx^3 + Dx^2 + (A + C)x + (B + D)$

$\therefore C = 0,\ D = 2,\ A + C = 0,\ B + D = 1$

$\therefore A = 0,\ B = -1,\ C = 0,\ D = 2$; and putting this into (2) yields

$$\frac{2x^2 + 1}{(x^2 + 1)^2} = \frac{-1}{(x^2 + 1)^2} + \frac{2}{x^2 + 1} \quad .$$

$\therefore$From (1)

$$\int \frac{x^4 dx}{(x^2 + 1)^2} = \int dx + \int \frac{dx}{(x^2 + 1)^2} - 2\int \frac{dx}{x^2 + 1} \quad .$$

From Exercise 6.1.2 $\int \frac{dx}{(x^2 + 1)^2} = \frac{1}{2}(\tan^{-1}x + \frac{x}{1 + x^2}) + C$.

We also know that

$$\int \frac{dx}{x^2 + 1} = \tan^{-1}x + C$$

$$\therefore \int \frac{x^4 dx}{(x^2 + 1)^2} = x + \frac{1}{2}(\tan^{-1}x + \frac{x}{1 + x^2}) - 2\tan^{-1}x + C$$

$$= x - \frac{3}{2}\tan^{-1}x + \frac{x}{2(1 + x^2)} + C \quad .$$

[6.2.6 cont'd]

(b) The key here is how we <u>don't</u> do this problem. Namely, we have already seen (Exercise 6.1.2) that

$$\int \frac{dx}{(x^2+1)^2} = \frac{1}{2}\left(\tan^{-1}x + \frac{x}{1+x^2}\right) + C \quad .$$

The point is that we <u>cannot</u> use partial fractions since $\frac{1}{(x^2+1)^2}$ is already decomposed.

Namely, if $\frac{1}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)^2} + \frac{Cx+D}{x^2+1}$.

We need only let B = 1 and A = C = D = 0 which is what we started with!

<u>6.2.7</u>

Here, we must make preliminary substitutions <u>before</u> we can use partial fractions.

(a) Let $u = \sin\theta$; then, $du = \cos\theta\, d\theta$

$$\therefore \int \frac{\cos\theta\, d\theta}{\sin^2\theta + 7\sin\theta + 12} = \int \frac{du}{u^2+7u+12} = \int \frac{du}{(u+4)(u+3)} \quad . \tag{1}$$

Now let $\frac{1}{(u+4)(u+3)} = \frac{A}{u+4} + \frac{B}{u+3}$.

Then

$$A + B\left(\frac{u+4}{u+3}\right) = \frac{1}{u+3}$$

$$A + B\left(\frac{u+4}{u+3}\right\rfloor_{u=-4} = \frac{1}{u+3}\rfloor_{u=-4}$$

$$\therefore A = -1 \quad .$$

[6.2.7 cont'd]

Similarly, $\left.\frac{1}{u + 4}\right]_{u=-3} = A\left(\frac{u + 3}{u + 4}\right)\Big]_{u=-3} + B$

$$B = 1$$

$$\therefore \frac{1}{(u + 4)(u + 3)} = \frac{1}{u + 3} - \frac{1}{u + 4}$$

$$\int \frac{du}{(u + 4)(u + 3)} = \int \frac{du}{x + 3} - \int \frac{du}{u + 4} = \ln |u + 3| - \ln |u + 4| + C$$

$$= \ln(\sin\theta + 3) - \ln(\sin\theta + 4) + C \quad .$$

Putting this into (1) yields

$$\int \frac{\cos\theta \; d\theta}{\sin^2\theta + 7\sin\theta + 12} = \ln(\sin\theta + 3) - \ln(\sin\theta + 4) + C \quad .$$

(b) Here, we let $u = e^x - 1$, or, $e^x = u + 1$.
Then $du = e^x dx = (u + 1)dx \quad \therefore dx = \frac{du}{u + 1}$

$$\therefore \int \frac{dx}{e^x - 1} = \int \frac{du}{u(u + 1)} \quad .$$

Now, if $\frac{1}{u(u + 1)} = \frac{A}{u} + \frac{B}{u + 1}$, we obtain $Au + A + Bu \equiv 1 \quad \therefore A + B = 0$, $A = 1$

$$\therefore A = 1, B = -1$$

$$\therefore \frac{1}{u(u + 1)} = \frac{1}{u} - \frac{1}{u + 1}$$

$$\int \frac{du}{u(u - 1)} = \int \frac{du}{u} - \int \frac{du}{u + 1}$$

[6.2.7 cont'd]

$$= \ln |u| - \ln |u + 1| + C$$

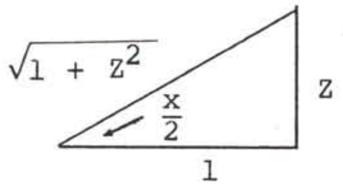
$$= \ln |e^x - 1| - \ln e^x + C$$

$$= \ln |e^x - 1| - x + C$$

$$\therefore \int \frac{dx}{e^x - 1} = \ln |e^x - 1| - x + C \quad .$$

<u>6.2.8</u>

Letting $Z = \tan \frac{x}{2}$



Then:

$$dZ = \frac{1}{2} \sec^2 \frac{x}{2}\, dx = \frac{1}{2}(1 + Z^2)\, dx$$

$$\therefore dx = \frac{2dZ}{1 + Z^2}$$

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = 2\left(\frac{Z}{\sqrt{1 + Z^2}}\right)\left(\frac{1}{\sqrt{1 + Z^2}}\right) = \frac{2Z}{1 + Z^2}$$

$$\therefore 1 + \sin x = 1 + \frac{2Z}{1 + Z^2} = \frac{1 + Z^2 + 2Z}{1 + Z^2} = \frac{(Z + 1)^2}{1 + Z^2}$$

$$\therefore \int \frac{dx}{1 + \sin x} = \int \frac{\frac{2dZ}{1 + Z^2}}{\frac{(Z + 1)^2}{1 + Z^2}} = \int \frac{2dZ}{(Z + 1)^2}$$

$$= -\frac{2}{Z + 1} + C$$

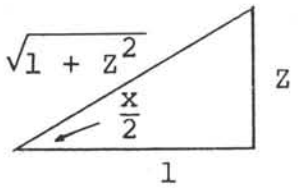
$$= \frac{-2}{1 + \tan\frac{x}{2}} + C \quad .$$

[6.2.8 cont'd]

The key here is that $Z = \tan \frac{x}{2}$ reduces a rational function of sin x and cos x into a rational function of x.

6.2.9

Again $Z = \tan \frac{x}{2}$



$dZ = \frac{1}{2} \sec^2 x \, dx$

$= \frac{1}{2}(1 + Z^2)dx$

$\therefore dx = \frac{2dZ}{1 + Z^2}$

$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \left(\frac{1}{\sqrt{1 + Z^2}}\right)^2 - \left(\frac{Z}{\sqrt{1 + Z^2}}\right)^2$

$= \frac{1 - Z^2}{1 + Z^2}$

$\therefore \sec x = \frac{1 + Z^2}{1 - Z^2}$

$$\int \sec x \, dx = \int \left(\frac{1 + Z^2}{1 - Z^2}\right) \frac{2dZ}{1 + Z^2} = 2\int \frac{dZ}{1 - Z^2} .$$

Now $\frac{1}{1 - Z^2} = \frac{1}{(1 + Z)(1 - Z)} = \frac{A}{1 + Z} + \frac{B}{1 - Z} = A - AZ + B + BZ$

$\therefore A + B = 1, \; B - A = 0 \quad \therefore A = B = \frac{1}{2}$

$\therefore \frac{1}{1 - Z^2} = \frac{1}{2}\left(\frac{1}{1 + Z} + \frac{1}{1 - Z}\right)$

[6.2.9 cont'd]

$$\int \frac{dZ}{1 - Z^2} = \frac{1}{2} \ell n(1 + Z) - \frac{1}{2} \ell n(1 - Z)$$

$$\therefore \int \sec x\, dx = \ell n\, |1 + Z| - \ell n\, |1 - Z| + C$$

$$= \ell n \left|\frac{1 + Z}{1 - Z}\right| + C$$

$$= \ell n \left|\frac{1 + \tan x/2}{1 - \tan x/2}\right| + C \quad .$$

UNIT 3: Integration by Parts

6.3.1. (L)

(a) Given $\int x^2 \cos x\, dx$, we observe that two differentiations would "get rid of" x^2. Thus we might be led to integrate by parts twice. For example:

$$\left.\begin{array}{ll} \text{let } u_1 = x^2 & dv_1 = \cos x\, dx \\ du_1 = 2x\, dx & v_1 = \sin x \end{array}\right\} \int u_1\, dv_1 = u_1 v_1 - \int v_1\, du_1$$

$$\therefore \int x^2 \cos x\, dx = x^2 \sin x - 2\int x \sin x\, dx \quad . \qquad (1)$$

Now, we can "reduce" $\int x \sin x\, dx$ by letting

$$\left.\begin{array}{ll} u_2 = x & dv_2 = \sin x\, dx \\ \therefore du_2 = dx & v_2 = -\cos x \end{array}\right\} \int u_2\, dv_2 = u_2 v_2 - \int v_2\, du_2$$

$$\therefore \int x \sin x\, dx = -x \cos x + \int \cos x\, dx$$

$$= -x \cos x + \sin x + C \quad . \qquad (2)$$

Putting (2) into (1) yields:

$$\int x^2 \cos x\, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C \quad .$$

(b) There may be several approaches in an exercise such as this. We shall explore a few. For one thing, while we may not be able to integrate $\sin\sqrt{x}$, we can certainly differentiate it. Thus, given $\int \sin\sqrt{x}\, dx$* we might be tempted to integrate by parts and let $u = \sin\sqrt{x}$, $dv = dx$.

---

*We hope that by this time there is no need for this warning, but just in case - observe that $\int \sin\sqrt{x}\, dx \neq -\cos\sqrt{x}$ since $\frac{d \cos\sqrt{x}}{dx} = -\sin\sqrt{x}\ \underline{\left(\frac{d\sqrt{x}}{dx}\right)}$.

[6.3.1 (L) cont'd]

Then $du = \cos\sqrt{x}\,\frac{d(\sqrt{x})}{dx} = \frac{\cos\sqrt{x}}{2\sqrt{x}}$ and $v = x$

$$\therefore \int \sin\sqrt{x}\,dx = x\sin\sqrt{x} - \int \frac{x(\cos\sqrt{x})}{2\sqrt{x}}\,dx$$

$$= x\sin\sqrt{x} - \frac{1}{2}\int \sqrt{x}\cos\sqrt{x}\,dx \qquad . \qquad (3)$$

The trouble is that $\int \sqrt{x}\cos\sqrt{x}\,dx$ seems no easier than $\int \sin\sqrt{x}\,dx$. This is all part of the nature of techniques of integration. Namely, changing the form of an integral is no guarantee of success. We often must resort to more than a single substitution. In any event $\int \sqrt{x}\cos\sqrt{x}\,dx$ seems to suggest a substitution such as $\underline{w = \sqrt{x}}$, whereupon $dw = \frac{dx}{2\sqrt{x}} = \frac{dx}{2w}$ or $dx = 2w\,dw$. Then:

$$\int \sqrt{x}\cos\sqrt{x}\,dx = \int w\cos w\,(2w\,dw)$$

$$= 2\int w^2\cos w\,dw \qquad . \qquad (4)$$

Putting (4) into (3), we obtain:

$$\int \sin\sqrt{x}\,dx = x\sin\sqrt{x} - \int w^2\cos w\,dw, \text{ where } w = \sqrt{x} \qquad . \qquad (5)$$

Now, not only does $\int w^2\cos w\,dw$ tend itself to integration by parts, but it is precisely the problem we solved in part a. That is:

$$\int w^2\cos w\,dw = w^2\sin w + 2w\cos w - 2\sin w + C \qquad .$$

Thus (5) becomes:

[6.3.1 (L) cont'd]

$$\int \sin\sqrt{x}\,dx = x\sin\sqrt{x} - w^2\sin w - 2w\cos w + 2\sin w + C$$

and since $w = \sqrt{x}$,

$$\int \sin\sqrt{x}\,dx = x\sin\sqrt{x} - x\sin\sqrt{x} - 2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$$

or:

$$\int \sin\sqrt{x}\,dx = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C \quad . \qquad (6)$$

Well, it was a lot of work, but we were at least fortunate that we got an answer. Could we, however, have proceeded differently and still obtained an answer?

For example, in the expression $\int \sin\sqrt{x}\,dx$ one might be immediately wary of $\sin\sqrt{x}$ and be tempted to remove the square root with the preliminary substitution $w = \sqrt{x}$ (notice that in our first approach to this problem we ultimately made this substitution). If we let $w = \sqrt{x}$, then, as before, $dx = 2w\,dw$ and we find:

$$\int \sin\sqrt{x}\,dx = \int \sin w\,(2w\,dw) = 2\int w\sin w\,dw \quad . \qquad (7)$$

In one step, we can handle $\int w\sin w\,dw$. In fact we have taken care of this also in part a. Although we used an x instead of a w,

$$\int w\sin w\,dw = -w\cos w + \sin w + C_1$$

and putting this into (7) yields:

$$\int \sin\sqrt{x}\,dx = 2[-w\cos w + \sin w + C_1],$$ and since

$w = \sqrt{x}$, $\int \sin\sqrt{x}\,dx = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$ (where $C = 2C_1$). (8)

[6.3.1 (L) cont'd]

Observe that (6) and (8) are identical, as they should be. What is very important to observe is that there are often several logical approaches at our command. In certain cases, one approach will be more advantageous than the others. In some cases, no approach will be helpful. For this reason we learn as many techniques as we can.

(c) We observe that e^x remains e^x whether we integrate it or differentiate it. As for sin x, after either two integrations or two differentiations this becomes -sin x. Thus, we might again be tempted to integrate by parts twice. For example, we might let $u = e^x$ and $dv = \sin x\, dx$, whereupon $du = e^x\, dx$, $v = -\cos x$ (similar results will be obtained in this case if we let $u = \sin x$ and $dv = e^x\, dx$ but this will not be pursued further here). We then have

$$\int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx \quad . \tag{9}$$

We now handle $\int e^x \cos x\, dx$ in a similar way, letting $u = e^x$ and $dv = \cos x\, dx$. Then $du = e^x\, dx$, $v = \sin x$, and

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx \quad . \tag{10}$$

Substituting (10) into (9) yields

$$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx \quad . \tag{11}$$

We now have the required integral in two places in (11), so we may combine terms to obtain:

[6.3.1 (L) cont'd]

$$2\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x$$

or
$$\int e^x \sin x\,dx = \frac{e^x(\sin x - \cos x)}{2} + C \qquad (12)$$

(where as usual we "tack on" the arbitrary constant in order to obtain the family of solutions).

6.3.2

Approach #1

Let $u = \sin(\ell n\ x)$, $dv = dx$. Then: $du = \frac{\cos(\ell n\ x)}{x}\,dx$, $v = x$

$$\therefore \int \sin(\ell n\ x)dx = x\sin(\ell n\ x) - \int \cos(\ell n\ x)\,dx \quad . \qquad (1)$$

We then handle $\int \cos(\ell n\ x)\,dx$ in the same way, letting $u = \cos(\ell n\ x)$ and $dv = dx$. Then $du = \frac{-\sin(\ell n\ x)}{x}\,dx$, $v = x$. Therefore

$$\int \cos(\ell n\ x)\,dx = x\cos(\ell n\ x) + \int \sin(\ell n\ x)\,dx \ (+C) \quad . \qquad (2)$$

Putting (2) into (1), we obtain

$$\int \sin(\ell n\ x)\,dx = x\sin(\ell n\ x) - x\cos(\ell n\ x) - \int \sin(\ell n\ x)\,dx + C$$

$$\therefore 2\int \sin(\ell n\ x)\,dx = x\sin(\ell n\ x) - x\cos(\ell n\ x) + C$$

$$\int \sin(\ell n\ x)\,dx = \tfrac{1}{2}x[\sin(\ell n\ x) - \cos(\ell n\ x)] + C \quad . \qquad (3)$$

[6.3.2 cont'd]

Approach #2

Let $w = \ell n\ x$ or $x = e^w$. Then $dx = e^w\ dw$

$$\therefore \int \sin(\ell n\ x)\ dx = \int \sin w(e^w\ dw) = \int e^w \sin w\ dw \quad . \qquad (4)$$

Now, by Exercise 6.3.1 (L) part c. (foresight, again) we have

$$\int e^w \sin w\ dw = \frac{e^w(\sin w - \cos w)}{2} + C \qquad (5)$$

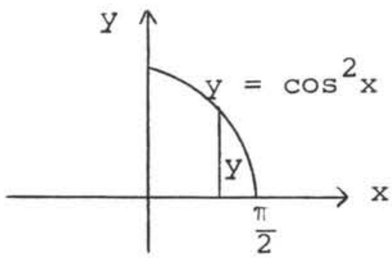
and since $w = \ell n\ x$ (or $x = e^w$), we may combine (4) and (5) to obtain

$$\int \sin(\ell n\ x)\ dx = \frac{x[\sin(\ell n\ x) - \cos(\ell n\ x)]}{2} + C \quad . \qquad (6)$$

Clearly, (3) and (6) agree!

6.3.3

We have:



$$V_y = \int_0^{\frac{\pi}{2}} 2\pi xy\ dx = 2\pi \int_0^{\frac{\pi}{2}} x \cos^2 x\ dx \quad . \qquad (1)$$

To handle $\int x \cos^2 x\ dx$*, it is wise to let $u = x$, $dv = \cos^2 x\ dx$ $= \frac{1 + \cos 2x\ dx}{2}$.

*Notice that $\int x \cos^2 x\ dx$ may "look like" $\int x^2 \cos x\ dx$ (Exercise 6.3.1 a.). Yet, switching the exponent makes the integrals of very different types.

[6.3.3 cont'd]

Then du = dx and $v = \frac{1}{2}\int(1 + \cos 2x)\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x$

$$\therefore \int x\cos^2 x\,dx = \frac{x^2}{2} + \frac{1}{4}x\sin 2x - \int\left(\frac{x}{2} + \frac{1}{4}\sin 2x\right)dx$$

$$= \frac{x^2}{2} + \frac{1}{4}x\sin 2x - \frac{1}{4}x^2 + \frac{1}{8}\cos 2x + C$$

$$= \frac{x^2}{4} + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + C \quad . \qquad (2)$$

Putting (2) into (1) yields:

$$V_y = 2\pi\left(\frac{x^2}{4} + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x\right|_0^{\frac{\pi}{2}} = 2\pi\left[\left(\frac{\pi^2}{16} + 0 + \frac{1}{8}\cos\pi\right) - \left(0 + 0 + \frac{1}{8}\cos 0\right)\right]$$

$$= 2\pi\left[\left(\frac{\pi^2}{16} - \frac{1}{8}\right) - \left(\frac{1}{8}\right)\right] = 2\pi\left[\frac{\pi^2}{16} - \frac{1}{4}\right] = \frac{\pi(\pi^2 - 4)}{8} \quad .$$

<u>6.3.4</u>

(a) Given $\int x^n e^x\,dx$, we "reduce" x^n by letting $u = x^n$, $dv = e^x\,dx$. Then $du = nx^{n-1}\,dx$, $v = e^x$.

$$\therefore \int x^n e^x\,dx = x^n e^x - \int e^x(nx^{n-1})\,dx$$

$$\text{or } \int x^n e^x\,dx = x^n e^x - n\int x^{n-1} e^n\,dx \quad . \qquad (1)$$

(b) To "reduce" $\int x^3 e^x\,dx$ we use (1) with n = 3. Then

$$\int x^3 e^x\,dx = x^3 e^x - 3\int x^2 e^x\,dx \quad . \qquad (2)$$

We then "reduce" $\int x^2 e^x\,dx$ by using (1) with n = 2 . Thus:

[6.3.4 cont'd]

$$\int x^2 e^x \, dx = x^2 e^x - 2\int x e^x \, dx \quad . \tag{3}$$

Putting (3) into (2) yields

$$\int x^3 e^x \, dx = x^3 e^x - 3x^2 e^x + 6\int x e^x \, dx \quad . \tag{4}$$

We then "reduce" $\int x e^x \, dx$ by using (1) with n = 1 . Thus

$$\int x e^x \, dx = x e^x - \int e^x \, dx$$

$$= x e^x - e^x + C \quad . \tag{5}$$

Putting (5) into (4) yields:

$$\int x^3 e^x \, dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C \quad . \tag{6}$$

(c)

$$A_R = \int_0^1 x^3 e^x \, dx$$

From (6) we have:

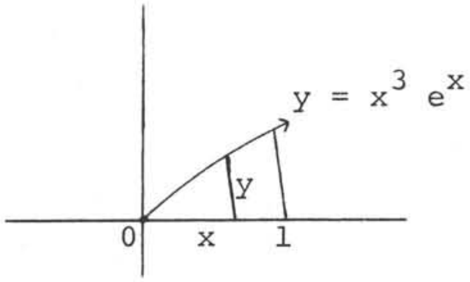
$$A_R = (x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x\Big|_0^1$$

$$= [(e - 3e + 6e - 6e) - (0 - 0 + 0 - 6e^0)]$$

$$= [(-2e)-(-6)]$$

$$= 6 - 2e \approx 0.56 \quad .$$

6.3.5



$$V_y = \int_0^1 2\pi xy\, dx = 2\pi \int_0^1 x(x^3 e^x)\, dx$$

$$= 2\pi \int_0^1 x^4 e^x\, dx \qquad (1)$$

Now, from Exercise 6.3.4 a. $\int x^4 e^x\, dx = x^4 e^x - 4\int x^3 e^x\, dx$ (2)

and from our solution to Exercise 6.3.4 b.,

$$\int x^3 e^x\, dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C \quad . \qquad (3)$$

Combining (2) and (3) yields:

$$\int x^4 e^x\, dx = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24e^x + C,$$

whereupon (1) becomes

$$V_y = 2\pi (x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24e^x \Big|_0^1$$

$$= 2\pi[(e - 4e + 12e - 24e + 24e) - 24]$$

$$= 2\pi[9e - 24] \approx 2.8 \quad .$$

6.3.6

(a) Let $u = \ell n^n x$ $(= [\ell n\ x]^n)$, $du = dx$.

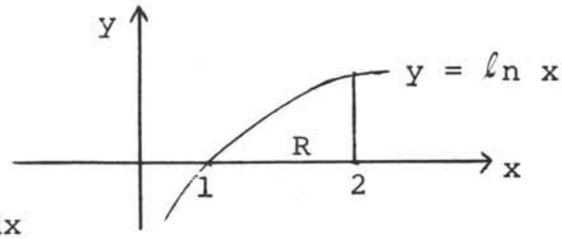
[6.3.6 cont'd]

Then $du = \frac{n\ \ell n^{n-1} x\ dx}{x}$, $v = x$

$$\therefore \int \ell n^n x\ dx = x\ \ell n^n x - \int x\left[\frac{n\ \ell n^{n-1} x}{x}\right] dx$$

$$= x\ \ell n^n x - n\int \ell n^{n-1} x\ dx \quad . \qquad (1)$$

(b)



(1) $$A_R = \int_1^2 \ell n\ x\ dx$$

$\int \ell n\ x\ dx$ may be obtained by letting n = 1 in (1) . Then

$$\int \ell n\ x\ dx = x\ \ell n\ x - \int \ell n^{\circ} x\ dx$$

$$= x\ \ell n\ x - \int dx$$

$$= x\ \ell n\ x - x + C \qquad (2)$$

$$\therefore A_R = x\ \ell n\ x - x\ \Big|_1^2 = (2\ \ell n\ 2 - 2) - (1\ell n\ 1 - 1)$$

$$= 2\ \ell n\ 2 - 2 + 1$$

$$= 2\ \ell n\ 2 - 1 \approx 0.4 \quad .$$

(2) $$V_x = \pi\int_1^2 \ell n^2 x\ dx$$

[6.3.6 cont'd]

Letting n = 2 in (1) yields

$$\int \ell n^2 x = x\,\ell n^2 x - 2\int \ell n\, x\, dx$$

and from (2), $\int \ell n\, x\, dx = x\,\ell n\, x - x + C$

$$\therefore \int \ell n^2 x = x\,\ell n^2 x - 2x\,\ell n\, x + 2x + C \quad .$$

Thus, $V_x = \pi[x\,\ell n^2 x - 2x\,\ell n\, x + 2x \Big|_1^2$

$$= \pi[(2\,\ell n^2 2 - 4\,\ell n\, 2 + 4) - 2]$$

$$= \pi[2(\ell n\, 2)^2 - 4\,\ell n\, 2 + 2]^*$$

$$= 2\pi[\ell n^2 2 - 2\,\ell n\, 2 + 1]$$

$$= 2\pi(\ell n\, 2 - 1)^2 \quad .$$

(3) $V_y = 2\pi\int_1^2 xy\, dx = 2\pi\int_1^2 x\,\ell n\, x\, dx$

Perhaps the simplest way to simplify $\int x\ \ell n\, x$ is to let

$$\left.\begin{array}{l} u = \ell n\, x,\ dv = x\ dx \\ du = \frac{dx}{x}\ ,\ v = \frac{1}{2}x^2 \end{array}\right\} \text{ then,}$$

$$\int x\,\ell n\, x\, dx = \frac{1}{2}x^2\,\ell n\, x - \frac{1}{2}\int x\, dx = \frac{1}{2}x^2\,\ell n\, x - \frac{1}{4}x^2 + C$$

---

*Note: Do not confuse $\ell n\, b^n$ with $(\ell n\, b)^n$. $\ell n\, b^n = n(\ell n\, b) \neq (\ell n\, b)^n$ unless n = 0 or 1 .

[6.3.6 cont'd]

$$\therefore V_y = 2\pi[\tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2|_1^2$$

$$= 2\pi[(2 \ln 2 - 1) - (-\tfrac{1}{4})]$$

$$= 2\pi[2 \ln 2 - \tfrac{3}{4}]$$

$$= \frac{\pi}{2}[8 \ln 2 - 3] \approx 4 \quad .$$

UNIT 4: Improper Integrals

6.4.1

(a) $(x - 2)^{-4}$ $\left(= \frac{1}{(x - 2)^4}\right)$ is infinite when $x = 2$.

$$\text{Hence} \int_0^3 (x - 2)^{-4}\,dx = \lim_{h\to 0^+}\left[\int_0^{2-h} (x - 2)^{-4}\,dx + \int_{2+h}^3 (x - 2)^{-4}\,dx\right]$$

$$= \lim_{h\to 0^+}\left[-\tfrac{1}{3}(x - 2)^{-3}\Big|_0^{2-h} + \left(-\tfrac{1}{3}(x - 2)^{-3}\Big|_{2+h}^{3}\right]$$

$$= \lim_{h\to 0^+}\left[\{-\tfrac{1}{3}(-h)^{-3} - [-\tfrac{1}{3}(-2)^{-3}] + [-\tfrac{1}{3}(3 - 2)^{-3} - \{-\tfrac{1}{3}(h)^{-3}\}]\right]$$

$$= \lim_{n\to 0^+}\left\{\frac{1}{3h^3} - \frac{1}{24} - \frac{1}{3} + \frac{1}{3h^3}\right\}$$

$$= \lim_{h\to 0^+}\left\{\frac{2}{3h^3}\right\} - \frac{9}{24} \qquad (1)$$

$$= \infty \quad . \qquad (2)$$

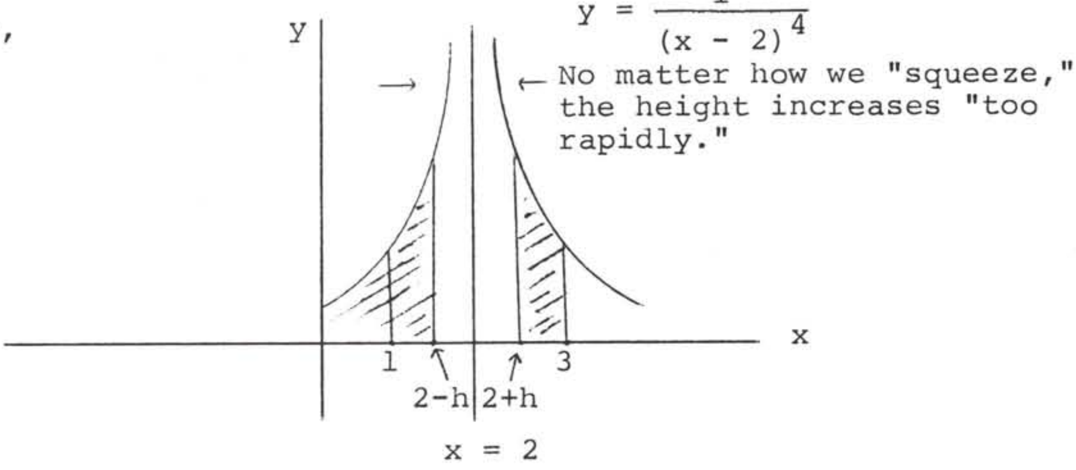
(Had we not noticed that $(x - 2)^{-4}$ was infinite at $x = 2$, we would have obtained

$$\int_0^3 (x - 2)^{-4}\,dx = -\tfrac{1}{3}(x - 2)^{-3}\Big|_0^3 = -\tfrac{1}{3} - [-\tfrac{1}{3}(-2)^{-3}] = -\tfrac{1}{3} - \tfrac{1}{24} = \tfrac{-9}{24}$$

which is the second term in (1).)

[6.4.1 cont'd]

Pictorially,

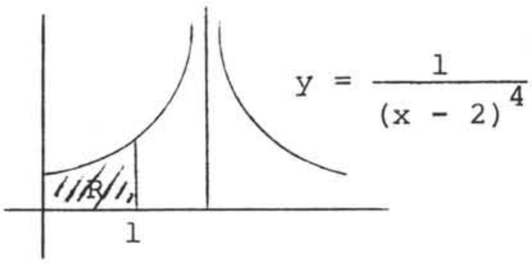


(b) $$\int_0^1 (x-2)^{-4}dx = -\frac{1}{3}(x-2)^{-3}\Big|_0^1 = [-\tfrac{1}{3}(1-2)^{-3}] - [-\tfrac{1}{3}(0-2)^{-3}]$$

$$= +\frac{1}{3} - \frac{1}{24}$$

$$= \frac{7}{24} \quad .$$

Here, we did not have to worry about an improper integral, since $(x-2)^{-4}$ is finite (in fact, continuous) on the given closed interval [0,1]. That is, $\int_0^1 (x-2)^{-4}\,dx$ is the area of R, where



[6.4.1 cont'd]

(c) Again, $(x - 2)^{-\frac{1}{4}}$ is infinite when x = 2. Hence:

$$\int_2^3 (x-2)^{-\frac{1}{4}}dx = \lim_{h\to 0^+}\left[\int_{2+h}^3 (x-2)^{-\frac{1}{4}}\,dx\right]$$

$$= \lim_{h\to 0^+}\left[+\frac{4}{3}(x-2)^{\frac{3}{4}}\Big|_{2+h}^{3}\right]$$

$$= \lim_{h\to 0^+}\left\{\left[\frac{4}{3}(3-2)^{\frac{3}{4}} - \frac{4}{3}(2+h-2)^{\frac{3}{4}}\right]\right\}$$

$$= \lim_{h\to 0^+}\left[\frac{4}{3} - \frac{4}{3}h^{\frac{3}{4}}\right] = \frac{4}{3}.$$

(Notice here that $\int_a^3 (x-2)^{-\frac{1}{4}}\,dx$ is imaginary unless $a \geqslant 2$.)

In this case, the height stays under control as we put the squeeze on.

<u>6.4.2</u>

$\int x^r\,dx$ suggests two cases. Namely $r = -1$ or $r \neq -1$.

<u>Case 1</u>: $r = -1$

$$\int x^r\,dx = \int \frac{dx}{x} = \ln|x| + C$$

$$\therefore \int_1^\infty x^r\,dx = \lim_{b\to\infty}\int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln|x|\Big|_1^b = \lim_{b\to\infty} \ln b = \infty$$

$\therefore \int_1^\infty x^r\,dx$ diverges when $r = -1$.

[6.4.2 cont'd]

<u>Case 2</u>: $r \neq -1$

If $r \neq -1$ then $\int x^r\, dx = \frac{1}{r+1} x^{r+1}$

$$\therefore \int_1^\infty x^r\, dx = \lim_{b\to\infty} \int_1^b x^r\, dx = \lim_{b\to\infty} \left[\frac{1}{r+1} x^{r+1} \Big|_1^b \right]$$

$$= \lim_{b\to\infty} \left[\frac{1}{r+1} (b^{r+1} - 1) \right] . \qquad (1)$$

In (1) if $r + 1 > 0$ (i.e., $r > -1$) then $b^{r+1} \to \infty$ as $b \to \infty$. Hence, $\int_1^\infty x^r\, dx = \infty$ if $r > -1$. Combining this with Case 1, we have

$$\int_1^\infty x^r\, dx \text{ diverges if } r \geqslant -1 \quad .$$

On the other hand, if $r + 1 < 0$ then $b^{r+1} \to \frac{1}{\infty}$ (that is, $b^{r+1} = \frac{1}{b^{-(1+r)}}$ and $-(1+r)$ is positive if $1+r$ is negative) as $b \to \infty$. That is, $r+1 < 0$ implies $\lim_{b\to\infty} b^{r+1} = 0$. Putting this into (1) we see:

$r < -1 \to \int_1^\infty x^r\, dx = \frac{-1}{r+1} = \frac{1}{1-r}$ which is a finite, positive number. That is, $\int_1^\infty x^r\, dx$ converges if $r < -1$.

<u>6.4.3</u>

Just as before we must consider the two cases $r = -1$, $r \neq -1$.

[6.4.3 cont'd]

Case 1: $r = -1$

Then $\int_0^1 x^r\,dx = \int_0^1 \frac{dx}{x} = \lim_{h\to 0^+}\int_h^1 \frac{dx}{x} = \lim_{h\to 0^+}[\ell n\ 1 - \ell n\ h]$

$$= \lim_{h\to 0}[-\ell n\ h] = -\ell n\ 0 = -(-\infty) = \infty$$

$\therefore \int_0^1 x^r\,dx$ diverges when $r = -1$.

Case 2: $r \neq -1$

Then $\int_0^1 x^r\,dx = \lim_{h\to 0^+}\int_h^1 x^r\,dx = \lim_{h\to 0^+}\left[\frac{x^{r+1}}{r+1}\Bigg|_h^1\right]$

$$= \lim_{h\to 0^+}\left[\frac{1}{r+1}(1 - h^{r+1})\right] \quad . \qquad (1)$$

Now, as $h \to 0$, $h^{r+1} \to 0$ if $r + 1 > 0$ (i.e., $r > -1$) .
That is, if $r > -1$:

$$\int_0^1 x^r\,dx = \lim_{h\to 0^+}\left[\frac{1}{r+1}(1 - h^{r+1})\right] = \frac{1}{r+1}$$

$\therefore \int_0^1 x^r\,dx$ converges if $r > -1$.

On the other hand, if $r + 1 < 0$ then as $h \to 0$, $h^{r+1} \to \frac{1}{0} = \infty$. Thus, from (1) we see:

[6.4.3 cont'd]

$$\text{If } r < -1, \int_0^1 x^r \, dx = \lim_{h \to 0} \left[\frac{1}{r+1}\left(1 - h^{r+1}\right)\right] = \frac{-\infty}{r+1} = \infty$$

$$\therefore r < -1, \rightarrow \int_0^1 x^r \, dx \text{ diverges .}$$

6.4.4 (L)

In a way, this is what improper integrals of the second kind are "all about." That is, we often do not care what the value of the integral is but are only interested in whether the integral converges.

For this particular example, recall that $\lim_{t \to \infty} \frac{t^n}{e^t} = 0$. Therefore, given a constant k, we can find a number M such that $t \geq M \rightarrow \frac{t^n}{e^t} < k$. For the sake of convenience, we might think of k as being 1. Then for "sufficiently large" t, $\frac{t^n}{e^t} < 1$ or $t^n < e^t$.

With n = 50 this leads to

$$\int_1^\infty t^{50} e^{-t} \, dt = \int_1^M t^{50} e^{-t} \, dt + \int_M^\infty t^{50} e^{-t} \, dt \quad . \qquad (1)$$

Now, there is no problem with $\int_1^M t^{50} e^{-t} \, dt$. At worst, it is a large but finite number which we may call A.

As for $\int_M^\infty t^{50} e^{-t} \, dt$, we chose M so that for $t \geq M$ $t^{50} < e^t$

$$\therefore \int_M^\infty t^{50} e^{-t} \, dt < \int_M^\infty e^t e^{-t} \, dt = \int_M^\infty dt = \infty \quad .$$

[6.4.4 (L) cont'd]

Of course, just because $\int_M^\infty t^{50} e^{-t}\,dt < \int_M^\infty dt$ we do not have any conclusive evidence as to the convergence of $\int_M^\infty t^{50} e^{-t}\,dt$.

The point is that we can improve our approach as follows. We know that $\int_M^\infty \frac{dt}{t^2}$ converges. With this in mind, we rewrite $\int t^{50} e^{-t}\,dt$ as $\int t^{-2}\, t^{52}\, e^{-t}\,dt$.

The idea is that for large t, $t^{52} < e^t$ just as before. Suppose $t > M_1$ implies that $t^{52} < e^t$. We then have

$$\int_1^\infty t^{50} e^{-t}\,dt = \underbrace{\int_1^{M_1} t^{50} e^{-t}\,dt}_{A_1} + \int_{M_1}^\infty t^{50} e^{-t}\,dt \qquad (2)$$

If we now write:

$$\int_{M_1}^\infty t^{50} e^{-t}\,dt = \int_{M_1}^\infty t^{-2}\, t^{52}\, e^{-t}\,dt < \int_{M_1}^\infty t^{-2}\, e^{t}\, e^{-t}\,dt = \int_{M_1}^\infty \frac{dt}{t^2}$$

$$= -\frac{1}{t}\Bigg|_{M_1}^{\infty} = \frac{1}{M_1}.$$

Putting this into (2), we obtain

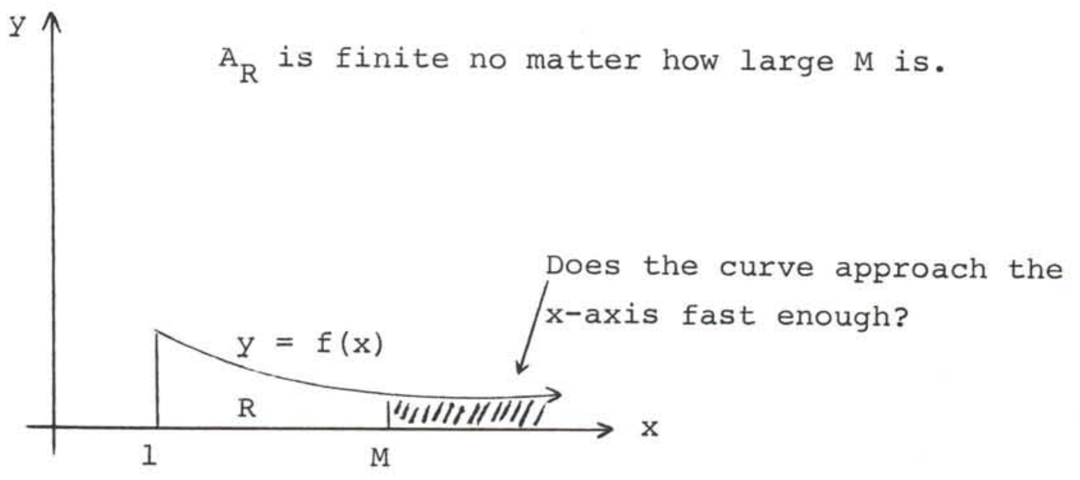
$$\int_1^\infty t^{50} e^{-t}\,dt \leqslant A_1 + \frac{1}{M_1} = \text{a finite number.}$$

[6.4.4 (L) cont'd]

Hence, $\int_1^\infty t^{50} e^{-t}\,dt$ converges, even without our knowing the value of the integral.

The key is that when investigating $\int_1^\infty f(t)\,dt$ we need only examine $\int_M^\infty f(t)\,dt$ for "large" M, for no matter how large M is $\int_1^M f(t)\,dt$ is finite if f(t) is. In essence, then, the convergence of $\int_a^\infty f(t)\,dt$ is determined by the behavior of $\int_M^\infty f(t)\,dt$ for arbitrarily large values of M.

Pictorially,



The convergence or divergence of the integral depends on

$$\lim_{b\to\infty}\int_M^b f(x)\,dx\ .$$

6.4.5

We are given $\int_1^\infty \frac{\ln x}{x^3}\,dx$.

We already know from our study of logarithms that $\lim_{x\to\infty} \frac{\ln x}{x} = 0$.

In particular, there exists a number M such that $x \geqslant M \to \frac{\ln x}{x} < 1$.

Therefore, for $x \geqslant M$ $\frac{\ln x}{x^3} = (\frac{\ln x}{x})\frac{1}{x^2} < \frac{1}{x^2}$.

Hence:

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left[\int_1^b \frac{\ln x\,dx}{x^3}\right]$$

$$= \lim_{b\to\infty}\left[\int_1^M \frac{\ln x\,dx}{x^3} + \int_M^b \frac{\ln x\,dx}{x^3}\right]$$

$$= \int_1^M \frac{\ln x\,dx}{x^3} + \lim_{b\to\infty}\int_M^b \frac{\ln x\,dx}{x^3}$$

$$\leqslant \int_1^M \frac{\ln x\,dx}{x^3} + \lim_{b\to\infty}\int_M^b \frac{dx}{x^2}$$

$$\therefore \int_1^\infty \frac{\ln x\,dx}{x^3} \leqslant A + \lim_{b\to\infty}\left[-\frac{1}{x}\Big|_M^b\right] \text{where } A = \int_1^M \frac{\ln x\,dx}{x^3}$$

$$\therefore \int_1^\infty \frac{\ln x\,dx}{x^3} \leqslant A + \lim_{b\to\infty}\left[-\frac{1}{b} - (-\frac{1}{M})\right] = A + \frac{1}{M} = \text{a finite number}$$

$$\therefore \int_1^\infty \frac{\ln x\,dx}{x^3} \text{ converges} .$$

6.4.6 (L)

Here we have what could be called the inverse of Exercise 6.4.4 . The idea is that for large values of t, e^{-t} "overides" t^n . On the other hand, when t is near 0, e^{-t} is finite (indeed, it's near 1) but t^n can become large if n is negative. In other words

$$\int_0^1 t^n e^{-t}\,dt$$

is an improper integral of the first kind if n is negative.

Thus, what we must do here is look at

$$\int_0^1 t^n e^{-t}\,dt = \lim_{h\to 0} \int_h^1 t^n e^{-t}\,dt \quad . \qquad (1)$$

Now, $\int_h^1 t^n e^{-t}\,dt$, just as in the previous exercise, is not too pleasant to evaluate. However, we can get upper and lower bounds for the integral without too much difficulty if we note that $0 < h < t < 1 \to e^0 < e^h < e^t < e^1$ or $1 < e^h < e^t < e$

$$\therefore 1 > \frac{1}{e^n} > \frac{1}{e^t} > \frac{1}{e} \quad .$$

In other words

$$\frac{1}{e} < e^{-t} < 1 \quad .$$

Thus,

$$\lim_{h\to 0^+} \frac{1}{e}\int_h^1 t^n\,dt \leq \lim_{h\to 0^+}\int_h^1 t^n e^{-t}\,dt \leq \lim_{h\to 0^+}\int_h^1 t^n\,dt \quad . \qquad (2)$$

[6.4.6 (L) cont'd]

Therefore, it is sufficient for us to study the behavior of

$$\lim_{h\to 0^+}\int_h^1 t^n\,dt .$$

<u>If n = -1</u> then $\lim_{h\to 0^+}\int_h^1 t^n\,dt = \lim_{h\to 0^+}\int_h^1 \frac{dt}{t} = \lim_{h\to 0^+}\left[\ln|t|\,\Big|_h^1\right]$

$$= \lim_{h\to 0^+}[\ln 1 - \ln h]$$

$$= \lim_{h\to 0^+}[0 + \ln \tfrac{1}{h}]$$

$$= "\ln \tfrac{1}{0}" = \infty$$

$\therefore \lim_{n\to 0^+} \frac{1}{e}\int_h^1 t^n\,dt = \infty$ and from (2) it follows that $\lim_{h\to 0^+}\int_h^1 t^n e^{-t}\,dt = \infty$

since $\lim_{h\to 0^+}\int_h^1 t^n e^{-t}\,dt \geq \lim_{h\to 0^+}\frac{1}{e}\int_h^1 t^n\,dt$ *

$$\therefore n = -1 \rightarrow \int_0^1 t^n e^{-t}\,dt \text{ diverges.}$$

---

*For those of us who are still uncomfortable working with the symbol ∞, notice that what we are really saying is that $\frac{1}{e}\int_h^1 t^{-1}\,dt$ increases without bound as h approaches 0. Then, since $\frac{1}{e}\int_h^1 t^{-1}\,dt <$ $\int_h^1 t^{-1}e^{-t}\,dt$, it follows that $\int_h^1 t^{-1}e^{-t}\,dt$ must also increase without bound as h approaches 0 .

[6.4.6 (L) cont'd]

<u>If $n \neq -1$</u> then $\displaystyle\int_h^1 t^n\,dt = \frac{t^{n+1}}{n+1}\Bigg|_h^1 = \frac{1}{n+1}\,[1 - h^{n+1}]$

$$\therefore \lim_{h\to 0^+} \int_h^1 t^n\,dt = \frac{1}{n+1}\,[1 - \lim_{h\to 0^+} (h^{n+1})] \quad . \qquad (3)$$

If $n + 1 > 0$ then $\lim_{h\to 0^+} h^{n+1} = 0$, but if $n + 1 < 0$ $\lim_{h\to 0^+} h^{n+1}$ $= \frac{1}{0} = \infty$. Putting this into (3),

$$\lim_{h\ 0^+} \int_h^1 t^n\,dt = \frac{1}{n+1} \quad \text{if} \quad n + 1 > 0; \text{ i.e. if } n > -1 \quad .$$

Thus if <u>$n > -1$</u>, (2) yields:

$$\frac{1}{e(n+1)} \leqslant \int_0^1 t^n e^{-t}\,dt \leqslant \frac{1}{n+1}$$

and in particular $\displaystyle\int_0^1 t^n e^{-t}\,dt$ converges.

If <u>$n < -1$</u> then $\displaystyle\frac{1}{e}\int_0^1 t^n\,dt = \int_0^1 t^n\,dt = \infty$ and since $\displaystyle\int_0^1 t^n e^{-t}\,dt$ is "squeezed between" these two integrals, it follows that $\displaystyle\int_0^1 t^n e^{-t}\,dt$ diverges.

In summary,

$\displaystyle\int_0^1 t^n e^{-t}\,dt$ is an improper integral of the first kind which converges if $n > -1$ but diverges if $n \leqslant -1$.

6.4.7 (L)

(a) Noticing that x is being treated as a constant* (t is the variable of integration) we may break up the given integral into two parts, each of which has already studied. Namely,

$$\int_0^\infty t^{x-1} e^{-t}\, dt = \int_0^1 t^{x-1} e^{-t}\, dt + \int_1^\infty t^{x-1} e^{-t}\, dt \quad . \qquad (1)$$

Now, in Exercise 6.4.4 we saw that $\int_1^\infty t^n e^{-t}\, dt$ converges for all n. (Actually, we worked with n = 50, but this is of little consequence; the key was that $\lim_{t\to\infty} \frac{t^n}{e^t} = 0$ for any n.) In particular, then, with x - 1 playing the role of n we see that

$$\int_1^\infty t^{x-1} e^{-t}\, dt \text{ converges for all } x \quad . \qquad (2)$$

As for $\int_0^1 t^{x-1} e^{-t}\, dt$, we saw in the last exercise that $\int_0^1 t^n e^{-t}\, dt$ converges if and only if $n > -1$.

---

*As an aside, notice that this exercise gives us still another way of defining a function as an integral. Previously, the variable occurred in the limits of integration. Namely,

$f(x) = \int_a^x g(t)\, dt$. In this exercise the limits of integration do not depend on x, but x appears as part of the integrand. In other words, we have the form: $f(x) = \int_a^b g(x,t)\, dt$ where x is fixed, once chosen, and t is the variable of integration.

[6.4.7 (L) cont'd]

If we now let n = x - 1 we obtain that:

$$\int_0^1 t^{x-1} e^{-t}\, dt \text{ converges } \underline{\text{if and only if}}\ x - 1 > -1 \text{ (i.e., } x > 0) \quad . \tag{3}$$

Putting the results of (2) and (3) into (1) we have

$$\int_0^\infty t^{x-1} e^{-t}\, dt \text{ converges if and only if } x > 0 \ .$$

(b) We let $u = e^{-t}$ and $dv = t^{x-1}\, dt$ where $x > 0$.

Then $du = -e^{-t}\, du$ and $v = \frac{1}{x} t^x$. Integration by parts then yields

$$\int_0^\infty t^{x-1} e^{-t}\, dt = \frac{1}{x} e^{-t} t^x \Big|_0^\infty + \frac{1}{x}\int_0^\infty t^x e^{-t}\, dt \quad . \tag{4}$$

Now $\lim\limits_{t\to\infty} \frac{1}{x} e^{-t} t^x = 0$ (since $\lim\limits_{t\to 0} \frac{t^x}{e^t} = 0$) and $\frac{1}{x} e^{-t} t^x \Big|_{t=0} = 0$ if $x > 0$.

Hence, (4) becomes:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt = \frac{1}{x}\int_0^\infty t^x e^{-t}\, dt \quad . \tag{5}$$

Notice now that $\int_0^\infty t^x e^{-t}\, dt$ is $\Gamma(x + 1)$.

Namely,

$$\Gamma([\ ]) = \int_0^\infty t^{[\ ]-1} e^{-t}\, dt \rightarrow$$

$$\Gamma([x + 1]) = \int_0^\infty t^{[x + 1]-1} e^{-t}\, dt = \int_0^\infty t^x e^{-t}\, dt \quad .$$

Putting this into (5) yields

$$\Gamma(x) = \frac{1}{x}\Gamma(x + 1)$$

or

$$\Gamma(x + 1) = x\,\Gamma(x) \quad .$$

[6.4.7 (L) cont'd]

(c) $\Gamma(1) = \int_0^\infty t^{1-1} e^{-t}\, dt = \int_0^\infty e^{-t}\, dt = -e^{-t}\Big|_0^\infty = -\frac{1}{\infty} - (-1) = 1$

By (b) $\Gamma(2) = 1\Gamma(1) = 1$

$\Gamma(3) = 2\Gamma(2) = 2$

$\Gamma(4) = 3\Gamma(3) = 3 \times 2 = 3!$

$\Gamma(5) = 4\Gamma(3) = 4 \times 3 \times 2 = 4!$

$\Gamma(6) = 5\Gamma(4) = 5 \times 4! = 5!$

Proceeding inductively;

$$\Gamma(n) = (n - 1)! \qquad (6)$$

In this sense, the gamma-function, among other things, extends the structure of factorials for all positive real numbers. In other words, for x real we mimick (6) and define $x! = \Gamma(x + 1) = \int_0^\infty t^x e^{-t}\, dt$.

In particular, for any whole number n,

$$\int_0^\infty t^n e^{-t}\, dt = n!\ .$$

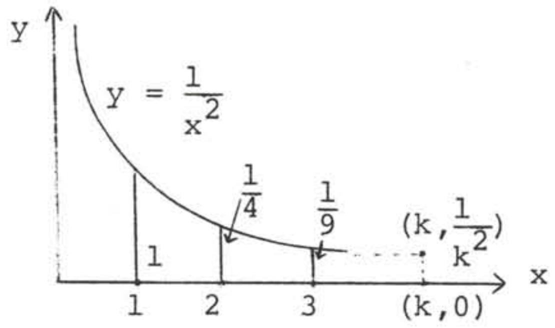
6.4.8 (L)

Here we see an interesting connection between infinite sums and improper integrals. To begin with, let us find the relationship

[6.4.8 (L) cont'd]

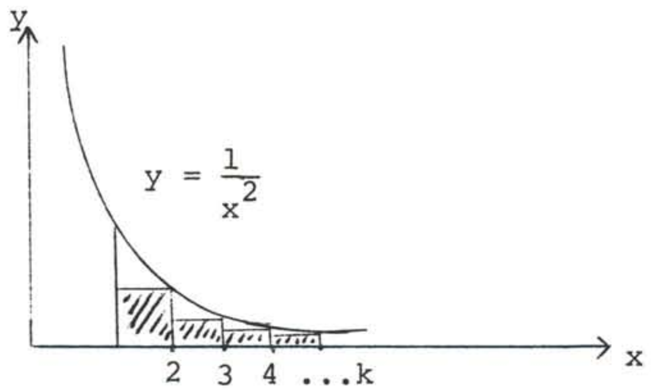
between $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2}$ $(= 1 + \frac{1}{4} + \frac{1}{9} + \ldots)$ and $\lim_{n\to\infty} \int_1^n \frac{dx}{x^2}$.

If we sketch $y = \frac{1}{x^2}$ $(x \geqslant 1)$, we find



In other words $1 + \frac{1}{4} + \frac{1}{9} + \ldots$ represents the length of the line segments connecting (k,0) and $k, \frac{1}{k^2})$. The length of these segments is also the area of a rectangle having the same height and base of 1 unit.

In other words $\frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n}$ is represented by the region shaded below:



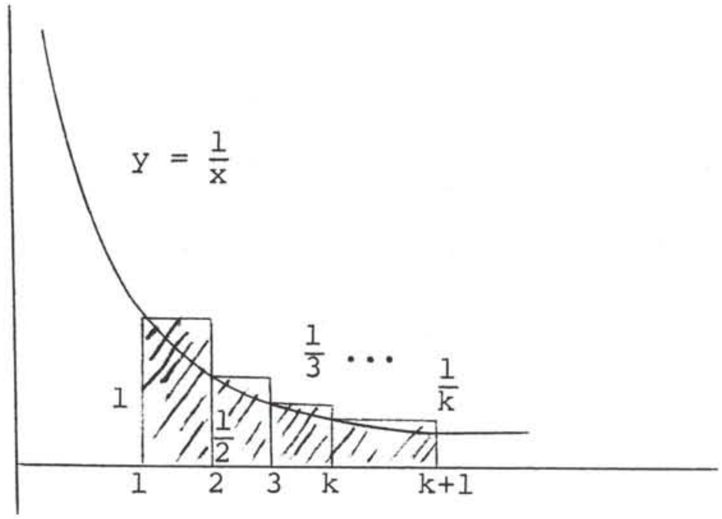
[6.4.8 (L) cont'd]

and this is clearly less that $\int_1^k \frac{dx}{x^2}$ (why ?) $= -\frac{1}{x}\Big|_1^k = 1 - \frac{1}{k}$

$$\therefore \lim_{k\to\infty} \left(\frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{k^2}\right) \leq \lim_{k\to\infty} \left(1 - \frac{1}{k}\right) = 1$$

$\therefore \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} + \ldots \leq 1 \qquad \therefore 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} + \ldots \leq 2$.

6.4.9



$$1 + \frac{1}{2} + \ldots + \frac{1}{k} > \int_1^{k+1} \frac{dx}{x} = \ln x \Big|_1^{k+1} = \ln(k+1)$$

$$\therefore \lim_{k\to\infty} \left(1 + \frac{1}{2} + \ldots + \frac{1}{k}\right) \geq \lim_{k\to\infty} \ln(k+1) = \infty$$

$\therefore \lim_{k\to\infty} \left(1 + \frac{1}{2} + \ldots + \frac{1}{k}\right)$ diverges (even though the terms in the sum approach 0 as a limit.)

QUIZ

1. Let $u = 1 + \sin 3x$. Then $du = 3 \cos 3x\, dx$

$$\frac{\cos 3x\, dx}{(1 + \sin 3x)^3} = \frac{\frac{1}{3}\, du}{u^3} = \frac{1}{3} \quad u^{-3}\, du$$

$$= \frac{1}{3}\,[-\frac{1}{2}\, u^{-2} + C_1]$$

$$= \frac{-1}{6u^2} + C$$

$$= \frac{-1}{6(1 + \sin 3x)^2} + C$$

$$f(x) = \frac{-1}{6(1 + \sin 3x)^2} + C \quad . \qquad (1)$$

Since $f(0) = 2$, (1) becomes:

$$2 = f(0) = \frac{-1}{6(1 + \sin 0)^2} + C, \text{ or:}$$

$$2 = -\frac{1}{6} + C$$

$C = \frac{13}{6}$, and (1) becomes:

$$f(x) = \frac{1}{6(1 + \sin 3x)^2} + \frac{13}{6} \quad .$$

2. We have:

$$\cos^2 x = \frac{1 + \cos 2x}{2} \qquad (1)$$

$$\cos^4 x = \frac{(1 + \cos 2x)^2}{4} = \frac{1 + 2 \cos 2x + \cos^2 2x}{4} \quad . \qquad (2)$$

Now using the same reasoning that led to (1),

$$\cos^2 2x = \frac{1 + \cos 4x}{2} \quad . \qquad (3)$$

Substituting (3) into (2):

$$\cos^4 x = \frac{1 + 2 \cos 2x + \left(\frac{1 + \cos 4x}{2}\right)}{4} = \frac{2 + 4 \cos 2x + 1 + \cos 4x}{8}$$

$$\cos^4 x = \frac{1}{8} (3 + 4 \cos 2x + \cos 4x) \quad . \qquad (4)$$

From (4),

$$\int \cos^4 x \, dx = \int \frac{1}{8} (3 + 4 \cos 2x + \cos 4x) \, dx = \frac{1}{8} \left[3x + 2 \sin 2x + \frac{1}{4} \sin 4x\right] + C$$

$$f(x) = \frac{3x}{8} + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C \quad . \qquad (5)$$

Since $f(0) = 1$, equation (5) leads to

$$1 = f(0) = 0 + 0 + 0 + C$$

$C = 1$, whereupon (5) becomes

$$f(x) = \frac{3x}{8} + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + 1 \quad .$$

3.

$$\frac{x^2 - 1}{x^3(x - 2)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x - 2} \qquad (1)$$

$$\equiv \frac{Ax^2(x - 2) + Bx(x - 2) + C(x - 2) + Dx^3}{x^3(x - 2)}$$

$$\equiv \frac{(A + D)x^3 + (-2A + B)x^2 + (-2B + C)x - 2C}{x^3(x - 2)}$$

$$\therefore \left.\begin{aligned} A + D &= 0 \\ -2A + B &= 1 \\ -2B + C &= 0 \\ -2C &= -1 \end{aligned}\right\} .$$

Since $-2C = -1$, $C = \frac{1}{2}$. Since $C = \frac{1}{2}$, $-2B + C = 0 \rightarrow -2B + \frac{1}{2} = 0 \rightarrow$ $B = \frac{1}{4}$. Since $B = \frac{1}{4}$, $-2A + B = 1 \rightarrow -2A + \frac{1}{4} = 1 \rightarrow -2A = \frac{3}{4} \rightarrow$ $A = -\frac{3}{8}$. Finally since $A + D = 0$, $A = -\frac{3}{8} \rightarrow D = \frac{3}{8}$.

Putting this information into (1) yields:

$\frac{x^2 - 1}{x^3(x - 2)} \equiv -\frac{3}{8x} + \frac{1}{4x^2} + \frac{1}{2x^3} + \frac{3}{8(x - 2)}$, whereupon $\int \frac{(x^2 - 1)\,dx}{x^3(x - 2)}$

$$= -\frac{3}{8}\ln|x| - \frac{1}{4x} - \frac{1}{4x^2} + \frac{3}{8}\ln|x - 2| + c \qquad (2)$$

$$\therefore f(x) = -\frac{3}{8}\ln|x| - \frac{1}{4x} - \frac{1}{4x^2} + \frac{3}{8}\ln|x - 2| + c$$

$$\therefore f(1) = -\frac{3}{8}\ln|1| - \frac{1}{4} - \frac{1}{4} + \frac{3}{8}\ln|1| + c = \frac{7}{2} \rightarrow -\frac{1}{2} + c = \frac{7}{2}$$

$\therefore c = 4$, whereupon (2) becomes

$$f(x) = -\frac{3}{8}\ln|x| - \frac{1}{4x} - \frac{1}{4x^2} + \frac{3}{8}\ln|x - 2| + 4 .$$

4. Let u = sin x.

Then $\displaystyle\int \frac{\cos x\,dx}{\sin^2 x - 5\sin x + 6} = \int \frac{du}{u^2 - 5u + 6} = \int \frac{du}{(u-3)(u-2)}$. (1)

Now

$$\frac{1}{(u-3)(u-2)} \equiv \frac{A}{u-3} + \frac{B}{u-2}$$

$$\equiv \frac{Au - 2A + Bu - 3B}{(u-3)(u-2)}$$

$$\equiv \frac{(A+B)u + (-2A-3B)}{(u-3)(u-2)}$$

$$\therefore \left.\begin{array}{l} A + B = 0 \\ -2A - 3B = 1 \end{array}\right\}$$

$\therefore A = -B$

$\therefore -2A - 3B = 1 \to -2A + 3A = 1 \to A = 1 \to B = -1$

$$\therefore \frac{1}{(u-3)(u-2)} = \frac{1}{u-3} - \frac{1}{u-2} \qquad (2)$$

Putting (2) into (1):

$$\int \frac{\cos x\,dx}{\sin^2 x - 5\sin x + 6} = \int \frac{du}{u-3} - \int \frac{du}{u-2} = \ln|u-3| - \ln|u-2| + C,$$

and since u = sin x, we have:

$$f(x) = \ln|\sin x - 3| - \ln|\sin x - 2| + C \qquad (3)$$

$\therefore f(0) = \ln\frac{3}{2} \to \ln\frac{3}{2} = f(0) = \ln|-3| - \ln|-2| + C = \ln 3 - \ln 2$
$+ C = \ln\frac{3}{2} + C$

[4. cont'd]

$\therefore C = 0$, and (3) becomes

$$f(x) = \ln(3 - \sin x) - \ln(2 - \sin x) \quad .$$

(We don't need the absolute value signs since both $3 - \sin x$ and $2 - \sin x$ are positive since $-1 \leq \sin x \leq 1$.)

5. We integrate $\int x^3 \cos x\, dx$ by parts. Namely, let $u = x^3$, $dv = \cos x\, dx$. Then $du = 3x^2\, dx$ while $v = \sin x$. Since $\int udv = uv - \int vdu$, we have:

$$\int x^3 \cos x\, dx = x^3 \sin x - \int 3x^2 \sin x\, dx \quad . \qquad (1)$$

We next integrate $\int 3x^2 \sin x\, dx$ by parts, letting $u = 3x^2$ and $dv = \sin dx$. Then $du = 6x\, dx$ while $v = -\cos x$. Therefore:

$$\int 3x^2 \sin x\, dx = -\, 3x^2 \cos x + \int 6x \cos x\, dx \quad . \qquad (2)$$

Putting (2) into (1) yields:

$$\int x^3 \cos x\, dx = x^3 \sin x + 3x^2 \cos x - \int 6x \cos x\, dx \quad . \qquad (3)$$

We next integrate $\int 6x \cos x\, dx$ by parts, letting $u = 6x$ and $dv = \cos dx$. Then $du = 6\, dx$ while $v = \sin x$. Therefore:

$$\int 6x \cos x\, dx = 6x \sin x - \int 6 \sin x\, dx = 6x \sin x + 6 \cos x + C_1 \quad . \qquad (4)$$

Putting (4) into (3) yields:

$$\int x^3 \cos x\, dx = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + C,$$

or

[5. cont'd]

$$f(x) = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + C \quad (5)$$

$\therefore f(0) = -6 + C$

$\therefore f(0) = 3 \rightarrow C = 9$, and putting this in (5) yields

$$f(x) = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + 9 \quad .$$

6. Let $w = \sqrt{x} = x^{1/2}$. Then $dw = \frac{1}{2}x^{-\frac{1}{2}}\,dx = \frac{1}{2\sqrt{x}}\,dx = \therefore 2\sqrt{x}\,dw$ $= dx$, and since $\sqrt{x} = w$:

$$2w\,dw = dx$$

$$\therefore \int \text{cox}\,\sqrt{x}\,dx = \int \cos w(2w\,dw) = 2\int w \cos w\,dw \quad . \quad (1)$$

We now integrate $\int w \cos w\,dw$ by parts. letting $u = w$ and $dv = \cos w\,dw$. Therefore $du = dw$ and $v = \sin w$. Hence:

$$\int w \cos w\,dw = w \sin w - \int \sin w\,dw \quad . \quad (2)$$

Substituting (2) into (1) yields:

$$\int \cos \sqrt{x}\,dx = 2w \sin w - 2\int \sin w\,dw$$

$$= 2w \sin w + 2 \cos w + C$$

$$= 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + C$$

$$\therefore f(x) = 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + C \quad (3)$$

$\therefore f(0) = 2 \cos 0 + C = 2 + C$

$\therefore f(0) = 3 \rightarrow C = 1$, and (3) becomes:

[6. cont'd]

$$f(x) = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + 1 \quad .$$

7. (a) Since $(x - 3)^{-2} = \infty$ $(= \frac{1}{0})$ when $x = 3$, $\int_0^5 (x - 3)^{-2}\,dx$ is an improper integral. That is:

$$\int_0^5 (x - 3)^{-2}\,dx = \lim_{h\to 0^+}\left[\int_0^{3-h} (x - 3)^{-2}\,dx + \int_{3+h}^5 (x - 3)^{-2}\,dx\right]$$

$$= \lim_{h\to 0^+}\left[-(x - 3)^{-1}\Big|_0^{3-h} + -(x - 3)^{-1}\Big|_{3+h}^{5}\right]$$

$$= \lim_{h\to 0^+}\left[\frac{-1}{x - 3}\Big|_0^{3-h} + \left(\frac{-1}{x - 3}\right)\Big|_{3+h}^{5}\right] \quad . \qquad (1)$$

Now $$\frac{-1}{x - 3}\Big|_0^{3-h} = \left[\frac{-1}{(3 - h)-3}\right] - \left[\frac{-1}{0 - 3}\right] = \frac{-1}{-h} - \frac{1}{3} = \frac{1}{h} - \frac{1}{3} \qquad (2)$$

while $$\frac{-1}{x - 3}\Big|_{3+h}^{5} = \frac{-1}{5 - 3} - \left[\frac{-1}{(3 + h)-3}\right] = \frac{-1}{2} - \left(\frac{-1}{h}\right) = \frac{-1}{2} + \frac{1}{h} \qquad (3)$$

Substituting (2) and (3) into (1) yields:

$$\int_0^5 (x - 3)^{-2}\,dx = \lim_{h\to 0^+}\left[\frac{1}{h} - \frac{1}{3} - \frac{1}{2} + \frac{1}{h}\right] = \lim_{h\to 0^+} \frac{2}{h} - \frac{5}{6} = \infty - \frac{5}{6} = \infty \quad .$$

[7. cont'd]

That is, $\int_0^5 (x - 3)^{-2}\,dx$ is a divergent improper integral.

Note that $\int_0^5 (x - 3)^{-2}\,dx = -(x - 3)^{-1}\Big|_0^5$ does not apply since $(x - 3)^{-2}$ is not continuous on [0,5]. In other words,

$\int_0^5 (x - 3)^{-2}\,dx = \frac{-5}{6}$ <u>is</u> <u>incorrect</u>.

(b) Since $f(x) = (x - 3)^{-2} = \frac{1}{(x - 3)^2}$, we see that f is discontinuous only if x = 3 . Hence f is continuous on [0,2] and we may therefore, write:

$$\int_0^2 (x - 3)^{-2}\,dx = -(x - 3)^{-1}\Big|_0^2 = \frac{-1}{x - 3}\Big|_0^2$$

$$= \left(\frac{-1}{2 - 3}\right) - \left(\frac{-1}{0 - 3}\right)$$

$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3} \quad .$$

(c) If $f(x) = (x - 3)^{-\frac{1}{2}} = \frac{1}{\sqrt{x - 3}}$, then f is discontinuous at x = 3 . Hence $\int_3^4 (x - 3)^{\frac{1}{2}}\,dx$ is an improper integral. Thus:

[7. cont'd]

$$\int_3^4 (x - 3)^{-\frac{1}{2}}\,dx = \lim_{h\to 0^+} \int_{3+h}^4 (x - 3)^{-\frac{1}{2}}\,dx$$

$$= \lim_{h\to 0^+} \left[2(x - 3)^{\frac{1}{2}} \Big|_{3+h}^{4} \right]$$

$$= \lim_{h\to 0^+} \left[2\sqrt{x - 3} \Big|_{3+h}^{4} \right]$$

$$= \lim_{h\to 0^+} \left[2\sqrt{1} - 2\sqrt{3 + h - 3} \right]$$

$$= \lim_{h\to 0^+} \left[2 - 2\sqrt{h} \right]$$

$$= 2 - 2 \lim_{h\to 0^+} \sqrt{h}$$

$$= 2 - 2(0)$$

$$= 2 \quad .$$

SOLUTIONS: Calculus of a Single Variable - Block VII: Infinite Series

PRETEST

1. (a) Diverges; n^{th} term doesn't go to 0.

   (b) Converges; ratio test.

   (c) Converges; comparison with $\sum \frac{1}{n^3}$

2. $\frac{1}{2} < c < \frac{3}{2}$

3. 0.3679

4. $|x| < c \underset{\neq}{<} \frac{1}{5}$

5. 0.1189

UNIT 1: Sequences and Series

7.1.1(L)

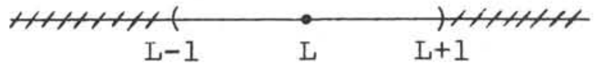
(a) This is a "rehash" of our discussion of boundedness. That is, we wish to reemphasize that every convergent sequence is bounded. Recall that when we have a finite number of terms, we can systematically determine the least and the greatest members by comparing the members in pairs until all have been tested. This system fails, of course, in the infinite case because we cannot test all pairs.

The procedure in the infinite case, then, is to pick a neighborhood of the limit L, say $(L-\varepsilon, L+\varepsilon)$. Since all terms beyond the Nth are in this neighborhood, our quest for upper and lower bounds is now limited to the finite set of numbers: $a_1, \ldots, a_N, L-\varepsilon, L+\varepsilon$.

In more detail, we choose any $\varepsilon>0$. Since the symbol ε may still strike you as being too abstract, let, for example, $\varepsilon=1$. We then have

Since $\lim_{n\to\infty} a_n = L$, we have that there exists an integer N such that $|a_n-L| < 1$ if $n > N$ [that is, $n > N \to a_n \varepsilon (L-1, L+1)$].

At most $a_1, \ldots, a_N$ lie outside $(L-1, L+1)$



[7.1.1(L) cont'd]

We now look at the N+2 numbers:

$$a_1, \ldots, a_N, L-1, L+1$$

We may then let

$$m = \min \{a_1, \ldots, a_N, L-1, L+1\}$$

and

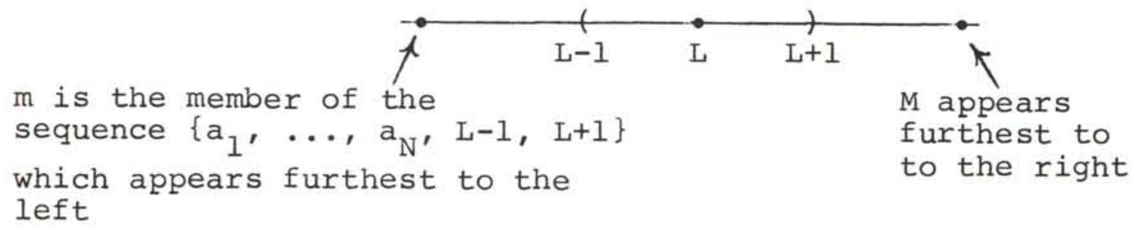
$$M = \max \{a_1, \ldots, a_N, L-1, L+1\}$$

(The key is that $\{a_1, \ldots, a_N, L-1, L+1\}$ is a finite set, and for finite sets we can determine the least and greatest members quite directly.)

Then for each n:

$$m \leqslant a_n \leqslant M$$

Again, pictorially,

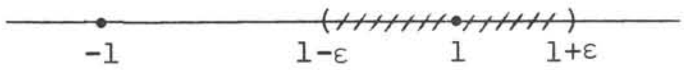


(b) Here we see what might seem like a peculiarity about our definition of limit. Observe that our sequence is given by $a_1 = (-1)^{1+1} = 1$, $a_2 = (-1)^{2+1} = -1$ etc. That is:

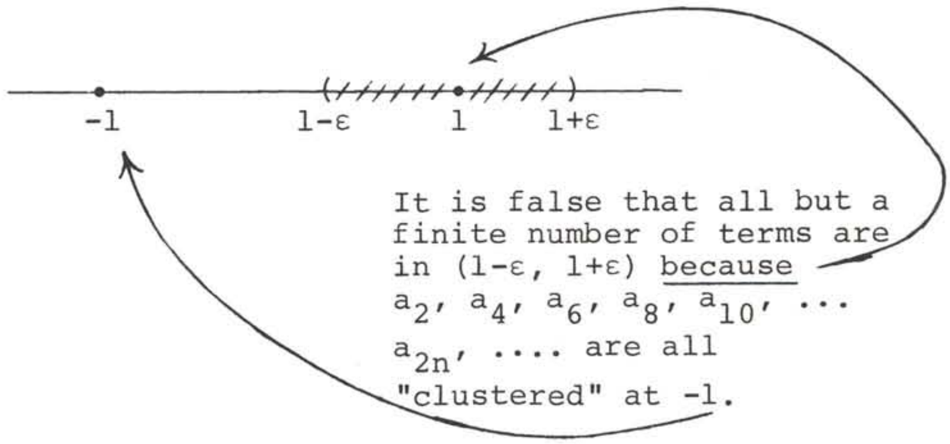
[7.1.1(L) cont'd]

$$\{a_n\} = \{1, -1, 1, -1, \ldots.\}$$

It might appear that both 1 and -1 are limits of this sequence. The fact is, however, that according to our definition of limit, neither 1 nor -1 is the limit*. We will demonstrate this fact pictorially. Suppose that we believed 1 was the limit. All we need do is choose a neighborhood of 1 that does <u>not</u> include <u>-1</u>. For example:



Now by <u>definition</u> if $\lim_{n\to\infty} a_n = 1$ then beyond a certain term each a_n would have to lie in $(1-\varepsilon, 1+\varepsilon)$ but this is impossible since no matter how far out we go we find terms $a_n = -1$ and $-1 \notin (1-\varepsilon, 1+\varepsilon)$. That is:



Similarly, it is easy to show that -1 can't be the limit either. Namely,

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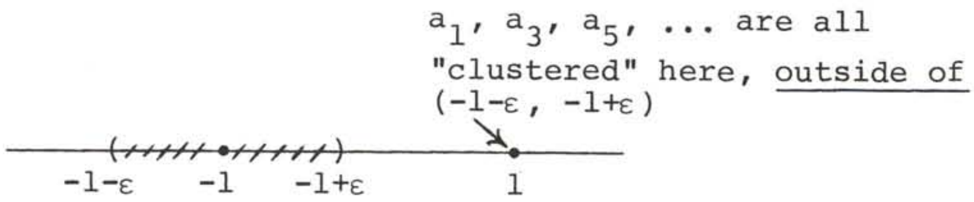
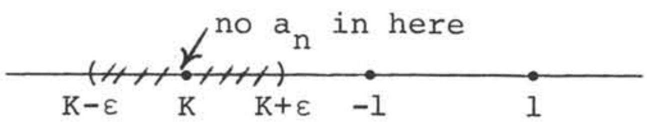
*This concept is generalized in the next exercise. What we shall show there is that a sequence cannot possess two or more limits.

[7.1.1(L) cont'd]

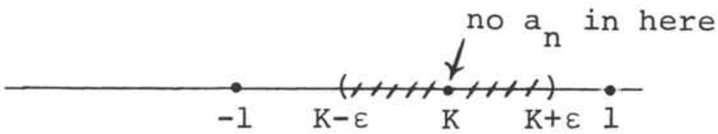


Finally, if K is different from 1 or -1, it can't be the limit of our sequence, since in this case we can always find an ε-neighborhood of K such that <u>no</u> a_n is in this neighborhood. Again, pictorially,

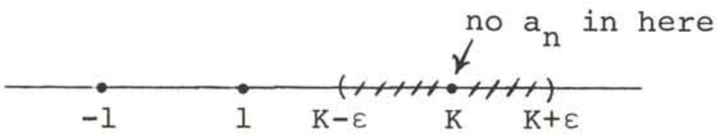
<u>Case 1</u>: $K < -1$



<u>Case 2</u>: $-1 < K < 1$



<u>Case 3</u>: $K > 1$



Putting all this together we have:

For any number L, <u>it is false that</u>

$$\lim_{n\to\infty} a_n = L, \text{ where } a_n = (-1)^{n+1};\ n = 1,2,3,...$$

[7.1.1(L) cont'd]

In this case, we say that $\{a_n\}$ diverges by oscillation. That is, it oscillates between -1 and 1. Accordingly, we never reach the stage where every remaining term stays in the same small neighborhood.

For those who might like to pursue this discussion further, we call -1 and 1 <u>limit points</u> (or <u>cluster points</u>) of the sequence.

A limit point is merely a limit of an appropriate subsequence of the given sequence. In our case, notice that $1 = \lim_{n\to\infty} a_{2n-1}$ while $-1 = \lim_{n\to\infty} a_{2n}$. In plainer language, the subsequence which consists of the odd numbered terms is $\{1,1,1,1,\ldots\}$ which clearly has 1 as a limit etc.

To generalize what we have shown in (b), we point out that if a sequence has more than one limit point it diverges (by oscillation), since infinitely many members of the sequence cluster about <u>each</u> limit point. More formally, the sequence $\{a_n\}$ converges if and only if it has exactly one limit point.

(c) Here we need only observe that for each n, $-1 \leqslant a_n \leqslant 1$. Thus, $\{a_n\}$ is bounded but does not converge.

<u>7.1.2(L)</u>

At first glance, it might seem that this exercise makes much ado about nothing. After all, we could, for example, argue that we learned in elementary geometry that things equal to the same thing are equal to each other. Thus, if $\lim_{n\to\infty} a_n = L_1$ and also $\lim_{n\to\infty} a_n = L_2$, isn't it therefore obvious that $L_1 = L_2$? The subtle point here is that it is not a priori obvious that $\lim_{n\to\infty} a_n$ is

[7.1.2(L) cont'd]

single-valued. That is, there might be more than one number that satisfies the criterion for being a limit of the sequence. In terms of the previous exercise, notice that one might have expected that both 1 and -1 were limits of the sequence 1,-1,1,-1,.....

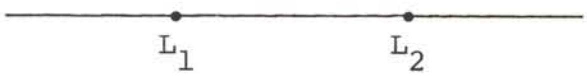
This exercise is asking us to show that this cannot happen - that if there is a limit of a sequence, it is unique. In this context, we showed with respect to the sequence 1,-1,1,-1,... that neither 1 nor -1 was the limit but that each was a limit (cluster) point.

At any rate, with this as motivation to show you that this exercise is more than an illustration of how an analytic proof is used to prove an "obvious" result, we proceed with the solution of this exercise.

One "neat" way of establishing the result is by the indirect proof. Namely, we will show that the assumption $L_1 \neq L_2$ leads to a contradiction.

The pictorial proof is as follows.

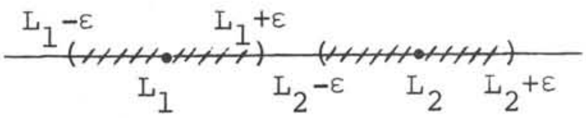
Assume $L_1 \neq L_2$. Without loss of generality, we may assume $L_2 > L_1$. Then,



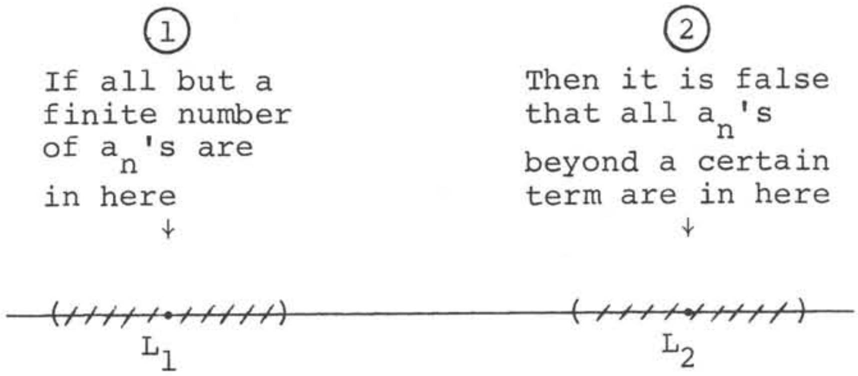
(If $L_1 > L_2$, simply reverse the subscripts in the diagram.)

We now choose an ε-neighborhood of L_1 which does not intersect the ε-neighborhood of L_2. For example, ε may be any number which is less than half the distance between L_1 and L_2. Thus,

[7.1.2(L) cont'd]



Since $\lim_{n\to\infty} a_n = L_1$, at most a finite number of a_n's can lie outside $(L_1-\varepsilon,\ L_1+\varepsilon)$. Hence, in particular, at most a finite number of a_n's lie inside $(L_2-\varepsilon,\ L_2+\varepsilon)$. Thus, L_2 cannot (by definition) be the limit, contrary to the given information. That is:



$$\therefore L_2 \neq L_1 \rightarrow \lim_{n\to\infty} a_n \neq L_2$$

Note that our proof hinged on the assumption that $L_1 \neq L_2$. If $L_1 = L_2$ then the distance between L_1 and L_2 is 0 and in this case ε would be 0. In other words, we would not be able to find <u>non-intersecting</u> neighborhoods of L_1 and L_2.

At this point, we would like to emphasize that pictorial proofs are more intuitive than the analytic proofs <u>but</u> that the analytic proofs exist. In fact, in many cases, the analytic proof

[7.1.2(L) cont'd]

is an appropriate translation of the pictorial proof. For instance, in this exercise we may proceed as follows.

The analytic counterpart of "the distance between L_1 and L_2" is $|L_1 - L_2|$ $(= |L_2 - L_1|)$. We may then say

(1) Assume $L_1 \neq L_2$ then $|L_1 - L_2| > 0$

(2) Let $\varepsilon = \frac{1}{2}|L_1 - L_2|$*. Hence $\varepsilon > 0$

(3) Since $\lim\limits_{n\to\infty} a_n = L_1$ there exists a number N_1 such that $n > N_1 \to |L_1 - a_n| < \varepsilon$

(4) Since $\lim\limits_{n\to\infty} a_n = L_2$, there exists a number N_2 such that $n > N_2 \to |a_n - L_2| < \varepsilon$

(5) Let $N = \max\{N_1, N_2\}$. Then $n > N \to |L_1 - a_n| < \varepsilon$ and $|a_n - L_2| < \varepsilon$

(6) $\therefore n > N \to |L_1 - a_n| + |a_n - L_2| < \varepsilon + \varepsilon = 2\varepsilon = |L_1 - L_2|$

(7) But $|L_1 - a_n| + |a_n - L_2| \geqslant |(L_1 - a_n) + (a_n - L_2)| = |L_1 - L_2|$

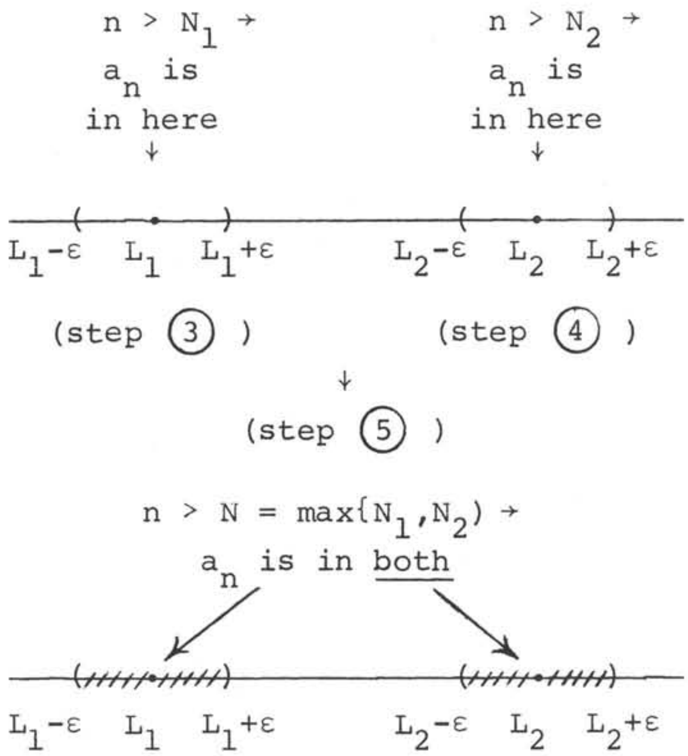
(8) Comparing (7) and (8) we have $|L_1 - L_2| > |L_1 - L_2|$ which is the desired contradiction

---

*Actually ε could be any number such that $0 < \varepsilon \leqslant \frac{1}{2}|L_1 - L_2|$. In terms of what's happening, pictorially, we merely want to guarantee that the ε-neighborhoods of L_1 and L_2 will not intersect.

[7.1.2(L) cont'd]

Again, relating our analytic statements to pictures, we have:



So, pictorially, the contradiction given by step ⑧ translates into "the same point must be in two different places at the same time."

7.1.3(L)

(a) There are two ways that we may establish this result. One way would use the fact that we knew specifically the limit to which $\{a_n\}$ converged while the other way would use the Cauchy criterion.

For example, suppose we know that $\lim_{n\to\infty} a_n = L$. It is then reasonable to suspect that $\lim_{n\to\infty} |a_n| = |L|$.

[7.1.3(L) cont'd]

To verify our conjecture, we must show that given $\varepsilon > 0$ we can find N such that

$$n > N \rightarrow \Big| |a_n| - |L| \Big| < \varepsilon \tag{1}$$

The key lies in the fact that

$$\Big| |a_n| - |L| \Big| \leqslant |a_n - L|^* \tag{2}$$

Namely, we now observe that for our given ε we can find N such that $n > N \rightarrow |a_n - L| < \varepsilon$ (since $\{a_n\}$ converges to L). Hence, from (2), $n > N \rightarrow \Big| |a_n| - |L| \Big| < \varepsilon$; and (1) is established.

If we use the Cauchy criterion, we have the following.

Given $\varepsilon > 0$ there exists N such that $n, m > N \rightarrow$

$$|a_n - a_m| < \varepsilon \tag{3}$$

However $\Big| |a_n| - |a_m| \Big| \leqslant |a_n - a_m|$; hence, from (3) we have $n, m > N \rightarrow \Big| |a_n| - |a_m| \Big| < \varepsilon$.

$\therefore \{|a_n|\}$ converges by the Cauchy criterion. (Notice that we did not have to know what value $\lim_{n\to\infty} a_n$ had to use this technique.)

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*Recall that we proved as a property of absolute values that $|a - b| \geqslant \Big| |a| - |b| \Big|$. An easy way to remember the result is that if a and b have opposite signs then a - b adds the magnitudes of a and b while $|a| - |b|$ always involves a subtraction since $|a|$ and $|b|$ are non-negative. For example, if a = 5 and b = -3 then $|a - b| = |5 - (-3)| = 8$ while $|a| - |b| = 5 - |-3| = 5 - 3 = 2$.

[7.1.3(L) cont'd]

(b) The fact that $\{a_n\}$ need not converge merely because $\{|a_n|\}$ does may be verified from Exercise 7.1.1(L), part (b). Namely, if $a_n = (-1)^{n+1}$ then we have seen that $\{a_n\}$ <u>diverged by oscillation</u>. On the other hand, if $a_n = (-1)^{n+1}$, then $|a_n| = |(-1)^{n+1}| = |\underline{+}1| = 1$ and $\lim_{n\to\infty} |a_n| = 1$.

Thus, $\{a_n\}$ diverges while $\{|a_n|\}$ converges.

Again, aside from providing drill in the use of the definition of limits, this exercise emphasizes, hopefully, the difference between boundedness and convergence. That is, convergence is "stronger than" boundedness in the sense that convergent sequences are bounded but bounded sequences need not converge.

(c) Since $\lim_{n\to\infty} |a_n| = 0$, given $\varepsilon > 0$ we can find N such that $n > N \to \left||a_n| - 0\right| < \varepsilon$. But $\left||a_n| - 0\right| = \left||a_n|\right| = |a_n| = |a_n - 0|$

$$\therefore n > N \to |a_n - 0| < \varepsilon$$

$$\therefore \lim_{n\to\infty} a_n = 0$$

Thus, in this particular case, the convergence of $\{|a_n|\}$ guarantees the convergence of $\{a_n\}$.

<u>7.1.4</u>

(We assume in this exercise that we already know that $\lim_{n\to\infty} \frac{1}{n} = 0$ and that the limit of a sum equals the sum of the limits, etc.)

[7.1.4 cont'd]

(a) $$\lim_{n\to\infty}\left[\frac{2n+7}{5n-6}\right] = \lim_{n\to\infty}\left[\frac{2+\frac{7}{n}}{5-\frac{6}{n}}\right] = \frac{\lim\limits_{n\to\infty} 2 + 7\lim\limits_{n\to\infty}\frac{1}{n}}{\lim\limits_{n\to\infty} 5 - 6\lim\limits_{n\to\infty}\frac{1}{n}} = \frac{2+0}{5-0} = \frac{2}{5}$$

(b) $$\lim_{n\to\infty}\left[\frac{2n+7}{5n^2-6}\right] = \lim_{n\to\infty}\left[\frac{\frac{2}{n}+\frac{7}{n^2}}{5-\frac{6}{n^2}}\right] = \frac{2\lim\limits_{n\to\infty}\frac{1}{n} + 7\lim\limits_{n\to\infty}\frac{1}{n^2}}{5-6\lim\limits_{n\to\infty}\frac{1}{n^2}}$$

but $\lim\limits_{n\to\infty}\frac{1}{n^2} = \lim\limits_{n\to\infty}\left(\frac{1}{n}\cdot\frac{1}{n}\right) = \left(\lim\limits_{n\to\infty}\frac{1}{n}\right)\left(\lim\limits_{n\to\infty}\frac{1}{n}\right) = (0)(0) = 0$

$$\therefore \lim_{n\to\infty}\left[\frac{2n+7}{5n^2-6}\right] = \frac{2(0)+7(0)}{5-6(0)} = \frac{0}{5} = 0$$

(c) $$\lim_{n\to\infty}\left[\frac{2n^2+7}{5n-6}\right] = \lim_{n\to\infty}\left[\frac{2+\frac{7}{n^2}}{\frac{5}{n}-\frac{6}{n^2}}\right] = \frac{2}{0} = \infty \text{ or } \{a_n\} \text{ diverges.}$$

(That is, $\frac{2n^2+7}{5n-6}$ increases without bound as n gets arbitrarily large.)

(d) $$1+(-1)^n = \begin{cases} 1+1=2, \text{ if n is even} \\ 1-1=0, \text{ if n is odd}\end{cases}$$

$\therefore \{a_n\} = \{0,2,0,2,0,2,...\}$

$\therefore \{a_n\}$ diverges by oscillation (0 and 2 are limit points)

[7.1.4 cont'd]

(e) $\lim\limits_{n\to\infty}\left[\frac{1+(-1)^n}{n}\right] = \lim\limits_{n\to\infty}\left[\frac{1}{n}+\frac{(-1)^n}{n}\right] = \lim\limits_{n\to\infty}\frac{1}{n} + \lim\limits_{n\to\infty}\frac{(-1)^n}{n} = 0 + 0 = 0$

(The convergence of $\frac{(-1)^n}{n}$ is an example of Ex. 7.1.3(L), part (c).)

7.1.5(L)

(a) The main aim here is to make sure that we are not confusing $\lim\limits_{n\to\infty} a_n$ with $\sum\limits_{n=1}^{\infty} a_n$. Namely, in part (a) of Ex. 7.1.4, we saw that

$$\lim_{n\to\infty}\frac{2n+7}{5n-6} = \frac{2}{5}$$

Loosely speaking, this means that in the sum $\sum\limits_{n=1}^{\infty}\frac{2n+7}{5n-6}$ the terms "behave like" $\frac{2}{5}$ when n is sufficiently large. That is, letting n = 1,2,3, etc. we have

$$\sum_{n=1}^{\infty}\frac{2n+7}{5n-6} = \frac{2+7}{5-6} + \frac{2(2)+7}{5(2)-6} + \frac{2(3)+7}{5(3)-6} + \frac{2(4)+7}{5(4)-6} + \ldots + \frac{2(100)+7}{5(100)-6} + \ldots$$

$$= -9 + \frac{11}{4} + \frac{13}{9} + \frac{15}{14} + \ldots + \frac{207}{494} \ldots.$$

These terms begin to resemble $\frac{2}{5}$

$$\therefore \sum_{n=1}^{\infty}\frac{2n+7}{5n-6} \approx -9 + \ldots + \frac{207}{494} + \left[\text{"}\frac{2}{5}\text{"} + \ldots + \text{"}\frac{2}{5}\text{"} + \ldots\right] = \infty$$

[7.1.5(L) cont'd]

More formally, we are reinforcing the idea that $\lim_{n\to\infty} a_n \neq 0 \rightarrow \sum_{n=1}^{\infty} a_n$ <u>diverges</u>.

(b) Here the danger is confusing "$\lim_{n\to\infty} a_n \neq 0 \rightarrow \sum_{n=1}^{\infty} a_n$ diverges" with "$\lim_{n\to\infty} a_n = 0 \rightarrow \sum_{n=1}^{\infty} a_n$ converges". The key point is that $\sum_{n=1}^{\infty} a_n$ may diverge even though $\lim_{n\to\infty} a_n = 0$.

In this example, it is clear that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right) = \ln 1 = 0$$

On the other hand, since $\ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln n$, we have

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} [\ln(n+1) - \ln n]$$

In this case, our k^{th} partial sum is given by

$$s_k = \sum_{n=1}^{k} [\ln(n+1) - \ln n]$$

and this is the "telescoping sum" $\ln(k+1) - \ln 1 = \ln(k+1)$

$$\left(\text{i.e., } \sum_{n=1}^{k} [\ln(n+1) - \ln n] = [\ln 2 - \ln 1] + [\ln 3 - \ln 2] + \ldots + [\ln k - \ln(k-1)] + [\ln(k+1) - \ln k]\right)$$

[7.1.5(L) cont'd]

$$\therefore \sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \lim_{k\to\infty} s_k = \lim_{k\to\infty} [\ln(k+1)] = \infty.$$

$$\therefore \sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) \text{ diverges}$$

even though $\lim_{n\to\infty} \left[\ln\left(\frac{n+1}{n}\right)\right] = 0.$

7.1.6

(a) Here we have a geometric series, namely

$$\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots + \left(\frac{2}{3}\right)^n + \ldots$$

In this type of example recall that we compute s_n and $\frac{2}{3}s_n$ (since the ratio is $\frac{2}{3}$ here). We obtain

$$s_n = \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots + \left(\frac{2}{3}\right)^{n-1} + \left(\frac{2}{3}\right)^n$$

$$\therefore \quad \frac{2}{3}s_n = \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \ldots + \left(\frac{2}{3}\right)^n + \left(\frac{2}{3}\right)^{n+1}$$

$$\therefore \quad s_n - \frac{2}{3}s_n = \frac{2}{3} - \left(\frac{2}{3}\right)^{n+1}$$

$$\frac{1}{3}s_n = \frac{2}{3} - \left(\frac{2}{3}\right)^{n+1}$$

$$\therefore \quad s_n = 2 - 3\left(\frac{2}{3}\right)^{n+1}$$

[7.1.6 cont'd]

$$\therefore \lim_{n\to\infty} s_n = 2 - 3 \lim_{n\to\infty} \left(\frac{2}{3}\right)^{n+1} = 2 - 0 = 2$$

$$\therefore \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \lim_{n\to\infty} s_n = 2$$

(b) Here we have a telescoping sum. Namely,

$$s_k = \sum_{n=1}^{k}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$\therefore \lim_{k\to\infty} s_k = 1 - \lim_{k\to\infty} \frac{1}{k+1} = 1 - 0 = 1$$

$$\therefore \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{n\to\infty} s_n = 1$$

(c) The sequence $\{a_n\}$ where $a_n = (-1)^{n+1}$ diverges by oscillation. (Recall that $\{(-1)^{n+1}\} = \{1,-1,1,-1,\ldots\}$.) In particular, $\lim_{n\to\infty} (-1)^{n+1} \neq 0$. Hence, $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges.

(In fact our sequence of partial sums here is

$s_1 = 1$

$s_2 = 1 + (-1) = 0$

$s_3 = s_2 + 1 = 1$

$s_4 = s_3 + (-1) = 1 + (-1) = 0$

$\vdots$

[7.1.6 cont'd]

In other words $\{s_n\} = \{1,0,1,0,1,0,\ldots\}$ which diverges by oscillation.)

7.1.7(L)

This is another illustration of geometric series but in a somewhat disguised form. For example,

$$0.513513\overline{513}\ldots = 513(10^{-3} + 10^{-6} + 10^{-9} + \ldots)$$

$$= .513(1 + 10^{-3} + (10^{-3})^2 + \ldots)$$

$$= .513\left[\frac{1}{1 - 10^{-3}}\right]$$

$$= \frac{.513}{.999}$$

$$= \frac{513}{999} = \frac{57}{111} = \frac{19}{37}$$

An alternative method which mimics our usual approach for geometric series is:

Let

$$N = 0.513513\overline{513}\ldots. \quad (1)$$

Hence,

$$1000\,N = 513.513513\overline{513}\ldots. \quad (2)$$

Subtracting (1) from (2) yields

$$999\,N = 513.0000\ldots$$

$\therefore\ N = \frac{513}{999} = \frac{19}{37}$ (A quick check, by long division, verifies that $19 \div 37 = 0.513513\overline{513}\ldots$.)

7.1.8

Again let

$$N = 0.51313\overline{13}\ldots$$

Since the repeating part is $\overline{13}$, we observe, for example, that

$$1000\ N = 513.1313\overline{13}\ldots$$

while

$$10\ N = 5.1313\overline{13}\ldots$$

$$\therefore\ 990\ N = 508 \text{ or } N = \frac{508}{990} = \frac{254}{495}$$

Alternatively,

$$\begin{aligned} N = 0.51313\overline{13}\ldots &= 0.5 + .013 + .00013 + .0000013 + \ldots \\ &= \frac{1}{2} + 13(10^{-3} + 10^{-5} + 10^{-7} + \ldots) \\ &= \frac{1}{2} + .013[1 + 10^{-2} + 10^{-4} + \ldots] \\ &= \frac{1}{2} + \frac{13}{1000}\left[\frac{1}{1 - 10^{-2}}\right] \\ &= \frac{1}{2} + \frac{13}{1000}\left[\frac{1}{.99}\right] \\ &= \frac{1}{2} + \frac{13}{990} = \frac{508}{990} \end{aligned}$$

UNIT 2: Positive Series

7.2.1 (L)

(a) This exercise affords us an excellent illustration of what we mean when we say that convergence depends on the "tail end" of the series. That is, if we look at the first "few" terms of $\sum_{n=1}^{\infty} \frac{1000^n}{n!}$, we obtain

$$\frac{1000}{1!} + \frac{1,000,000}{2!} + \frac{1,000,000,000}{3!} + \frac{1,000,000,000,000}{4!} + \cdots$$

It appears that our terms are quite large and that they get even larger as n increases. Yet, this is not really the case - as the ratio test will show.

To apply the ratio test we let $a_n = \frac{1000^n}{n!}$, whence $a_{n+1} = \frac{1000^{n+1}}{(n+1)!}$.

$$\text{Then } \frac{a_{n+1}}{a_n} = \frac{1000^{n+1}}{(n+1)!} \div \frac{1000^n}{n!} = \frac{(1000)^{n+1}\, n!}{(n+1)!\,(1000)^n}$$

$$= \frac{(1000)^{n+1}}{(1000)^n} \frac{n!}{(n+1)!}$$

$$= \frac{1000}{n+1}\ ^* \qquad (1)$$

If we now let $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$, (1) tells us that $\rho = 0$.

Since $\rho < 1$ we have, by the ratio test, that $\sum_{n=1}^{\infty} \frac{(1000)^n}{n!}$ converges.

---

*Recall that $(n+1)! = (n+1)(n)(n-1)\ \ldots\ (2)(1) = (n+1)(n!)$. Hence $\frac{n!}{(n+1)!} = \frac{n!}{(n+1)n!} = \frac{1}{n+1}$.

[7.2.1 (L) cont'd]

We would now like to make the following observations:

(1) Equation (1) shows us that the terms in our series increase as long as n is less than 1,000 . That is, $\frac{a_{n+1}}{a_n} = \frac{1000}{n+1}$ indicates that $a_{n+1} \geq a_n \longleftrightarrow \frac{a_{n+1}}{a_n} \geq 1 \longleftrightarrow \frac{1000}{n+1} \geq 1 \longleftrightarrow n \leq 999$.

Thus, the first 999 terms in $\sum_{n=1}^{\infty} \frac{1000^n}{n+1}$ have the property that they are not only large but also that each term exceeds the one(s) that came before. After this, the terms decrease in size (even though the terms will remain large for quite some time) rapidly enough so that the series converges. In other words $\sum_{n=1}^{\infty} \frac{(1000)^n}{n!}$ converges but the sum to which it converges is quite enormous.

(2) From another point of view, what we are saying is that, for large values of n, n! "dwarfs" 1000^n. That is, since the convergence of $\sum a_n$ implies that $\lim_{n\to\infty} a_n = 0$, we have that $\lim_{n\to\infty} \frac{1000^n}{n!} = 0$, but more to the point the limit approaches 0 fast enough so that $\sum \frac{(1000)^n}{n!}$ converges. (Recall that $\lim_{n\to\infty} a_n = 0$ is not sufficient to guarantee that $\sum_{n=1}^{\infty} a_n$ converges.)

(3) 1000 was chosen for dramatic appeal. Aside from that, any number would work as well. That is, the ratio test will show that $\sum_{n=1} \frac{b^n}{n!}$ converges for any real number b.

[7.2.1 (L) cont'd]

The magnitide of the sum is affected by the choice of b, but not the fact that the series converges.

(b) Since $(n + 1)! = (n + 1)n!$, it is quite natural to think of the ratio test for any series that involves factorials. In this case, we have

$a_n = \frac{n!}{n^n}$, $a_{n+1} = \frac{(n + 1)!}{(n+1)^{(n+1)}}$ (remember that in forming a_{n+1} we replace n by n+1 wherever n appears in a_n).

Hence
$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \div \frac{n!}{n^n}$$

$$= \frac{(n+1)!\ n^n}{(n+1)^{n+1}\ n!}$$

$$= \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}}$$

$$= (n+1) \frac{n^n}{(n+1)^{n+1}}$$

$$= \frac{n^n}{(n+1)^n}$$

$$= \left(\frac{n}{n+1}\right)^n \quad . \qquad (1)$$

Using (1), we have:

$$\rho = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \quad . \qquad (2)$$

[7.2.1 (L) cont'd]

At first glance, we might be tempted to conclude from (2) that $\rho = 1$, arguing that $\frac{n}{n+1} \to 1$ as $n \to \infty$ and that $1^{\infty} = 1$. We hasten to caution against this approach.

In fact, we chose this problem to re-emphasize a previous result. Namely,

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \quad . \tag{3}$$

(We derived this result in our supplementary notes during the discussion of the natural logarithm.)

In any event, (3) can be incorporated into (2) as follows:

Dividing numerator and denominator by n, shows us that

$$\frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}}$$

$$\left(\frac{n}{n+1}\right)^n = \left(\frac{1}{1 + \frac{1}{n}}\right)^n = \frac{1^n}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{\left(1 + \frac{1}{n}\right)^n} \quad .$$

Hence:

$$\lim_{n\to\infty} \left[\left(\frac{n}{n+1}\right)^n\right] = \lim_{n\to\infty} \left[\frac{1}{\left(1 + \frac{1}{n}\right)^n}\right] = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} \left[\left(1 + \frac{1}{n}\right)^n\right]} = \frac{1}{e}$$

and since $e = 2.7^{+}$, $\frac{1}{e} < 1$.

$\therefore$ From (2), $\rho = \frac{1}{e} < 1$.

$\therefore$ By the ratio test $\sum_{n=0}^{\infty} \frac{n!}{n^n}$ converges.

[7.2.1 (L) cont'd]

(c) The most general result concerning $\sum_{n=1}^{\infty} \frac{1}{n^{\rho}}$ is obtained from the integral test (see the note at the end of this example).

Namely, we compare $\sum_{n=1}^{\infty} \frac{1}{n^{\rho}}$ with $\int_1^{\infty} \frac{dx}{x^{\rho}} = \int_1^{\infty} x^{-\rho} dx$. Since

$$\int x^{-\rho} dx = \begin{cases} \frac{1}{-\rho+1} x^{-\rho+1} + C, \text{ if } \rho \neq 1 \\ \ln x + C \text{ if } \rho = 1 \end{cases}$$

two cases are suggested.

Case 1:

$\rho = 1$

Then: $$\sum_{n=1}^{\infty} \frac{1}{n^{\rho}} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ and } \int_1^{\infty} x^{-\rho}\, dx = \int_1^{\infty} x^{-1}\, dx = \int_1^{\infty} \frac{dx}{x} = \ln \infty - \ln 1$$

$$= \ln \infty = \infty \quad .$$

Since $\int_1^{\infty} \frac{dx}{x}$ diverges so does $\sum_{n=1}^{\infty} \frac{1}{n}$.

In other words, from Case 1 we conclude $\sum_{n=1}^{\infty} \frac{1}{n^{\rho}}$ diverges if $\rho = 1$.

Case 2:

$\rho \neq 1$

Then $$\int_1^b x^{-\rho}\, dx = \frac{1}{-\rho+1} x^{-\rho+1} \Big|_1^b = \frac{1}{-\rho+1} \left[b^{-\rho+1} - 1 \right] = \frac{1}{1-\rho} \left[\frac{1}{b^{\rho-1}} - 1 \right]$$

[7.2.1 (L) cont'd]

$$\therefore \int_1^\infty x^{-\rho}dx = \lim_{b\to\infty} \left[\frac{1}{1-\rho}\left(\frac{1}{b^{\rho-1}} - 1\right)\right]$$

$$= \frac{1}{1-\rho}\left[\frac{1}{\lim_{b\to\infty} b^{\rho-1}} - 1\right] \qquad . \qquad (1)$$

The key now lies in the fact that if $\rho > 1$, $\lim_{b\to\infty} b^{\rho-1} = \infty$ or $\frac{1}{\lim_{b\to\infty} b^{\rho-1}} = 0$

$\therefore \rho > 1 \to \int_1^\infty x^{-\rho}\, dx = \frac{1}{1-\rho}\left[0 - 1\right] = \frac{1}{\rho-1}$, and since $\frac{1}{\rho-1}$ is a finite (positive) number, $\int_1^\infty x^{-\rho}\, dx$ converges if $\rho > 1$.

Hence, the integral test tells us that if $\rho > 1$, $\sum_{n=1}^\infty \frac{1}{n^\rho}$ converges.

On the other hand, if $\rho < 1$, then $\lim_{b\to\infty} b^{\rho-1} = 0$ since we then get a "large" number to a negative power, meaning that the denominator is large. At any rate, then, if $\rho < 1$, equation (1) yields

$$\int_1^\infty x^{-\rho}\, dx = \frac{1}{1-\rho}\ [\infty - 1] = \infty \qquad .$$

Hence $\sum_{n=1}^\infty \frac{1}{n^\rho}$ diverges if $\rho < 1$.

Putting the results of Case 1 and Case 2 together, we have:

$$\sum_{n=1}^\infty \frac{1}{n^\rho} \text{ converges if } \rho > 1, \text{ and diverges if } \rho \leqslant 1 .$$

In particular, since $3 > 1$, $\sum_{n=1}^\infty \frac{1}{n^3}$ converges

[7.2.1 (L) cont'd]

Note:

This exercise offers us a good illustration of when the ratio test is inconclusive. Namely, if we were to apply the ratio test to $\sum \frac{1}{n^3}$ we would have:

$$a_n = \frac{1}{n^3}, \quad a_{n+1} = \frac{1}{(n+1)^3}$$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{n^3}{(n+1)^3} \quad . \qquad (2)$$

But $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n^3}{(n+1)^3} = 1$. Notice that $(\frac{n}{n+1})^3 \to 1$ as $n \to \infty$; the difficulty in part (b) stemmed from the fact that our exponent was not a constant but rather n itself.)

Notice from (2) that $\frac{a_{n+1}}{a_n} < 1$ for each n; yet $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. That is, $\rho = 1$ even though $\frac{a_{n+1}}{a_n} < 1$ for each n. Since $\rho = 1$, the ratio test fails.

In fact, if we use the ratio test for $\sum \frac{1}{n}$, we would obtain $\frac{a_{n+1}}{a_n} = \frac{n}{n+1}$, whence ρ again equals 1 even though $\frac{n}{n+1} < 1$ for each n.

In other words, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, yet $\rho = 1$ in both cases. This offers conclusive proof that $\rho = 1$ allows us to make no definite conclusion concerning convergence.

(d) The main aim of this part is to emphasize the meaning of $\rho > 1$ in the result of (c). Namely, $1.000001 > 1$; hence, $\sum_{n=1}^{\infty} \frac{1}{n^{1.000001}}$ converges by part (c). Yet n and $n^{1.000001}$ seem

[7.2.1 (L) cont'd]

"pretty much" equal for "reasonable" values of n. The point, however, is that for sufficiently large values of n, $n^{1.000001}$ is much larger than n*; and convergence depends on what is happening for <u>sufficiently large values</u> of n.

(e) Here we have a good place to apply the <u>comparison test</u>. The integral test is awkward because it is at best difficult to evaluate $\int \frac{\sin^2 x\,dx}{x^3}$. The ratio is awkward because of $\lim_{n\to\infty} \frac{\sin^2(n+1)}{\sin^2 n}$.

The key behind the comparison test is that $\sin^2 n \leqslant 1$ for all n, hence $0 \leqslant \frac{\sin^2 n}{n^3} \leqslant \frac{1}{n^3}$.

Since, from part (c), $\sum \frac{1}{n^3}$ converges, it follows from the comparison test that

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$$

also converges.

<u>7.2.2</u>

(a) Let $a_n = \frac{2n+7}{3n+2}$. Then $\lim_{n\to\infty} a_n = \frac{2}{3} \neq 0$. Hence $\sum_{n=1}^{\infty} a_n$ $\left(= \sum_{n=1}^{\infty} \frac{2n+7}{3n+2}\right)$ diverges since $\lim_{n\to\infty} a_n \neq 0$. In other words, since the n'th term does not get arbitrarily small in magnitude, the series diverges.

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*For example, let $n = 10^{10,000,000,000}$. Then $n^{1.000001}$
$= (10^{10,000,000,000})^{1.000001}$
$= 10^{10,000010,000}$
$= 10^{10,000}\ 10^{10,000,000,000}$
$= 10^{10,000}\ n$.

[7.2.2 cont'd]

(b) (We start with $n = 2$ since $\ell n\ 1 = 0$ and this would make $\frac{1}{\ell n\ n}$ infinite.) The key here lies in the fact that $\ell n\ n < n$ for $n > 0$. Since $\ell n\ n < n$, it follows that:

$$\frac{1}{\ell n\ n} > \frac{1}{n} \quad .$$

Then since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges so also does $\sum_{n=2}^{\infty} \frac{1}{\ell n\ n}$.

(c) We compare $\sum_{n=2}^{\infty} \frac{1}{n\ \ell n\ n}$ with $\int_2^{\infty} \frac{dx}{x\ \ell n\ x}$. Letting $u = \ell n\ x$, where $\frac{dx}{x} = du$, we see that

$$\int_2^{\infty} \frac{dx}{x \ell n\ x} = \int_{\ell n\ 2}^{\infty} \frac{du}{u} = \ell n\ \infty - \ell n(\ell n\ 2) = \infty \quad .$$

Since $\int_2^{\infty} \frac{dx}{x\ \ell n\ x}$ diverges, the integral test tells us that $\sum_{n=2}^{\infty} \frac{1}{n\ \ell n\ n}$ also diverges.

(d) We use the ratio test with $a_n = \frac{(n+2)!}{n!\ 3^n}$. Then,

$$a_{n+1} = \frac{(n+3)!}{(n+1)!\ 3^{n+1}} \quad .$$

Therefore,

$$\frac{a_{n+1}}{a_n} = \frac{(n+3)!}{(n+1)!\ 3^{n+1}} \div \frac{(n+2)!}{n!\ 3^n}$$

$$= \frac{(n+3)!\ n!\ 3^n}{(n+1)!\ 3^{n+1}\ (n+2)!}$$

[7.2.2 cont'd]

$$= \left[\frac{(n+3)!}{(n+2)!}\right]\left[\frac{n!}{(n+1)!}\right]\left[\frac{3^n}{3^{n+1}}\right]$$

$$= (n+3)\left(\frac{1}{n+1}\right)\left(\frac{1}{3}\right)$$

$$= \frac{1}{3}\left(\frac{n+3}{n+1}\right)$$

$$\therefore \rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{3}\lim_{n\to\infty}\left(\frac{n+3}{n+1}\right) = \frac{1}{3} < 1 \quad .$$

Since $\rho < 1$, $\sum_{n=1}^{\infty} \frac{(n+2)!}{n!\ 3^n}$ converges by the ratio test.

<u>7.2.3 (L)</u>

(a) This exercise, hopefully, will show us how the comparison test often comes up in theoretical applications, and at the same time it will teach us how to get a feeling for what convergence means. The main intuitive idea is that if $\sum_{n=1}^{\infty} a_n$ converges, the terms a_n, themselves must approach zero "fast enough." Then, since the square of a positive number which is less than 1 is less than the number itself, it follows that if the a_n's approach zero sufficiently fast, then so also do the $a_n{}^2$'s .

More mathematically, we argue as follows:
Since $\lim_{n\to\infty} a_n = 0$ (since $\sum_{n=1}^{\infty} a_n$ converges), it follows that we can find N such that $n > N$ implies $0 \leqslant a_n < 1$. If we now multiply our inequality through by a_n, we obtain:

$$0 \leqslant a_n{}^2 < a_n \quad . \qquad (1)$$

[7.2.3 (L) cont'd]

Then since $\sum_{n=1}^{\infty} a_n$ converges, equation (1) allows us to conclude, by the comparison test, that $\sum_{n=1}^{\infty} a_n^2$ also converges. (Notice here a more general use of the comparison test. Namely we have that $a_n^2 < a_n$ for $n > N$ rather than for all n. In other words, we are again emphasizing that convergence depends only on what is happening "sufficiently far out" in the series.)

(b) Here we wish to emphasize, as usual, the delicacy of our various results, and how important it is to remember the proper "order" of our information.

In particular, recall that we have already seen that $\sum \frac{1}{n}$ diverges while $\sum \frac{1}{n^2}$ converges. Noting that $\frac{1}{n^2} = (\frac{1}{n})^2$ we see that if we let $a_n = \frac{1}{n}$, $\sum_{n=1}^{\infty} a_n$ is a positive divergent series while $\sum_{n=1}^{\infty} a_n^2$ converges.

Thus, the convergence,of $\sum a_n^2$ need not imply the convergence of a_n.*

From a more intuitive point of view, what we are saying is that since $a_n^2 \to 0$ more rapidly than does a_n, the fact that a_n^2 goes to zero "fast enough" is no guarantee that a_n does. Thus $\sum a_n^2$ may converge while $\sum a_n$ doesn't.

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*Certainly $\sum a_n$ may converge if $\sum a_n^2$ does. For example, $\sum \frac{1}{n^4}$ converges and so does $\sum \frac{1}{n^2}$. The point is that it is not an inescapable conclusion that $\sum a_n$ converges merely because $\sum a_n^2$ converges.

[7.2.3 (L) cont'd]

Notice that by understanding what is happening, we not only get a hint as to how to set up formal proofs, but we also find that we do not have to rely on memory to keep the subtleties straightened out.

7.2.4 (L)

The main part of this learning exercise is part (b). Part (a) simply presents a computational observation that is essential in proving part (b), while part (c) is simply a specific illustration of the result proved in part (b). From a learning point of view, what we want to emphasize is that our three tests for positive series (comparison, ratio, and integral) which are stressed in the text are by no means the only available tests. Rather, they are the most common (perhaps because they are the most elementary and least computational). There are many other tests, not described in our course, which are employed in more subtle series. All we wish to do in this exercise is give some inkling as to how other tests are proved and to present the reminder that other tests are often useful in more advanced examples.

(a) $$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

consists of n terms (namely, $\frac{1}{n+\underline{1}}, \frac{1}{n+\underline{2}}, \ldots, \frac{1}{n+\underline{n}}$), each of which is at least as great as $\frac{1}{2n}$ (since $\frac{1}{2n}$ has the greatest denominator of our n fractions).

Hence,

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq \underbrace{\frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}}_{n \text{ terms}} \quad . \qquad (1)$$

But, $$\underbrace{\frac{1}{2n} + \cdots + \frac{1}{2n}}_{n \text{ terms}} = n\left(\frac{1}{2n}\right) = \frac{1}{2} \, . \qquad (2)$$

[7.2.4 (L) cont'd]

Combining (1) and (2) we have that

$$\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \geqslant \frac{1}{2} \quad . \tag{3}$$

(By the way, (3) offers us another proof that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Namely, we group the terms in our series as follows:

$$1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + (\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}) + \ldots + (\frac{1}{2^n + 1} + \frac{1}{2^n + 2} + \ldots + \frac{1}{2^n + 2^n}) + \ldots \tag{4}$$

and each of the sets of parentheses represents a number at least as great as $\frac{1}{2}$ by (3). In other words we can keep adding on "at least $\frac{1}{2}$" as many times as we desire just by continuing the process long enough. Notice that the number of terms we must take to get "at least $\frac{1}{2}$" increases quite drastically as we get out further in the series.)

(b) Here we will use (a) to construct another test for convergence (divergence). Notice that the given information says more than that $\sum_{n=1}^{\infty} a_n$ diverges. Rather we are told that a rather "thin" sub-series is enough to conclude that the series diverges. That is, we need only look at the sum:

$$a_2 + a_4 + a_8 + a_{16} + a_{32} + a_{64} + \ldots$$

to determine that

$$a_1 + a_2 + a_3 + a_4 + \ldots$$

diverges.

[7.2.4 (L) cont'd]

The conclusion that we wish to arrive at is that under such "spectacular" conditions, dividing each term by its position in the sequence (and for terms way out in the sequence this makes a big difference; for example, a_{100} is replaced by $\frac{a_{100}}{100}$ etc.) is not enough of a reduction to make the resulting series converge.

In other words, under the conditions of this exercise, we are asking to show that not only does $\sum_{n=1}^{\infty} a_n$ diverge but so also does $\sum_{n=1}^{\infty} \frac{a_n}{n}$.

Our proof proceeds as follows:

We group the terms of

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \frac{a_4}{4} + \cdots$$

to form

$$a_1 + \frac{a_2}{2} + \left(\frac{a_3}{3} + \frac{a_4}{4}\right) + \left(\frac{a_5}{5} + \frac{a_6}{6} + \frac{a_7}{7} + \frac{a_8}{8}\right) + \cdots \qquad (5)$$

Notice that (5) is modelled after the format of (4). Now, since $a_3 \geqslant a_4$ we have:

$$\left(\frac{a_3}{3} + \frac{a_4}{4}\right) \geqslant \left(\frac{a_4}{3} + \frac{a_4}{4}\right) = a_4\left(\frac{1}{3} + \frac{1}{4}\right) \geqslant \frac{1}{2}\, a_4 \quad . \qquad (6)$$

Since $a_1 \geqslant 0$,

$$\left(a_1 + \frac{a_2}{2}\right) \geqslant \frac{a_2}{2} \quad . \qquad (6')$$

Similarly, $a_5 \geqslant a_6 \geqslant a_7 \geqslant a_8 \rightarrow$

$$\left(\frac{a_5}{5} + \frac{a_6}{6} + \frac{a_7}{7} + \frac{a_8}{8}\right) \geqslant a_8\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \geqslant \frac{1}{2}\, a_8 \quad . \qquad (6'')$$

[7.2.4 (L) cont'd]

Putting (6), (6') and (6") into (5), we obtain:

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \ldots = (a_1 + \frac{a_2}{2}) + (\frac{a_3}{3} + \frac{a_4}{4}) + (\frac{a_5}{5} + \frac{a_6}{6} + \frac{a_7}{7} + \frac{a_8}{8}) + \ldots$$

$$\geqslant \frac{a_2}{2} + \frac{a_4}{4} + \frac{a_8}{8} + \ldots$$

That is:

$$\sum_{n=1}^{\infty} \frac{a_n}{n} \geqslant \frac{1}{2}(a_2 + a_4 + a_8 + \ldots) = \frac{1}{2}\sum_{k=1}^{\infty} a_{2^k} \quad . \quad (7)$$

In any event, since $a_2 + a_4 + a_8 + \ldots$ diverges so does $\frac{1}{2}(a_2 + a_4 + a_8 + \ldots)$, and the result follows.

(c) To apply (b) here we observe that $\sum \frac{1}{n \ell n\ n}$ has the form $\sum \frac{a_n}{n}$ if $a_n = \frac{1}{\ell n\ n}$, in which case $a_{2^k} = \frac{1}{\ell n\ 2^k} = \frac{1}{k\ \ell n\ 2}$.

Hence by (b) $\sum_{n=2}^{\infty} \frac{1}{n\ \ell n\ n}$ diverges as soon as $\sum_{k=2}^{\infty} = \frac{1}{k\ \ell n\ 2}$ diverges. But $\sum_{k=2}^{\infty} \frac{1}{k\ \ell n\ 2} = \frac{1}{\ell n\ 2}\sum_{k=2}^{\infty} \frac{1}{k}$.

Since $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges, so does $\frac{1}{\ell n\ 2}\sum_{k=2}^{\infty} \frac{1}{k}$. Therefore $\sum_{n=2}^{\infty} \frac{1}{n\ \ell n\ n}$ diverges.

While we obtained the same result in Exercise 7.2.2 (c), our aim here was to apply a new test even though we already **knew** the correct answer.

[7.2.4 (L) cont'd]

As a final aside, let's make sure it's clear that the divergence of $\sum_{n=1}^{\infty} a_n$ is not enough to guarantee that $a_2 + a_4 + a_8 + \ldots$ also diverges.

For example, let $a_n = \frac{1}{n}$. Then, as we know $\sum_{n=1}^{\infty} a_n$ diverges; but in this case

$$a_2 + a_4 + a_8 + \ldots + a_{2^k} + \ldots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^k} + \ldots$$

which is the convergent geometric series $\sum_{k=1}^{\infty} (\frac{1}{2})^k = 1$.

7.2.5

Actually this is essentially a rewording of 7.2.4 (L), part (a). Our aim here is to help you reinforce the ideas of that exercise.

Our procedure is to write:

$$a_1 + a_2 + a_3 + a_4 + \ldots =$$

$$a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) \quad . \tag{1}$$

Now since $a_3 \geqslant a_4$,

$$(a_3 + a_4) \geqslant 2a_4 \tag{2}$$

and since $a_5 \geqslant a_6 \geqslant a_7 \geqslant a_8$,

$$(a_5 + a_6 + a_7 + a_8) \geqslant a_8 + a_8 + a_8 + a_8 = 4a_8 \quad . \tag{3}$$

[7.2.5 cont'd]

Putting (2) and (3) into (1), we have

$$a_1 + a_2 + a_3 + a_4 + \dots \geq a_1 + a_2 + 2a_4 + 4a_8 + \dots \quad (4)$$

Then since $a_1 \geq \frac{a_1}{2}$, (4) becomes

$$\sum_{n=1}^{\infty} a_n \geq \frac{1}{2} a_1 + a_2 + 2a_4 + 4a_8 + \dots$$

$$\therefore 2 \sum_{n=1}^{\infty} a_n \geq a_1 + 2a_2 + 4a_4 + 8a_8 + \dots \quad (5)$$

From (5) we conclude that $2 \sum_{n=1}^{\infty} a_n$ diverges since we compare it with the given divergent series $a_1 + 2a_2 + 4a_4 + 8a_8 + \dots$

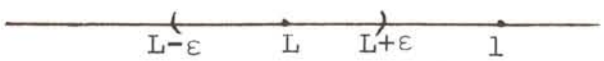
$$\therefore \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

<u>7.2.6 (L)</u>

(a) This test is known as the <u>root test</u> and has application where our other tests fail (see part (b)). Aside from its practical value, we include it as an exercise because its proof closely parallels the proof of the ratio test and, as a result, we have another opportunity to "practice" the proof.

The key idea is as follows:

If $L < 1$, we can choose $\varepsilon > 0$ such that $L + \varepsilon < 1$. That is:



[7.2.6 (L) cont'd]

Hence by definition of $\lim_{n\to\infty} \sqrt[n]{a_n} = L$, we have that for the given ε we can find N such that

$$n > N \rightarrow \sqrt[n]{a_n} < L + \varepsilon$$

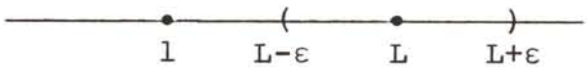
That is

$$n > N \rightarrow a_n < (L + \varepsilon)^n$$

$$\therefore \quad \sum_{n=N+1}^{\infty} a_n < \sum_{n=N+1}^{\infty} (L + \varepsilon)^n$$

But since $\sum (L + \varepsilon)^n$ is a geometric series with positive ratio $L + \varepsilon < 1$, $\sum_{n=N+1}^{\infty} (L + \varepsilon)^n$ converges. Hence, by the comparison test $\sum_{n=N+1}^{\infty} a_n$ also converges. Finally since adding on a finite number of finite terms does not affect convergence, it follows that $\sum_{n=1}^{\infty} a_n$ converges.

On the other hand, if $L > 1$ we choose $\varepsilon > 0$ so that



[7.2.6 (L) cont'd]

Then, there exists N such that $n > N \to \sqrt[n]{a_n} > L - \varepsilon > 1$ $\therefore n > N \to a_n > (L - \varepsilon)^n > 1$, and now $\sum a_n$ diverges (by the comparison test) since $\sum (L - \varepsilon)^n$ diverges.

(b) Here $a_n = (\frac{n+1}{3n})^n$ and it follows <u>immediately</u> that $\sqrt[n]{a_n} = \frac{n+1}{3n}$

$\therefore \lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} (\frac{n+1}{3n}) = \frac{1}{3} < 1$.

Hence, from (a), $\sum\limits_{n=1}^{\infty} (\frac{n+1}{3n})^n$ converges (since $L = \frac{1}{3} < 1$).

Notice that the ratio test here would have yielded:

$$a_n = (\frac{n+1}{3n})^n, \quad a_{n+1} = \left[\frac{n+2}{3(n+1)}\right]^{n+1} = \frac{(n+2)^{n+1}}{3^{n+1}(n+1)^{n+1}}$$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{(n+2)^{n+1}}{3^{n+1}(n+1)^{n+1}}\left[\frac{3^n(n)^n}{(n+1)^n}\right]$$

$$= \frac{(n+2)^{n+1}\, n^n}{3\ (n+1)^{2n+1}}$$

in which case, determining $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ can be a bit messy.

In other words, (b) is an extreme illustration of how the root test can be much more palatable than the ratio test.

<u>7.2.7</u>

We use the root test to obtain:

[7.2.7 cont'd]

$$a_n = \left(\frac{n}{n+1}\right)^{n^2}$$

$$\therefore \sqrt[n]{a_n} = \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{\frac{1}{n}} = \left(\frac{n}{n+1}\right)^n$$

$$= \left[\frac{1}{\left(1 + \frac{1}{n}\right)^n}\right]$$ (see, e.g., solution of Exercise 7.2.1 (L), part (b))

$$\therefore L = \lim_{n\to\infty} \sqrt[n]{a^n} = \frac{1}{\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

$$\therefore \sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} \text{ converges.}$$

<u>7.2.8 (L)</u>

(a) If $\sum a_n$ converges, let $s_n = a_1 + \ldots + a_n$. Then $\sum a_n = S = \lim_{n\to\infty} s_n$.

Now let $t_n = ca_1 + \ldots + ca_n$. Then

$$\sum ca_n = \lim_{n\to\infty} t_n = \lim_{n\to\infty} c(a_1 + \ldots + a_n) = c \lim_{n\to\infty} (a_1 + \ldots + a_n)$$

$$= c \lim_{n\to\infty} s_n$$

$$= cS$$

$$\therefore \sum ca_n \text{ converges.}$$

[7.2.8 (L) cont'd]

If $\sum a_n$ diverges, then $\sum a_n = \infty$. That is, a positive series either converges or else diverges to infinity.

If we again let $t_n = ca_1 + \dots + ca_n$ we have:

$$\sum_{n=1}^{\infty} ca_n = \lim_{n\to\infty} t_n = \lim_{n\to\infty} c(a_1 + \dots + a_n) = c \lim_{n\to\infty} (a_1 + \dots + a_n).$$

Since $\lim_{n\to\infty} (a_1 + \dots + a_n) = \infty$, $c \lim_{n\to\infty} (a_1 + \dots + a_n) = \pm\infty$, depending on whether $c < 0$ or $c > 0$. (If $c = 0$, $c(a_1 + \dots + a_n) = 0$ for all n; whence $\sum ca_n = 0$.)

(b) The key idea, now, is that we can study negative series as a corollary of positive series. Namely suppose $a_n \leqslant 0$ for all n; then $-a_n \geqslant 0$. Using (a) with $c = -1$ we have that

$$\sum_{n=1}^{\infty} -a_n \text{ converges} \longleftrightarrow \sum_{n=1}^{\infty} (-1)(-a_n) \text{ converges}$$

but $\sum_{n=1}^{\infty} (-1)(-a_n) = \sum_{n=1}^{\infty} a_n$.

Thus we may determine the convergence of the negative series $\sum a_n$ by studying the positive series $\sum -a_n$.

For example, since $\sum \frac{1}{n^2}$ converges so also does $\sum -\frac{1}{n^2}$, etc.

In the next block we will consider the situation of a series which is neither positive nor negative. That is, we will consider series, some of whose terms are positive and some negative.

7.2.9 (L)

Since $1 - \frac{1}{n} < 1$, both $\sum \ln(1 - \frac{1}{n})$ and $\sum \ln(1 - \frac{1}{n^2})$ are examples of negative series. The main aim of this exercise is to emphasize that despite the number of tests for convergence at our disposal, we must often actually compute the limit of the sequence of partial sums in order to see whether the series converges. At any rate,

(a) $$\sum_{n=2}^{\infty} \ln(1 - \frac{1}{n}) = \sum_{n=2}^{\infty} \ln(\frac{n-1}{n})$$

$$= \sum_{n=2}^{\infty} [\ln(n-1) - \ln n]$$

but this is a telescoping sum. Namely,

$$\sum_{n=2}^{\infty} \ln(1-\frac{1}{n}) = [\ln 1 - \ln 2] + [\ln 2 - \ln 3] + [\ln 3 - \ln 4] + \ldots$$

$$= -\ln \infty$$

$$= -\infty .$$

(That is, let $s_n = [\ln 1 - \ln 2] + \ldots + [\ln(n-1) - \ln n]$ then everything except $\ln 1$ and $-\ln n$ "cancel" since each term appears twice but with opposite signs. Therefore:

$$s_n = \ln 1 - \ln n$$

$$= 0 - \ln n$$

$$= -\ln n$$

$$\therefore \lim_{n\to\infty} s_n = \lim_{n\to\infty} [-\ln n] = -\infty .)$$

[7.2.9 (L) cont'd]

In other words, even though $\lim_{n\to\infty} [\ln(1 - \frac{1}{n})] = 0$ the convergence is not "rapid enough" to cause $\sum_{n=2}^{\infty} \ln(1 - \frac{1}{n})$ to converge.

(b) Here we must be a bit more careful in keeping track of terms than we were in (a), but the idea is basically the same. Namely,

$$\sum_{n=2}^{\infty} \ln(1 - \frac{1}{n^2}) = \sum_{n=2}^{\infty} \ln(\frac{n^2 - 1}{n^2})$$

$$= \sum_{n=2}^{\infty} [\ln(n^2 - 1) - \ln n^2]$$

$$= \sum_{n=2}^{\infty} [\ln(n + 1)(n - 1) - 2 \ln n]$$

$$= \sum_{n=2}^{\infty} \{\ln(n + 1) + \ln(n - 1) - 2 \ln n\} \quad . \tag{1}$$

The computational key behind simplifying (1) lies in the observation that except for the first term and the last two terms in the partial sum $\sum_{n=2}^{k} \{\ln(n + 1) + \ln(n - 1) - 2 \ln n\}$ all other terms are completely cancelled. (For example, $\ln 4$ occurs when we look at $\ln(n + 1)$ with $n = 3$. It also occurs when we look at $\ln(n - 1)$ with $n = 5$. On the other hand with $n = 4$ we obtain $-2 \ln 4$ which cancels our two $\ln 4$'s .)

[7.2.9 (L) cont'd]

In more detail,

$$
\begin{aligned}
\sum_{n=2}^{7} \{\ell n(n+1) + \ell n(n-1) - 2\,\ell n\, n\} = \; & \cancel{\ell n\, 3} + \cancel{\ell n\, 1}^{\circ} - 2\,\ell n\, 2 \\
& + \cancel{\ell n\, 4} + \ell n\, 2 - \cancel{2\,\ell n\, 3} \\
& + \cancel{\ell n\, 5} + \cancel{\ell n\, 3} - \cancel{2\,\ell n\, 4} \\
& + \cancel{\ell n\, 6} + \cancel{\ell n\, 4} - \cancel{2\,\ell n\, 5} \\
& + \ell n\, 7 + \cancel{\ell n\, 5} - \cancel{2\,\ell n\, 6} \\
& + \ell n\, 8 + \cancel{\ell n\, 6} - 2\,\ell n\, 7 \\
\hline
& \ell n\, 8 - \ell n\, 7 - \ell n\, 2
\end{aligned}
$$

More generally,

$$
\begin{aligned}
\sum_{n=2}^{k} \{\ell n(n+1) + \ell n(n-1) - 2\,\ell n\, n\} &= \ell n(k+1) - \ell n\, k - \ell n\, 2 \\
&= \ell n\left(\frac{k+1}{k}\right) - \ell n\, 2 \\
&= \ell n\left(1 + \frac{1}{k}\right) - \ell n\, 2 \quad .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sum_{n=2}^{\infty} \ell n\left(1 - \frac{1}{n^2}\right) &= \sum_{n=2}^{\infty} \{\ell n(n+1) + \ell n(n-1) - 2\,\ell n\, n\} \\
&= \lim_{k\to\infty} \left[\ell n\left(1 + \frac{1}{k}\right) - \ell n\, 2\right] \\
&= 0 - \ell n\, 2 \\
&= -\,\ell n\, 2 \quad .
\end{aligned}
$$

[7.2.9 (L) cont'd]

Thus $\sum_{n=2}^{\infty} \ell n(1 - \frac{1}{n^2})$ converges.

(The more alert among us may be a bit dismayed at first glance about our results. Namely $1 - \frac{1}{n} < 1 - \frac{1}{n^2}$, yet $\sum \ell n(1 - \frac{1}{n})$ diverges while $\sum \ell n(1 - \frac{1}{n^2})$ converges. This does not contradict the comparison test, since $\ell n(1-\frac{1}{n})$ and $\ell n(1 - \frac{1}{n^2})$ are both negative. In other words the fact that $n(1 - \frac{1}{n}) < \ell n(1 - \frac{1}{n^2})$ means that the <u>magnitude</u> of $\ell n(1 - \frac{1}{n^2})$ is less than that of $\ell n(1 - \frac{1}{n})$. In other words, for negative numbers, the smaller the magnitude the larger the number. In still other words, $\ell n(1 - \frac{1}{n^2})$ is closer to zero than is $\ell n(1 - \frac{1}{n})$.)

UNIT 3: Absolute Convergence

7.3.1

$$\left(1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16}\right) + \left(\frac{1}{5} - \frac{1}{18} - \frac{1}{20} - \frac{1}{22} - \frac{1}{24}\right) + \ldots$$

$$+ \ldots + \left(\frac{1}{2n-1} - \frac{1}{8n-6} - \frac{1}{8n-4} - \frac{1}{8n-2} - \frac{1}{8n}\right) + \ldots$$

7.3.2

$$a_n^{\ 2} = \frac{\left[(-1)^{n+1}\right]^2}{(\sqrt{n})^2} = \frac{1}{n}$$

But we already know that $\sum \frac{1}{n}$ diverges.

$\therefore \sum a_n^{\ 2}$ diverges.

On the other hand $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ is an alternating series whose terms are non-increasing in magnitude, and the terms tend to 0 in the limit. Thus, in line with our theorem concerning alternating series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} a_n$ converges.

Therefore $\sum a_n^{\ 2}$ diverges while $\sum a_n$ converges. According to Ex. 7.2.3, this couldn't happen if $\sum a_n$ were a positive series.

7.3.3(L)

Our purpose here is to indicate how we use the tests for positive series to test a non-positive series for absolute convergence.

[7.3.3(L) cont'd]

In our present example we note that $\sum_{n=0}^{\infty}(-2)^n(n+1)(c-1)^n$ need not be a positive series. For example if c > 1 then each factor is positive except $(-2)^n$ which alternates in sign. In other words, if c > 1, $\sum(-2)^n(n+1)(c-1)^n$ is an alternating series.

In any event, to study the absolute convergence of $\sum_{n=0}^{\infty}(-2)^n(n+1)(c-1)^n$ we study the convergence of the <u>positive</u> series $\sum_{n=0}^{\infty}|(-2)^n(n+1)(c-1)^n|$.

Now,

$$\begin{aligned}|(-2)^n(n+1)(c-1)^n| &= |(-2)^n|\ |n+1|\ |(c-1)^n| \\ &= 2^n(n+1)|c-1|^n\end{aligned}$$

Hence, we must investigate for what values of c, $\sum_{n=0}^{\infty}2^n(n+1)|c-1|^n$ converges. The most natural test is the ratio test. We have

$$a_n = 2^n(n+1)|c-1|^n$$

$$a_{n+1} = 2^{n+1}(n+2)|c-1|^{n+1}$$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{2^{n+1}(n+2)|c-1|^{n+1}}{2^n(n+1)|c-1|^n} = \frac{2(n+2)|c-1|}{(n+1)}$$

$$\therefore \rho = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left[2\left(\frac{n+2}{n+1}\right)|c-1|\right] = 2|c-1| \lim_{n\to\infty}\frac{n+2}{n+1} = 2|c-1| \qquad (1)$$

[7.3.3(L) cont'd]

But for convergence we must have $\rho < 1$. (We must check the case $\rho = 1$ separately since the ratio test fails if $\rho = 1$.)

Letting $\rho < 1$ in (1) yields

$$2|c-1| < 1$$

$$\therefore |c-1| < \frac{1}{2}$$

$$\therefore -\frac{1}{2} < c-1 < \frac{1}{2}$$

$$\therefore \frac{1}{2} < c < \frac{3}{2}$$

Hence

$$\sum_{n=0}^{\infty} |(-2)^n(n+1)(c-1)^n| \text{ converges when } \frac{1}{2} < c < \frac{3}{2}$$

Checking the "end points" $c = \frac{1}{2}$ and $c = \frac{3}{2}$ (which corresponds to $\rho = 1$) we find

$$\sum_{n=0}^{\infty} |(-2)^n(n+1)(\tfrac{1}{2} - 1)^n| = \sum_{n=0}^{\infty} (-2)^n(-\tfrac{1}{2})^n(n+1)$$

$$= \sum_{n=0}^{\infty} \left[(-2)(-\tfrac{1}{2})\right]^n(n+1)$$

$$= \sum_{n=0}^{\infty} (1)^n(n+1)$$

$$= \sum_{n=0}^{\infty} (n+1)$$

$$= \infty$$

[7.3.3(L) cont'd]

$$\therefore \sum_{n=0}^{\infty} |(-2)^n (n+1)(c-1)^n| \text{ diverges when } c = \frac{1}{2}$$

Similarly when $c = \frac{3}{2}$, we obtain

$$\sum_{n=0}^{\infty} |(-2)^n (n+1)(c-1)^n| = \sum_{n=0}^{\infty} |(-2)^n (n+1)(\tfrac{3}{2} - 1)^n| = \sum_{n=0}^{\infty} |(-2)^n (n+1)(\tfrac{1}{2})^n|$$

$$= \sum_{n=0}^{\infty} |(-\tfrac{2}{2})^n (n+1)| = \sum_{n=0}^{\infty} |(-1)^n (n+1)|$$

$$= \sum_{n=0}^{\infty} (n+1)$$

$$= \infty$$

$$\therefore \sum_{n=0}^{\infty} |(-2)^n (n+1)(c-1)^n| \text{ diverges when } c = \frac{3}{2}$$

$$\therefore \sum_{n=0}^{\infty} |(-2)^n (n+1)(c-1)^n| \text{ converges} \leftrightarrow \frac{1}{2} < c < \frac{3}{2}$$

$$\therefore \sum_{n=0}^{\infty} (-2)^n (n+1)(c-1)^n \text{ converges absolutely} \leftrightarrow \frac{1}{2} < c < \frac{3}{2}$$

7.3.4

(a) We study the convergence of $\sum_{n=1}^{\infty} |nc^n|$

Now: $$\sum_{n=1}^{\infty} |nc^n| = \sum_{n=1}^{\infty} |n|\,|c^n| = \sum_{n=1}^{\infty} n|c|^n$$

We then invoke the ratio test with $a_n = n|c|^n$, $a_{n+1} = (n+1)|c|^{n+1}$. Then:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)|c|^{n+1}}{n|c|^n} = \frac{n+1}{n}|c|$$

$$\therefore \rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)|c| = |c|$$

$$\rho < 1 \leftrightarrow |c| < 1$$

$$\rho = 1 \leftrightarrow |c| = 1$$

$$\rho > 1 \leftrightarrow |c| > 1$$

$$\therefore \quad \sum_{n=1}^{\infty} n|c|^n \begin{cases} \text{converges if } |c| < 1 \\ \text{diverges if } |c| > 1 \end{cases}$$

The doubtful case ($\rho = 1$) occurs when $|c| = 1$. In this case $\sum_{n=1}^{\infty} n|c|^n = \sum_{n=1}^{\infty} n = \infty$

$$\therefore \sum_{n=1}^{\infty} n|c|^n \begin{cases} \text{converges if } |c| < 1 \\ \text{diverges if } |c| \geqslant 1 \end{cases}$$

[7.3.4 cont'd]

Hence $\sum_{n=1}^{\infty} nc^n$ converges absolutely $\leftrightarrow |c| < 1 \leftrightarrow -1 < c < 1$

(b) We look at $\sum_{n=0}^{\infty} |n!c^n| = \sum_{n=0}^{\infty} n!|c|^n$

Then: $a_n = n!|c|^n$, $a_{n+1} = (n+1)!|c|^{n+1}$

$$\therefore \rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{(n+1)!|c|^{n+1}}{n!|c|^n}\right] = \lim_{n\to\infty} [(n+1)|c|]$$

$\therefore \rho = \infty$ if $c \neq 0$

$\rho = 0$ if $c = 0$

$\therefore \rho < 1 \leftrightarrow c = 0$

$$\therefore \sum n!c^n \text{ converges absolutely} \leftrightarrow c = 0$$

<u>Note</u>:

If $c = 0$, obviously $\sum_{n=0}^{\infty} a_n c^n = 0$ for any finite numbers $a_1, a_2, a_3, \ldots$. What we have seen in this example is that $\sum a_n c^n = 0$ <u>only</u> if $c = 0$. In other words, in this problem $a_n (=n!)$ grew so rapidly that it dwarfed c^n for large n no matter how small in magnitude c was (as long as $c \neq 0$). More specifically, in this exercise we are reaffirming that n! goes to infinity more rapidly than does c^n.

(c) Let $a_n = \frac{|c+5|^n}{n+1}$. Then $a_{n+1} = \frac{|c+5|^{n+1}}{n+2}$

[7.3.4 cont'd]

$$\therefore \rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right) |c+5| = |c+5|$$

$$|c+5| < 1 \to \sum_{n=0}^{\infty} \frac{|c+5|^n}{n+1} \text{ converges}$$

$$|c+5| = 1 \to \sum_{n=0}^{\infty} \frac{|c+5|^n}{n+1} = \sum_{n=0}^{\infty} \frac{1^n}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

$$\therefore \sum_{n=0}^{\infty} \frac{(c+5)^n}{n+1} \text{ converges absolutely} \leftrightarrow |c+5| < 1$$

Removing absolute value signs we have:

$$\sum_{n=0}^{\infty} \frac{(c+5)^n}{n+1} \text{ converges absolutely} \leftrightarrow -1 < c+5 < 1$$

$$\leftrightarrow -6 < c < -4$$

<u>Note:</u>

If $c = -6$, then $\sum \frac{(c+5)^n}{n+1} = \sum \frac{(-1)^n}{n+1}$, which converges. In other words $\sum \frac{(c+5)^n}{n+1}$ converges when $c = -6$ <u>but not absolutely</u>.

(d) Letting $a_n = \frac{|c|^n}{n!}$, $a_{n+1} = \frac{|c|^{n+1}}{(n+1)!}$ we have

$$\rho = \lim_{n\to\infty} \frac{|c|^{n+1}\, n!}{(n+1)!\,|c|^n} = \lim_{n\to\infty} \frac{|c|}{n+1} = 0$$

[7.3.4 cont'd]

$\therefore \rho = 0 < 1$ for every c

$\therefore \sum_{n=0}^{\infty} \frac{c^n}{n!}$ converges for all real numbers, c, that is, for $-\infty < c < \infty$

Note:

In a way, this is the converse of (b). Namely, our denominator grows so fast that it can't be "undone" by our numerator, no matter how large (but finite, of course) c is.

7.3.5(L)

Actually, the major learning experience of this exercise involves the given information - that is, the concept of how one multiplies series. In line with our theme that operations for infinite sets are adapted from the corresponding operations with finite sets, we get a hint of how to define the product of two series by looking at how we take the product of two finite sums - say

$$(a_0 + a_1 + \dots + a_m)(b_0 + b_1 + b_2 + \dots + b_k) \qquad (1)$$

Now we know that the product consists of a sum of terms each of which consists of two factors, one from each set of parentheses. A convenient way of making sure that we get all the terms is to keep track of the subscripts. For example, the only way that the sum of the subscripts of an a and a b can be 0 is if each subscript is 0. That is, we must have a_0b_0. On the other hand, there are two ways that the sum of the subscripts can be 1, namely a_0b_1 and a_1b_0. Reasoning in this way, the product in (1) may be written as

[7.3.5(L) cont'd]

$$a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \dots$$

This idea is particularly convenient when we think of multiplying polynomials. In other words,

$$(a_0 + a_1x + \dots + a_mx^m)(b_0 + b_1x + \dots + b_kx^k)$$

is equal to

$$a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \dots.$$

(where in each term the sum of the subscripts equals the exponent). This "coding" system gives us a very quick way to write down any particular term in the product. For example, the coefficient of x^6 in the product of the two polynomials would be

$$(a_0b_6 + a_1b_5 + a_2b_4 + a_3b_3 + a_4b_2 + a_5b_1 + a_6b_0),$$

provided, of course, that we keep in mind the fact that if, for example, our first polynomial is only of degree 4, then a_5 and a_6 will be zero.

With this as background, notice that our definition of the product of two series (and this type of product is more specifically called the Cauchy product) is precisely the analog of what we did in the finite case. That is, we define the product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ to be

$$(a_0b_0) + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \dots$$

where the nth term is the sum of those terms, the sum of whose subscripts is n.

[7.3.5(L) cont'd]

That is, if c_n denotes the nth term in the product, then

$$c_0 = a_0b_0$$
$$c_1 = a_0b_1 + a_1b_0$$
$$\vdots$$
$$c_n = a_0b_n + a_1b_{n-1} + \cdots + a_nb_0$$
$$\vdots$$

While it might not look like it, this is exactly the same thing as saying that $c_n = \sum_{k=0}^{n} a_kb_{n-k}$. In other words, if we fix n and let k vary from 0 to n, we have from the definition of the sigma-notation

$$\sum_{k=0}^{n} a_kb_{n-k} = a_0b_n + a_1b_{n-1} + \cdots + a_kb_{n-k} + \cdots + a_nb_0$$

We are now ready to tackle this exercise. Namely,

$$\left(\sum_{n=0}^{\infty} \frac{1}{n+1}\right)^2 = \left(\sum_{n=0}^{\infty} \frac{1}{n+1}\right)\left(\sum_{n=0}^{\infty} \frac{1}{n+1}\right)$$

In this case we have $\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right)$ with $a_n = b_n = \frac{1}{n+1}$.

[7.3.5(L) cont'd]

Therefore, $a_k b_{n-k} = \left(\frac{1}{k+1}\right)\left(\frac{1}{n-k+1}\right)^*$

$$\therefore\ c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} \frac{1}{(k+1)(n-k+1)} = \frac{1}{n+1} + \frac{1}{2n} + \frac{1}{3(n-1)} + \ldots + \frac{1}{n+1}$$

For example

$$c_0 = \sum_{k=0}^{0} \frac{1}{(k+1)(0-k+1)} = 1$$

$$c_1 = \sum_{k=0}^{1} \frac{1}{(k+1)(1-k+1)} = \sum_{k=0}^{1} \frac{1}{(k+1)(2-k)} = \frac{1}{2} + \frac{1}{2} = 1$$

$$c_2 = \sum_{k=0}^{2} \frac{1}{(k+1)(2-k+1)} = \sum_{k=0}^{2} \frac{1}{(k+1)(3-k)} = \frac{1}{3} + \frac{1}{4} + \frac{1}{3} = \frac{11}{12}$$

$$c_3 = \sum_{k=0}^{3} \frac{1}{(k+1)(3-k+1)} = \sum_{k=0}^{3} \frac{1}{(k+1)(4-k)} = \frac{1}{4} + \frac{1}{6} + \frac{1}{6} + \frac{1}{4} = \frac{5}{6}$$

Hence, according to our definition,

$$\left(\sum_{n=0}^{\infty} \frac{1}{n+1}\right)^2 = \left(1 + \frac{1}{2} + \frac{1}{3} + ..\right)\left(1 + \frac{1}{2} + \frac{1}{3} + ..\right) = 1 + \quad + \frac{11}{12} + \frac{5}{6} + \ldots. + \sum_{k=0}^{n} \frac{1}{(k+1)(n-k+1)} + \ldots$$

*Recall that $b_n = \frac{1}{n+1}$ is read as $b_{(\ )} = \frac{1}{(\ ) + 1}$ since n is a dummy variable etc.

[7.3.5(L) cont'd]

If we prefer the completely abstract $\sum$-notation we write our answer as

$$\left(\sum_{n=0}^{\infty} \frac{1}{n+1}\right)^2 = \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{1}{(k+1)(n-k+1)}\right)$$

The nth term in the series is a sum of n+1 numbers; n=0,1,2,...

Clearly, the product of two series is more complicated than the sum. We must also beware, in terms of our discussion of absolute convergence, that if we change the order of the terms in our product we may change the product. Of course, this problem doesn't exist when we deal with finite sums. In other words, the Cauchy product is but one of many ways to define the product of two series.

In fact in Exercises 7.3.6 and 7.3.7 we shall illustrate the perhaps surprising result that the product of two convergent series need not be a convergent series.

7.3.6

Here $a_n = b_n = \dfrac{(-1)^n}{\sqrt{n+1}}$

$$\left.\begin{array}{l} \therefore a_k = \dfrac{(-1)^k}{\sqrt{k+1}} \\[2ex] b_{n-k} = \dfrac{(-1)^{n-k}}{\sqrt{n-k+1}} \end{array}\right\} \therefore a_k b_{n-k} = \frac{(-1)^k(-1)^{n-k}}{\sqrt{k+1}\,\sqrt{n-k+1}} = \frac{(-1)^n}{\sqrt{(k+1)(n-k+1)}}$$

$$\therefore c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} \frac{(-1)^n}{\sqrt{(k+1)(n-k+1)}}$$

[7.3.6 cont'd]

$$\therefore \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}\right)^2 = \sum_{n=0}^{\infty} c_n, \text{ where } c_n = \sum_{k=0}^{n} \frac{(-1)^n}{\sqrt{(k+1)(n-k+1)}}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{(-1)^n}{\sqrt{(k+1)(n-k+1)}}\right)$$

$$= 1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}\sqrt{2}} + \frac{1}{\sqrt{3}}\right) -$$

$$\left(\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{3}\sqrt{2}} + \frac{1}{\sqrt{2}\sqrt{3}} + \frac{1}{\sqrt{4}}\right) + \ldots +$$

$$+ (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}} + \ldots$$

$$= 1 - \sqrt{2} + \frac{4\sqrt{3} + 3}{6} - \frac{\sqrt{6} + 3}{3} + \ldots$$

7.3.7(L)

To get a proper overview of this exercise, what we are going to do is show that if $c_n = \sum_{k=0}^{n} \frac{(-1)^n}{\sqrt{(k+1)(n-k+1)}}$ then $\lim_{n\to\infty} c_n \neq 0$.

This, in turn, will mean that $\sum c_n$ diverges. But $\sum c_n$ is the square of the convergent series $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ (since this is an alternating series which meets the necessary criteria for convergence). Thus, we will have proved that the product of two convergent series (note that the square of a convergent series <u>is</u> a product of two convergent series) can be a divergent series*.

At any rate,

(a) $$(k+1)(n-k+1) = kn - k^2 + k + n - k+1 = kn - k^2 + n + 1$$

$$= -(k^2 - kn) + n+1$$

$$= -(k^2 - kn + \frac{n^2}{4} - \frac{n^2}{4}) + n+1$$

$$= -(k^2 - kn + \frac{n^2}{4}) + \frac{n^2}{4} + n+1$$

$$\therefore (k+1)(n-k+1) = -(k - \frac{n}{2})^2 + (\frac{n}{2} + 1)^2$$

$$= (\frac{n}{2} + 1)^2 - (k - \frac{n}{2})^2 \qquad (1)$$

and since $(k - \frac{n}{2})^2 \geqslant 0$ (since its the square of a real number), it follows that

$$(\frac{n}{2} + 1)^2 - (k - \frac{n}{2})^2 \leqslant (\frac{n}{2} + 1)^2 \qquad (2)$$

Combining (1) and (2), we obtain

$$(k+1)(n-k+1) = (\frac{n}{2} + 1)^2 - (k - \frac{n}{2})^2 \leqslant (\frac{n}{2} + 1)^2$$

---

*While we will not prove it here, there is yet another interesting property of absolute convergence. Namely, if <u>at least one</u> of the two series is <u>absolutely</u> convergent then the <u>(Cauchy) product</u> will also be a convergent series. Note that in our example, $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ is <u>conditionally</u> convergent.

[7.3.7(L) cont'd]

(b)

$$c_n = \sum_{k=0}^{n} \frac{(-1)^n}{\sqrt{(k+1)(n-k+1)}} \text{ or } |c_n| = \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}} \qquad (3)$$

Now from (a)

$$(k+1)(n-k+1) \leqslant \left(\frac{n}{2} + 1\right)^2$$

$$\therefore \sqrt{(k+1)(n-k+1)} \leqslant \frac{n}{2} + 1 = \frac{n+2}{2}$$

$$\therefore \frac{1}{\sqrt{(k+1)(n-k+1)}} \geqslant \frac{2}{n+2} \qquad (4)$$

Putting (4) into (3), we have

$$|c_n| = \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}} \geqslant \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2}{n+2} \sum_{k=0}^{n} 1$$

$$\therefore |c_n| \geqslant \frac{2}{n+2} (\underbrace{1 + \ldots + 1}_{n+1 \text{ times}}) = \frac{2}{n+2}(n+1)$$

$$\therefore \lim_{n\to\infty} |c_n| \geqslant \lim_{n\to\infty} \frac{2(n+1)}{n+2} = 2 \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right) = 2$$

Since $\lim_{n\to\infty} |c_n| \geqslant 2$, it follows that $\lim_{n\to\infty} c_n \neq 0$. Finally, since $\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}\right)^2 = \sum_{n=0}^{\infty} c_n$ and $\lim_{n\to\infty} c_n \neq 0$, we have that $\sum_{n=0}^{\infty} c_n$; hence, $\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}\right)^2$, diverges.

[7.3.7(L) cont'd]

(c) This just reinforces the remarks in our overview. Namely, $\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}\right)$ represents the product of two (conditionally) convergent series - which is, itself, divergent.

UNIT 4: Polynomial Approximations

7.4.1 (L)

(a) We have seen that the conditions $P_n(0) = f(0)$, $P_n'(0) = f'(0)$, $P_n^{(n)}(0) = f^{(n)}(0)$ lead to:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)\,x^2}{2!} + \frac{f'''(0)\,x^3}{3!} + \dots + \frac{f^{(n)}(0)\,x^n}{n!} .$$

In this problem:

$$\begin{aligned} f(x) &= x^3 - 6x^2 + 9x + 1 & \therefore\ f(0) &= 1 \\ f'(x) &= 3x^2 - 12x + 9 & f'(0) &= 9 \\ f''(x) &= 6x - 12 & f''(0) &= -12 \\ f'''(x) &= 6 & f'''(0) &= 6 \\ f^{(4)}(x) &= 0 & f^{(4)}(0) &= 0 . \end{aligned}$$

Therefore:

$$P_0(x) = f(0) = \underline{1}$$

$$P_1(x) = f(0) + f'(0)x = \underline{1 + 9x}$$

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)\,x^2}{2!} = \underline{1 + 9x - 6x^2}$$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)\,x^2}{2!} + \frac{f'''(0)\,x^3}{3!} = \underline{1 + 9x - 6x^2 + x^3}$$

$$P_4(x) = f(0) + \dots + \frac{f'''(0)\,x^3}{3!} + \frac{f^{(4)}(0)\,x^4}{4!} = \underline{1 + 9x - 6x^2 + x^3}$$

$$(\text{since } f^{(4)}(0) = 0) .$$

[7.4.1 (L) cont'd]

(b) Continuing as we did in (a), we see that $f^{(n)}(0) = 0$ for $n \geqslant 4$ since, in particular, $f^{(n)}(x) \equiv 0$ for $n \geqslant 4$.

Thus, for $n \geqslant 4$,

$$P_n(x) = 1 + 9x - 6x^2 + x^3 \quad .$$

(This is precisely what happened in (a) with n = 4.)

In any event, since $1 + 9x - 6x^2 + x^3$ does not depend on n, if follows that

$$P(x) = \lim_{n\to\infty} P_n(x) = 1 + 9x - 6x^2 + x^3 = f(x) \quad .$$

In this case P(x) is a perfect approximation for f(x) since $P(x) \equiv f(x)$.

A key observation is that in the event f(x) is a polynomial P(x) will always equal f(x). To see this, let $f(x) = a_0 + a_1x + \ldots + a_kx^k + \ldots + a_nx^n$. Then $f^{(k)}(x) = k!\, a_k + x(\ldots)$. That is, each time we differentiate another term drops out so that the k'th derivatives begin with the coefficient a_k. In any case, $f^{(k)}(0) = k!\, a_k + 0(\ldots) = k!\, a_k$. Hence,

$$\frac{f^{(k)}(0)}{k!} = a_k \quad .$$

In this sense, then, P(x) is a generalization of the polynomial concept. That is, if f(x) is already a polynomial, then it is its own power series.

<u>7.4.2 (L)</u>

Observing that $P_n(x)$ is determined by the behaviour of f, f', ..., $f^{(n)}$ at x = 0 we see that $P_n(x)$ turns out to be exactly the same as in Exercise 7.4.1 (L). In other words the fact that

[7.4.2 (L) cont'd]

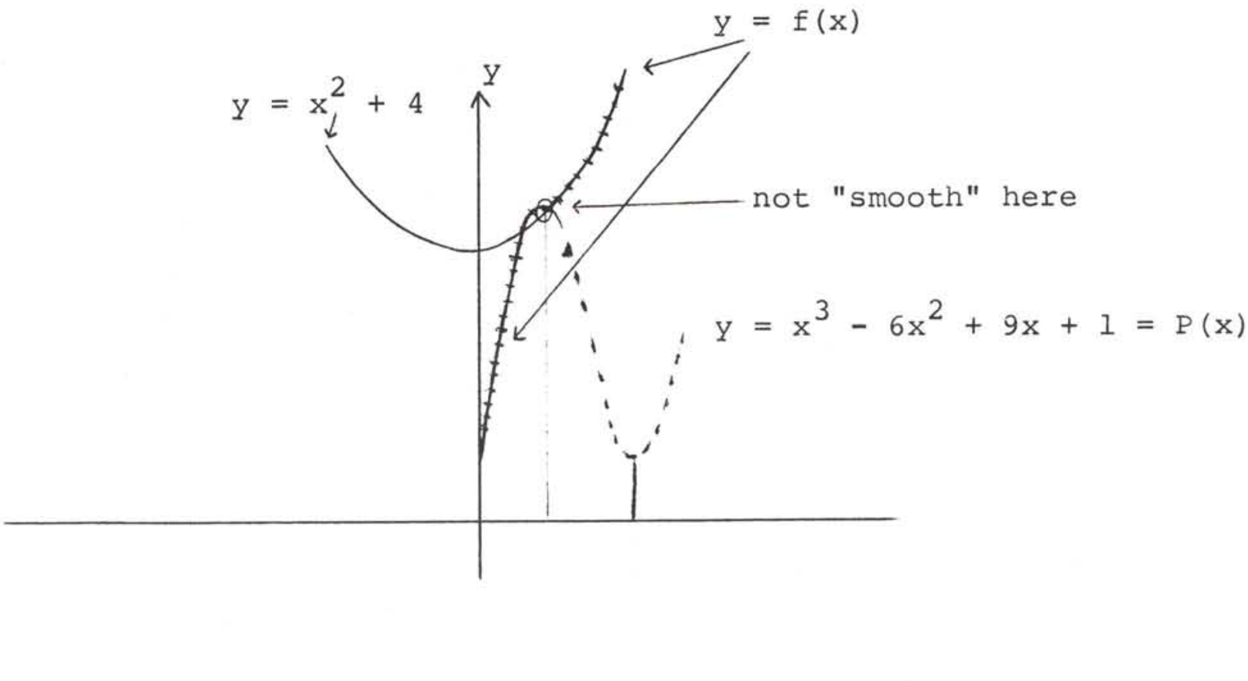
$f(x) = x^2 + 4$ does not apply unless $x > 1$, and for $x = 0$, $f(x) = x^3 - 6x^2 + 9x + 1$ -- which is exactly what f(x) was in Exercise 7.4.1 (L) .

Again, as before, we find that

$$P(x) = \lim_{n\to\infty} P_n(x) = x^3 - 6x^2 + 9x + 1 \quad .$$

In this case $P(x) \equiv f(x)$ provided that $0 \leq x \leq 1$. If $x > 1$ then $P(x) = x^3 - 6x^2 + 9x + 1$ while $f(x) = x^2 + 4$.

The key point here is that f(x) must be smooth for our curve-fitting technique to work and, in this exercise, f(x) is continuous but not smooth (differentiable) at x = 1. Pictorially,



7.4.3 (L)

(1)

$f(x) = \sin x$	$f^{(4)}(x) = \sin x$
$f'(x) = \cos x$	$f^{(5)}(x) = \cos x$
$f''(x) = -\sin x$	$f^{(6)}(x) = -\sin x$
$f'''(x) = -\cos x$	$f^{(7)}(x) = -\cos x$

In other words the derivatives of sin x repeat the cycle sin x, cos x, -sin x, -cos x .

More specifically,

$$f(x) = f^{(4)}(x) = f^{(8)}(x) = \ldots = f^{(4n)}(x) = \sin x$$

$$f'(x) = f^{(5)}(x) = f^{(9)}(x) = \ldots = f^{(4n+1)}(x) = \cos x$$

$$f''(x) = f^{(6)}(x) = f^{(10)}(x) = \ldots = f^{(4n+2)}(x) = -\sin x$$

$$f'''(x) = f^{(7)}(x) = f^{(11)}(x) = \ldots = f^{(4n+3)}(x) = -\cos x \quad .$$

Then since sin 0 = 0 and cos 0 = 1, we have:

$$\left.\begin{array}{l} f^{(n)}(0)=0 \text{ if } n \text{ is even} \\ f^{(n)}(0)=1 \text{ if } n=1,5,9,\ldots,4n+1 \\ f^{(n)}(0)=-1 \text{ if } n=3,7,11,\ldots,4n+3 \end{array}\right\} \therefore a_n = \frac{f^{(n)}(0)}{n!} = \begin{cases} 0, \text{ if } n \text{ is even} \\ \frac{1}{n!} \text{ if } n=1,5,9,13,\ldots 4n+1 \\ -\frac{1}{n!} \text{ if } n=3,7,11,15\ldots 4n+3 \end{cases} \quad .$$

[7.4.3 (L) cont'd]

∴ (a)

$P_0(x) = f(0) = 0$

$P_1(x) = f(0) + f'(0)x = 0 + (1)x = x$

these are equal

$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 0 + x + 0x^2 = x$

$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = x + \frac{(-1)x^3}{3!}$
$= x - \frac{x^3}{3!}$

these are equal

$P_4(x) = f(0) + \ldots + \frac{f'''(0)\ x^3}{3!} + \frac{f^{(4)}(0)}{4!}x^4 = x - \frac{x^3}{3!}$
$+ 0x^4 = x - \frac{x^3}{3!}$

In other words, if n is odd:

$$P_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \pm \frac{x^n}{n!} \qquad (1)$$

while if n is even, $P_n(x) = P_{n-1}(x)$. (That is, as shown above, $\frac{f^{(n)}(0)}{n!}x^n = 0$ when n is even.) Again, utilizing $(-1)^k$ as a sign alternater and recognizing that the odd numbers have the form $2n + 1$, $n = 0, 1, 2, \ldots$, (1) becomes

$$P_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k\ x^{2k+1}}{(2k+1)!}\ ;\ \text{and}\ P_{2n+1}(x) = P_{2n+2}(x) \quad .$$

[7.4.3 (L) cont'd]

(b) $$P(x) = \lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} P_{2n+1}(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$ (recalling that $\sum_{n=0}^{\infty} a_n$ is an abbreviation for $\lim_{n\to\infty} \sum_{k=0}^{n} a_k$)

(c) We look at $$\sum_{n=0}^{\infty} \left| \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right| = \sum_{n=0}^{\infty} \frac{|(-1)^n||x|^{2n+1}}{|(2n+1)!|} = \sum_{n=0}^{\infty} \frac{|x|^{2n+1}}{(2n+1)!}.$$

We now use the ratio test with $a_n = \frac{|x|^{2n+1}}{(2n+1)!}$

$$\therefore a_{n+1} = \frac{|x|^{2(n+1)+1}}{[2(n+1)+1]!} = \frac{|x|^{2n+3}}{(2n+3)!}$$

$$\therefore P = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{|x|^{2n+3}(2n+1)!}{(2n+3)!\ |x|^{2n+1}} \right] = \lim_{n\to\infty} \left[\frac{|x|^{2n+3}}{|x|^{2n+1}} \frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!} \right]$$

$$= \lim_{n\to\infty} \left[\frac{|x|^2}{(2n+3)(2n+2)} \right] = \frac{x^2}{\infty} = 0, \text{ for any finite } x .$$

$\therefore P(x)$ converges absolutely for any (finite) real number x .

[7.4.3. (L) cont'd]

(2) $f(x) = \cosh x$ $f''(x) = \cosh x$

$f'(x) = \sinh x$ $f'''(x) = \sinh x$, etc.

Since sinh 0 = 0 and cosh 0 = 1, we have:

$$\left.\begin{cases} f^{(n)}(0) = 1 & \text{if } n \text{ is even} \\ f^{(n)}(0) = 0 & \text{if } n \text{ is odd} \end{cases}\right\} \therefore a_n = \frac{f^{(n)}(0)}{n!} = \begin{cases} \frac{1}{n!} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}.$$

(a) $$\left.\begin{array}{l} \therefore P_{2n}(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} \\ P_{2n+1}(x) = P_{2n}(x) \end{array}\right\}$$

(b) $$P(x) = \lim_{n\to\infty} P_n(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

(c) We look at $\sum_{n=0}^{\infty} \left| \frac{x^{2n}}{2n!} \right| = \sum_{n=0}^{\infty} \frac{|x|^{2n}}{(2n)!}$ and we use the ratio test with $a_n = \frac{|x|^{2n}}{(2n)!}$. Thus $a_{n+1} = \frac{|x|^{2(n+1)}}{[2(n+1)]!} = \frac{|x|^{2n+2}}{(2n+2)!}$

$$\therefore P = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{|x|^{2n+2}(2n)!}{(2n+2)!\ |x|^{2n}} \right]$$

$$= \lim_{n\to\infty} \left[\frac{|x|^2}{(2n+2)(2n+1)} \right] = \frac{|x^2|}{\infty} = 0$$

$\therefore P(x)$ converges absolutely for all real x .

[7.4.3 (L) cont'd]

(3) $f(x) = \frac{1}{1 - x} = (1 - x)^{-1}$

$$f'(x) = -1(1 - x)^{-2}(-1)^{*} = (1 - x)^{-2}$$

$$f''(x) = -2(1 - x)^{-3}(-1) = 2(1 - x)^{-3}$$

$$f'''(x) = -3!(1 - x)^{-4}(-1) = 3!(1 - x)^{-4}$$

$$f^{(4)}(x) = -4!(1 - x)^{-5}(-1) = 4!(1 - x)^{-5}$$

We now see (either inductively or otherwise) that

$$f^{(n)}(x) = n!(1 - x)^{-(n+1)}$$

$$\therefore f^{(n)}(0) = n!(1 - 0)^{-(n+1)} = n!(1) = n!$$

$$\therefore \frac{f^{(n)}(0)}{n!} = \frac{n!}{n!} = 1 \quad \text{for each } n$$

$\therefore$ (a)

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} x^k$$

$$= \underline{1 + x + x^2 + \ldots + x^n}$$

---

*Lest we have forgotten, we are using the chain rule here. Namely,

$\frac{d}{dx}(1 - x)^{-1} = \frac{d}{d(1 - x)}(1 - x)^{-1}\,\frac{d(1 - x)}{dx} = -1(1 - x)^{-2}\,\frac{d(1 - x)}{dx}$

$= (-1)(1 - x)^{-2}(-1)$.

[7.4.3 (L) cont'd]

(b) $P(x) = \lim_{n\to\infty} P_n(x) = \sum_{n=0}^{\infty} x^n$

(c) We look at $\sum_{n=0}^{\infty} |x^n|$. Then $a_n = |x|^n$, $a_{n+1} = |x|^{n+1}$.

By the ratio test:

$$P = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} = \lim_{n\to\infty} |x| = |x|$$

$\therefore P < 1 \longleftrightarrow |x| < 1$.

Moreover if $|x| = 1$ then $\sum_{n=0}^{\infty} |x|^n = \sum_{n=0}^{\infty} 1 = 1 + 1 + 1 + \dots = \infty$

$\therefore P(x)$ converges absolutely $\longleftrightarrow |x| < 1$.

(4) Hopefully, this is a review of the ideas contained in exercise 7.2.1 (L) . Namely,

If f(x) is the <u>polynomial</u> $6x^4 + 3x^2 + 7x - 5$ then:

(a) $P_0(x) = -5$

$P_1(x) = 7x - 5$

$P_2(x) = 3x^2 + 7x - 5$

$P_3(x) = 0x^3 + 3x^2 + 7x - 5 = 3x^2 + 7x - 5$

$P_4(x) = 6x^4 + 3x^2 + 7x - 5$

$P_n(x) = 6x^4 + 3x^2 + 7x - 5 \qquad n \geq 4$

[7.4.3 (L) cont'd]

(b) $P(x) = \lim_{n\to\infty} P_n(x) = 6x^4 + 3x^2 + 7x - 5$

(c) Actually, the question of convergence is essentially irrelevant here, since, as we mentioned in our introduction to infinite sequences and series, convergence is replaced by the more elementary notion of "last term" when we deal with finite collections.

In other words, P(x) in this case automatically converges for all x, since P is a recipe that determines the number P(x) in a finite number of arithmetic steps.

7.4.4 (L)

In a manner of speaking, this is the first exercise that gets to the crux of the matter from a pragmatic point of view. That is, up to now, we have considered, primarily, how to compute $P_n(x)$ and P(x) for a given function f(x) although Exercise 7.4.1 (L), part (b), and Exercise 7.4.2 (L) mentioned (or at least introduced) a more complex problem that we shall now discuss in more detail.

From a practical point of view, once we have computed P(x) and know the interval for which P(x) is absolutely convergent, we are then interested in finding for what interval P(x) is actually equal to f(x). It is in this context that Exercise 7.4.2 (L) emphasizes in a rather elementary way the difference between $P_n(x)$ converging to P(x) and P(x) converging to f(x). For example, in terms of Exercise 7.4.2 (L), we saw that P(x) was exactly $x^3 - 6x^2 + 9x + 1$ for every real number, x, but that P(x) equalled f(x) only when $0 \leqslant x \leqslant 1$.

In summary, then, one crucial question is whether the sequence of polynomial approximations converges to a limit function at all. A more crucual question, from an applied point of view especially,

[7.4.4 (L) cont'd]

is whether the limit function, even assuming that it does exist, actually represents the given function f(x) .

In fact, it is precisely for this reason that one must come to grips with Taylor's Theorem with remainder which tells us the error we get if we replace f(x) by $P_n(x)$.

More specifically, since $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x,a)$ and $P_n(x)^* = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$, it follows that:

$$P(x) \quad (= \lim_{n\to\infty} P_n(x)) = f(x) \longleftrightarrow \lim_{n\to\infty} R_n(x,a) = 0 \quad .$$

The point is that: (1) in the given form, $R_n(x,a)$ is often difficult to compute exactly, and (2) in many cases all we really need to know, once P(x) = f(x), is when $|R_n(x,a)|$ is less than a certain prescribed amount. At any rate, this is what the present exercise is concerned with; namely, finding bounds for $R_n(x,a)$.

---

*Actually, our treatment up to now has considered the special case a = 0 . In other words, the more general definition of $P_n(x)$ allows for the degree of contact to be treated at points other than x = 0 . If we let x = a, we obtain $P_n(x) = \sum_{k=0}^{n} \frac{f^{(n)}(a)}{k!}(x-a)^k$. Our special case arises by virtue of the fact that we can label the point of contact as the origin (though, of course, f(x) must also be changed if we change the location of the coordinate axes). The point is that while, in theory, proofs remain unaltered by choosing x = 0 rather than x = a, there are times when it is more convenient to pick a ≠ 0, expecially in such cases where f(x) or some of its derivatives are not defined at x = 0, for example, if f(x) = ℓn x .

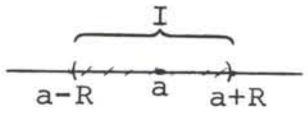
[7.4.4 (L) cont'd]

(a) We will utilize the following two properties of the definite integral:

(1) $\left|\int_a^b f(t)\,dt\right| \leqslant \int_a^b |f(t)|\,dt$ (that is, if $a \leqslant b$; if $b \leqslant a$, then $\left|\int_a^b f(t)\,dt\right| < \int_b^a |f(t)|\,dt$.)

(2) If $|g(t)| \leqslant M$ for all $t \in [a,b]$, then:

$$\int_a^b |g(t)|\,|f(t)|\,dt \leqslant M \int_a^b |f(t)|\,dt \qquad (a \leqslant b) \quad .$$

Since $I = \{x: |x - a| < R\}$ (pictorially, 

) there are two cases to consider.

<u>Case 1</u>: $x \geqslant a$

Then: $|R_n(x,a)| = \left|\frac{1}{n!}\int_a^x (x - t)^n f^{(n+1)}(t)\,dt\right|$

$$= \frac{1}{n!}\left|\int_a^x (x - t)^n f^{(n+1)}(t)\right| dt$$

$$\leqslant \frac{1}{n!}\int_a^x |x - t|^n\,|f^{(n+1)}(t)|\,dt \quad \text{(by (1) above)}$$

$$\leqslant \frac{M}{n!}\int_a^x |x - t|^n\,dt \quad \text{(by (2) above)} \quad .$$

Now since $x \geqslant a$ and t is between a and x, we have

$$a \leqslant t \leqslant x, \text{ whence } |x - t| = x - t \quad .$$

[7.4.4 (L) cont'd]

Hence (1) leads to:

$$|R_n(x,a)| \leqslant \frac{M}{n!}\int_a^x (x-t)^n\,dt = \frac{M}{n!}\left[\frac{-(x-t)^{n+1}}{n+1}\right|_{t=a}^{t=x} \qquad (2)$$

(Observe that we are integrating with respect to t <u>not</u> x, which accounts for the introduction of the minus sign in (2).)

$$\therefore |R_n(x,a)| \leqslant \frac{M}{n!}\left[\frac{-(x-x)^{n+1}}{n+1} - \left\{\frac{-(x-a)^{n+1}}{n+1}\right\}\right] = \frac{M}{n!}\left[0 + \frac{(x-a)^{n+1}}{n+1}\right]$$

$$\therefore |R_n(x,a)| \leqslant \frac{M(x-a)^{n+1}}{n!(n+1)} = \frac{M\,|x-a|^{n+1}}{(n+1)!} \quad .$$

<u>Case 2</u>: $x \leqslant a$

In this case since t is between x and a, we have $x \leqslant t \leqslant a$, whence $x - t < 0$

$$\therefore |x-t| = t - x, \text{ and:}$$

$$|R_n(x,a)| = \frac{1}{n!}\left|\int_a^x (x-t)^n f^{(n+1)}(t)\,dt\right|$$

$$\leqslant \frac{1}{n!}\int_x^a |x-t|^n\,|f^{(n+1)}(t)|\,dt$$

$$\leqslant \frac{M}{n!}\int_x^a |x-t|^n\,dt = \frac{M}{n!}\int_x^a (t-x)^n\,dt = \frac{M}{n!}\left[\frac{(t-x)^{n+1}}{n+1}\right|_{t=x}^{t=a}$$

$$\therefore |R_n(x,a)| \leqslant \frac{M}{n!}\left[\frac{(a-x)^{n+1}}{n+1} - 0\right] = \frac{M(a-x)^{n+1}}{(n+1)!} \quad .$$

[7.4.4 (L) cont'd]

Finally, since $x \leqslant a$, $(a - x) = |x - a|$, and we obtain:

$$|R_n(x,a)| \leqslant \frac{M|x - a|^{n+1}}{(n+1)!} \quad .$$

Combining our two cases, we have

$$x \in I \to |R_n(x,a)| \leqslant \frac{M|x - a|^{n+1}}{(n+1)!}$$

(b) Using (a) we can now proceed as follows:
We have:

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + R_n(x,0), \text{ where } R_n(x,0)$$

$$= \frac{1}{n!}\int_0^x (x - t)^n e^t \, dt \quad . \qquad (3)$$

Now if $I = \{x: |x| < r\}$ we have that for any $x \in I$: $|e^x| \leqslant e^r = M$ and we may apply part (a) to obtain

$$|R_n(x,0)| < \frac{e^r}{n!}\left|\int_0^x (x - t)^n \, dt\right| \quad .$$

Again, mimicking the procedure in (a) of using two cases ($x \leqslant 0$ or $x \geqslant 0$), we obtain:

$$|R_n(x,0)| \leqslant \frac{e^r|x|^{n+1}}{(n + 1)!}$$

and since $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n + 1)!} = 0$ (this is our earlier result that $n!$ "dwarfs" x^n), we see that $\lim_{n\to\infty} R_n(x,0) = 0$ if $-r < x < r$, but since r was an arbitrary real number, $\lim_{n\to\infty} R_n(x,0) = 0$ for all real x .

[7.4.4 (L) cont'd]

Hence, from (3), we have

$$\lim_{n\to\infty} e^x = \lim_{n\to\infty} \left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}\right) + \lim_{n\to\infty} R_n(x,0)$$

$$\therefore e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (+0) \quad . \qquad (4)$$

(Notice that (b) generalizes in terms of (a). Namely, the condition that $\lim_{n\to\infty} R_n(x,a) = 0$ is guaranteed once $|f^{(n+1)}(t)| \leqslant M$ for every t between a and x.)

As a final note to (b), let us again emphasize that without (a) we could still conclude that if $f(x) = e^x$, then $P(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. The "new," information allows us to deduce that P(x) indeed equals f(x) .

(c) From (b), letting x = -1 we have:

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

$$= \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \ldots + \frac{(-1)^n}{n!} + \ldots$$

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{(-1)^n}{n!} + \ldots \qquad (1)$$

We would like to determine how many terms in (1) are necessary if we wish $\frac{1}{e}(= e^{-1})$ to be correct to three decimal places.

[7.4.4 (L) cont'd]

Since (1) is a convergent alternating series the error we obtain by truncating the series at $\frac{(-1)^n}{n!}$ cannot exceed the magnitude of the term, $\frac{(-1)^{n+1}}{(n+1)!}$.

That is:

$$\left|\frac{1}{e} - \left(\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{[-1]^n}{n!}\right)\right| \leqslant \frac{1}{(n+1)!} .$$

Now if n = 7 then $(n+1)! = 8! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8 = 40,320$, and therefore, $\frac{1}{(n+1)!} = \frac{1}{40,320} = 0.00002^+$.

That is, the maximum error we can have if we approximate $\frac{1}{e}$ by $\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!}$ is 0.00002^+. Thus $\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!}$ is <u>at least</u> accurate to three decimal places as an approximation for $\frac{1}{e}$.

Carrying out the remaining details, we have:

$$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} =$$

$$\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} = \frac{2520 - 840 + 210 - 42 + 7 - 1}{5040}$$

$$= \frac{1854}{5040} = \frac{103}{280}$$

$$= 0.36785^+ .$$

Since we stopped after a negative term, our approximation is less than the correct answer by an amount not in excess of 0.0002^+.

[7.4.4 (L) cont'd]

In other words

$$0.36785^{+} \leqslant \frac{1}{e} \leqslant 0.36785^{+} + 0.00002^{+}$$

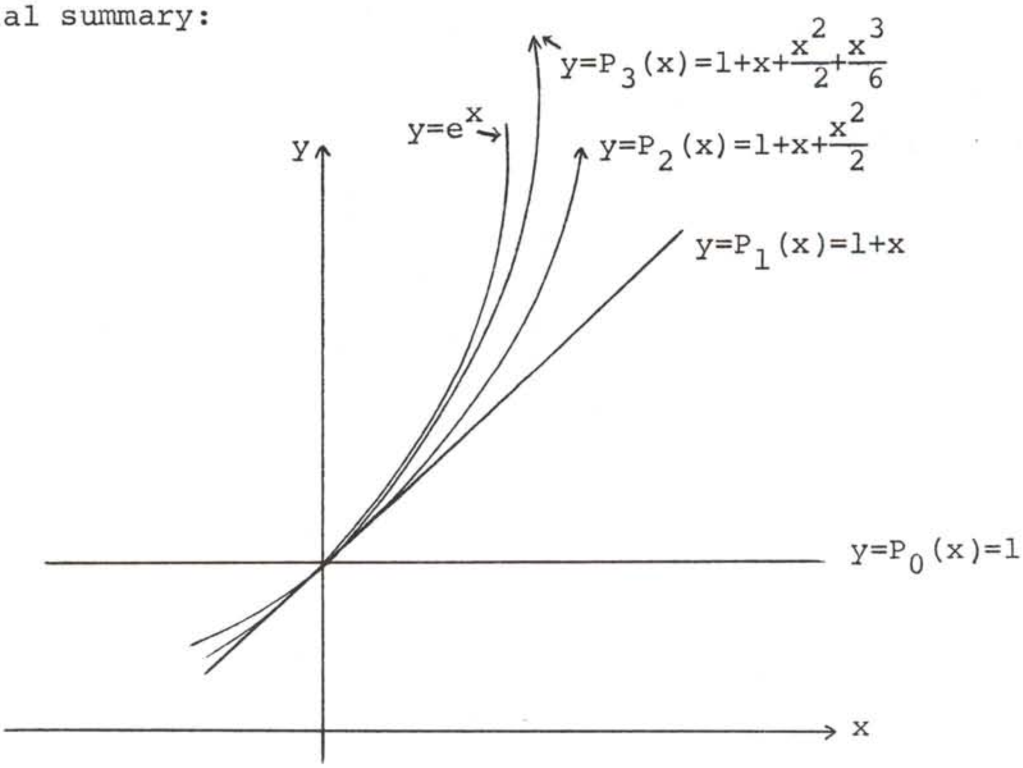
$$0.36785^{+} \leqslant \frac{1}{e} \leqslant 0.36787^{+\ (*)} \quad . \qquad (2)$$

Equation (2) gives us an even sharper result than we were looking for.

Namely, (2) tells us that rounded off to 4 decimal places,

$$\underline{\frac{1}{e} = 0.3679} \quad .$$

Pictorial summary:



An n increases $y = P_n(x)$ "fits" $y = e^x$ better over a longer interval.

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*Note that in addition to truncation errors, which arise from our "chopping off" a series after a finite number of terms, there is also a rounding-off error which occurs when we replace common fractions by their decimal equivalents.

7.4.5

Since $|f^{(n)}(x)| \leqslant 1$ for all n and x if f(x) = sin x, the condition of Exercise 7.4.4 (L) part (a) applies and we can conclude from exercise 7.4.3 (1) that $P(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \; x^{2n+1}}{(2n+1)!}$ converges to f(x) = sin x for all real x.

In other words:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots \qquad (1)$$

Now if $x = \frac{1}{2}$, $\frac{x^5}{5!} = \frac{(1/2)^5}{120} = \frac{1}{32(120)} = \frac{1}{3840} = 0.0002^+$. Thus it is possible that $\frac{1}{2} - \frac{(1/2)^3}{120}$ is correct to 3 decimal places as an approximation for $\sin \frac{1}{2}$.

What we are sure of is that

(1) $\frac{1}{2} - \frac{(1/2)^3}{3!}$ is less than $\sin \frac{1}{2}$, and

(2) $\frac{1}{2} - \frac{(1/2)^3}{3!} + 0.0002^+$ is greater than $\sin \frac{1}{2}$.

In any event, we find

$$\frac{1}{2} - \frac{(1/2)^3}{3!} = \frac{1}{2} - \frac{1}{8(3!)} = \frac{1}{2} - \frac{1}{48} = \frac{23}{48} = 0.4791^+$$

$$\therefore 0.4791^+ \leqslant \sin \frac{1}{2} \leqslant 0.4791^+ + 0.0002^+$$

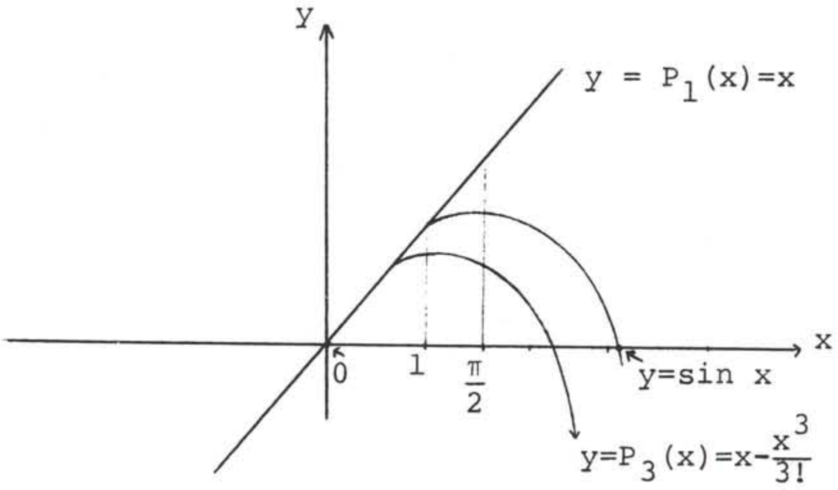
$$\therefore 0.4791^+ \leqslant \sin \frac{1}{2} \leqslant 0.4793^+$$

$\therefore \sin \frac{1}{2} = 0.479$ to 3 decimal places.

[7.4.5 cont'd]

(Note: if you decide to check this result by using trig tables, etc. notice that in terms of angular measure $\sin \frac{1}{2}$ means $\sin \frac{1}{2}$ radian not $\sin \frac{1}{2}°$, in accordance with our discussion concerning the fact that x is a pure number in the expression sin x .)

Again pictorially:



$y = P_3(x)$ approximates $y = \sin x$ very well if $0 \leqslant x < 1$.

7.4.6 (L)

One of our aims of this exercise is to introduce alternative methods for determining $R_n(x,a)$. In this case f(x) is the quotient of two polynomials, and as a result we can express f(x) as a polynomial plus a remainder term simply by the operation of division. Namely,

(Step 1)

$$\begin{array}{r|l} 1 & 1 + x \\ \underline{1 + x} & \overline{1} \\ -x & \end{array}$$

$$\therefore \frac{1}{1 + x} = 1 - \frac{x}{1 + x}$$

[7.4.6 (L) cont'd]

(Step 2)

$$\begin{array}{l} 1 \\ \underline{1 + x} \\ \quad - x \\ \quad \underline{- x - x^2} \\ \qquad\quad x^2 \end{array} \qquad \begin{array}{|l} 1 + x \\ \hline 1 - x \\ \hline \end{array}$$

$$\therefore \frac{1}{1 + x} = 1 - x + \frac{x^2}{1 + x}$$

(Step 3)

$$\begin{array}{l} 1 \\ \underline{1 + x} \\ \quad - x \\ \quad \underline{- x - x^2} \\ \qquad\quad x^2 \\ \qquad\quad \underline{x^2 + x^3} \\ \qquad\qquad - x^3 \end{array} \qquad \begin{array}{|l} 1 + x \\ \hline 1 - x + x^2 \\ \hline \end{array}$$

$$\therefore \frac{1}{1 + x} = 1 - x + x^2 - \frac{x^3}{1 + x}$$

Part (a) now follows by induction. The key is that

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n + \frac{(-1)^{n+1}\, x^{n+1}}{1 + x} \quad . \tag{1}$$

is exact as long as we retain $(-1)^{n+1} \frac{x^{n+1}}{1 + x}$ which in this case is $R_n(x,0)$. (In fact if we apply the technique of Exercise 7.4.3 to $f(x) = \frac{1}{1 + x}$ we find that $P_n(x) = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n$, and therefore, from (1)

$$\frac{1}{1 + x} = P_n(x) + \frac{(-1)^{n+1}\, x^{n+1}}{1 + x}$$

$$\therefore \frac{1}{1 + x} - P_n(x) = \frac{(-1)^{n+1}\, x^{n+1}}{1 + x} = R_n(x,0).)$$

[7.4.6 (L) cont'd]

(b)

$$\ell n(1+x) = \int_0^x \frac{dt}{1+t} = \int_0^x \left[1 - t + t^2 - t^3 + \ldots + (-1)^n t^n + \frac{(-1)^{n+1} t^{n+1}}{t+1}\right] dt$$

$$= t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \ldots + \frac{(-1)^n t^{n+1}}{n+1}\Bigg|_0^x + \int_0^x \frac{(-1)^{n+1} t^{n+1}}{t+1} dt$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots + \frac{(-1)^n x^{n+1}}{n+1} + R_n(x,0) \qquad (2)$$

where $R_n(x,0) = \int_0^x \frac{(-1)^{n+1} t^{n+1}}{t+1} dt$.

Now if $x \geqslant 0$, then

$$|R_n(x,0)| \leqslant \int_0^x \frac{t^{n+1}}{t+1} dt \leqslant \int_0^x t^{n+1} dt = \frac{1}{n+2} t^{n+2}\Bigg|_0^x = \frac{x^{n+2}}{n+2}$$

$$\therefore x \leqslant 1 \rightarrow \lim_{n\to\infty} |R_n(x,0)| = 0 \quad . \qquad (3)$$

Combining (2) and (3), we see that

$\therefore 0 \leqslant x \leqslant 1 \rightarrow$

$$\ell n(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \quad . \qquad (4)$$

[7.4.6 (L) cont'd]

(Again, we could use the technique of Exercise 7.4.3 with $f(x) = \ln(1 + x)$ to show that $P(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$.)

(c) We use (4) with $x = .2 = \frac{1}{5}$

$$\ln(1 + \tfrac{1}{5}) = \ln 1.2 = \frac{1}{5} - \frac{1}{2}(\frac{1}{5})^2 + \frac{1}{3}(\frac{1}{5})^3 - \frac{1}{4}(\frac{1}{5})^4 + \frac{1}{5}(\frac{1}{5})^5$$

$$= \frac{1}{5} - \frac{1}{50} + \frac{1}{375} - \frac{1}{2500} + \frac{1}{15,625}$$

$$= 0.2 - 0.02000 + 0.00267 - 0.00040 + 0.00006^+$$

$$= 0.18227 \text{ with error } \leqslant 0.00006$$

i.e., $0.18227 \leqslant \ln 1.2 \leqslant 0.18233^+$

In fact, we thus see that to four places

$$\ln 1.2 = 0.1823 \quad .$$

(d) Letting $x = 1$ in (4),

$$\ln(1 + 1) = \ln 2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots$$

In other words the conditional convergent series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$, which we have seen can have any sum under a suitable rearrangement of terms, converges to $\ln 2$ if the terms are added in the given order.

7.4.7

While we could use division on $\frac{1}{1 + t^2}$, it is easier to recall that we already know that

$$\frac{1}{1+u} = 1 - u + u^2 - u^3 + \ldots + (-1)^n u^n + \frac{(-1)^{n+1} u^{n+1}}{1+u} \quad . \tag{1}$$

Since u symbolically denotes any real number, we may replace u by t^2 in (1) to obtain:

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \ldots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}$$

$$\therefore \frac{\pi}{4} = \tan^{-1} 1 = \int_0^1 \frac{dt}{1+t^2}$$

$$= \int_0^1 (1 - t^2 + t^4 - t^6 + \ldots + (-1)^n t^{2n})\, dt + \int_0^1 \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} \tag{2}$$

Now:

$$\left| \int_0^1 \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\, dt \right| \leqslant \int_0^1 \frac{t^{2n+2}}{1+t^2} \leqslant \int_0^1 |t|^{2n+2} = \frac{1}{2n+3} t^{2n+3} \Big|_0^1$$

$$\therefore \left| \int_0^1 \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\, dt \right| \leqslant \frac{1}{2n+3}$$

$$\therefore \lim_{n\to\infty} \int_0^1 \frac{(-1)^{n+1} t^{2n+2}\, dt}{1+t^2} = 0 \quad . \tag{3}$$

[7.4.7 cont'd]

Using the result of (3) in (2) we have

$$\frac{\pi}{4} = \lim_{n\to\infty} \int_0^1 (1 - t^2 + \ldots + (-1)^n t^{2n})\, dt$$

or:

$$\frac{\pi}{4} = \lim_{n\to\infty} \left[t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \ldots + \frac{(-1)^n t^{2n+1}}{2n+1} \right|_0^1$$

$$= \lim_{n\to\infty} \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + \frac{(-1)^n}{2n+1} \right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$$\therefore \pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 4}{2n+1}$$

$$= 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \ldots + \frac{(-1)^n 4}{2n+1} + \ldots$$

(This is an impractical way to compute π since the nth term approaches 0 rather slowly, although rapidly enough to insure convergence. One tries to find other series for π in which the error term approaches 0 more rapidly. A short discussion of this idea occurs in Thomas 18.6 .)

Unit 5: Uniform Convergence

7.5.1(L)

Our aim here is to show in computational detail the difference between pointwise convergence and uniform convergence.

(a) If $|x| < 1$ we have already seen that $\lim_{n\to\infty} x^n = 0$. Hence $\{x^n\}$ converges pointwise to 0 if $0 < x < b < 1$. That is,

$$\lim_{n\to\infty} x^n = 0 \text{ for each } x \in (0,b);\ b < 1. \tag{1}$$

The computational aspects of pointwise convergence become clearer if we use the quantitative ε-definition of limit. Namely, if we apply this definition to (1), we obtain:

For a given $x \in (0,b)$ and a given $\varepsilon > 0$, we can find N such that

$$n > N \to |x^n - 0| < \varepsilon \tag{2}$$

The key point is that the N in (2) appears to depend on both ε and x. In other words, from (1) it would appear that while the required N exists for each choice of ε, N will vary with x.

Now if we investigate (2) in more detail we find:

(i) $|x^n - 0| = |x^n| = |x|^n$. Hence,

$$|x^n - 0| < \varepsilon \leftrightarrow |x|^n < \varepsilon \tag{3}$$

(ii) In handling inequalities (or even equalities) in which the "unknown" appears only as an exponent (in our problem, x and ε are assumed to be given quantities; hence, n is the "unknown"),

[7.5.1(L) cont'd]

it is wise to invoke the logarithmic properties that (a) $\log u^r = r \log u$ and (b) $u < v \rightarrow \log u < \log v$ (that is, log is a monotonic increasing function). Notice that these logarithmic properties are true for any base, although we will usually use base e if only because most of our formulas are geared to it.

At any rate, if we apply these considerations to (3), we obtain

$$|x|^n < \varepsilon \leftrightarrow \log|x|^n < \log \varepsilon$$

$$\leftrightarrow n \log |x| < \log \varepsilon \qquad (4)$$

Since $0 < x < b < 1$, it follows that $|x| = x$; hence, (4) becomes

$$|x|^n < \varepsilon \leftrightarrow n \log x < \log \varepsilon \qquad (5)$$

In (5), we now divide through by log x to obtain

$$|x|^n < \varepsilon \leftrightarrow n > \frac{\log \varepsilon}{\log x} \qquad (6)$$

Notice that in (6) we reversed the direction of the inequality when we divided through by log x in (5). The reason is that since $0 < x < 1$, log x is negative, and we already know that dividing (or multiplying) both sides of an inequality by a negative number reverses the inequality.

Equation (6) also gives us a hint as to why we made $b < 1$. Namely, since $\log 1 = 0$, $\frac{\log \varepsilon}{\log b}$ would be infinite when $b = 1$. Indeed, this will be the crucial point in part (b) of this problem.

[7.5.1(L) cont'd]

Also, notice that we excluded 0 from our domain. Obviously $0^n = 0$ for all n and hence x = 0 causes us no trouble, except that we can't take the log of 0. If, however, we accept the notion that $\log 0 = \lim_{x\to 0^+} \log x = -\infty$, $\frac{\log \varepsilon}{\ln x} = 0$ and then (6) merely says that if $n > 0$, $0^n < \varepsilon$ which is certainly true.

Let us return to the crux of our problem. From (6) we now know that

$$\text{If } n > \frac{\log \varepsilon}{\log x} \text{ then } x^n < \varepsilon \text{ where } 0 < x < b < 1$$

In line with our earlier remarks, it thus seems clear that $N \left(= \frac{\log \varepsilon}{\log x}\right)$ depends on both x and ε.

What we want is an expression for N which depends on ε but not x. For example, we would prefer, if we could, to express N in terms of ε and b, since b is a fixed number in this exercise.

Before developing this point further it might be wise to call attention to the fact that since $0 < x < 1$, $\ln x$ is negative. As a result $n > \frac{\ln \varepsilon}{\ln x}$ is trivially fulfilled by any whole number, n, if $\ln \varepsilon$ is positive since in this case $\frac{\ln \varepsilon}{\ln x}$ would be negative and thus exceeded by any whole number. Thus, the only "interesting" case occurs if $\ln \varepsilon$ is negative and this occurs if $0 < \varepsilon < 1$. Notice that $0 < \varepsilon < 1$ is a very realistic condition in practical problems since we are almost always interested in "small" values of ε.

So, without loss of generality, we may assume that both $\ln \varepsilon$ and $\ln x$ are negative and hence that $0 < \varepsilon < 1$ and $0 < x < 1$. Thus,

[7.5.1(L) cont'd]

$$\frac{\log \varepsilon}{\log x} = \frac{-|\log \varepsilon|}{-|\log x|} = \frac{|\log \varepsilon|}{|\log x|} \qquad (7)$$

Then since $0 < x < b < 1$ we have, since the log is a monotonically increasing function, that $\ln x < \ln b$. Notice again, however, that both $\ln x$ and $\ln b$ are negative (since both x and b are between 0 and 1). Therefore, $|\ln x|$ is greater than $|\ln b|$ (that is, $\ln x$ has a greater magnitude than $\ln b$). This, in turn, implies that

$$\frac{|\log \varepsilon|}{|\log x|} < \frac{|\log \varepsilon|}{|\log b|} \qquad (8)$$

Combining (7) and (8) with (6) we see that

$$n > \frac{|\log \varepsilon|}{|\log b|} \rightarrow n > \frac{|\log \varepsilon|}{|\log x|} \rightarrow n > \frac{\log \varepsilon}{\log x} \rightarrow |x|^n < \varepsilon \qquad (9)$$

Since $\frac{|\log \varepsilon|}{|\log b|}$ is independent of x, we have found the required N.

From (9) we see that we can find, for a given ε, an N, which works for all $x \in (0,b)$ at once. Namely we pick N, to be any integer which is at least as great as $\left|\frac{\log \varepsilon}{\log b}\right|$. The key is that b is defined independently of x; hence, $\left|\frac{\log \varepsilon}{\log b}\right|$ depends on ε but not x.

Thus $\{x^n\}$ converges uniformly to 0 in the interval (0,b) where $b < 1$.

(b) Up to now, one might not see the key step behind uniform convergence, since in (a) we saw that both types of convergence were present. The key difference comes when we look at (6) as $x \rightarrow 1^-$.

[7.5.1(L) cont'd]

Namely as $x \to 1^-$, $|\log x| \to 0$, hence $\left|\dfrac{\log \varepsilon}{\log x}\right| \to \infty$. Now log x is <u>no longer bounded away from</u> 0. That is, the crucial step in establishing (a) was that $|\log x| > |\log b| \neq 0$ and hence that $\left|\dfrac{\log \varepsilon}{\log x}\right| < \left|\dfrac{\log \varepsilon}{\log b}\right|$ = a <u>finite</u> number.

Our claim now is that if b = 1 and $0 < x < b = 1$ then the choice of N such that $n > N_1 \to x^n < \varepsilon$ depends on x as well as ε.

One way of seeing this is that from (6) or (7), if we choose x near 1 then for a given ε we must have

$$n > \left|\frac{\log \varepsilon}{\log x}\right|$$

if

$$x^n < \varepsilon$$

We now let $N_1 = \dfrac{\log \varepsilon}{\log x}$ and we see that as $x \to 1$, $N_1 \to \infty$. That is, for a given ε, N_1 increases without bound as $x \to 1$. This clearly means that N_1 depends on x.

To see this from a more quantitative point of view, let us find N if $\varepsilon = e^{-12}$ and $x = 1 - 10^{-15}$. In this case $\ln \varepsilon = -12$ while $\ln x \approx -10^{-15}$*

$$\therefore \frac{\ln \varepsilon}{\ln x} \approx \frac{-12}{-10^{-15}} = 12 \times 10^{15} \qquad (10)$$

---

*We could look this up in the tables if the tables were "delicate" enough. (Namely $1 - 10^{-15} = 0.999999999999999$ and it would take comprehensive tables to include this.) On the other hand, we can put our knowledge of series to work to see that

$$\ln(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{x^n}{n} + \ldots\right), \quad -1 \leqslant x < 1 \qquad \text{(i)}$$

Hence, with $x = 10^{-15}$, (i) shows that $\ln(1 - 10^{-15}) = -10^{-15}$ is good to about 30 decimal places. (Notice that we picked base e here to utilize the series for ln x.)

[7.5.1(L) cont'd]

From (10) we see that N_1 must be at least as great as 12×10^{15}.

However, if we let $N_1 = 12 \times 10^{15}$ all we can say for sure is that if $n > N_1$ then $x^n < \varepsilon$ provided $0 < x < 1 - 10^{-15}$.

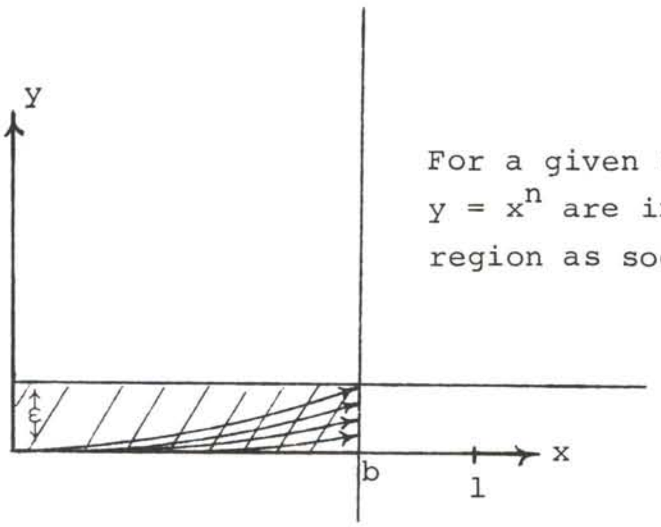
If $x > 1 - 10^{-15}$ but still less that 1 then we will need a larger N if x^n is to be less than ε.

For example, if $\varepsilon = e^{-12}$ and $x = 1 - 10^{-20}$ then $\frac{\ln \varepsilon}{\ln x} \approx \frac{-12}{-10^{20}} = 12 \times 10^{20}$ (=N).

In summary then, $\{x^n\}$ converges uniformly to 0 on (0,b] if $b < 1$. If, however, $b = 1$ then $\{x^n\}$ still converges pointwise to 0 on (0,b] but no longer uniformly.

Pictorially,

(a)

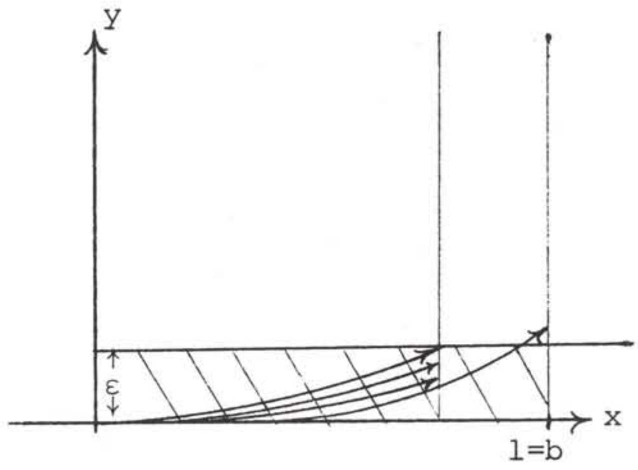


For a given ε, all curves $y = x^n$ are in the shaded region as soon as $n > \frac{\ln \varepsilon}{\ln b}$

But if we now let $b = 1$, we see that for values of x near 1, our curves escape the shaded region.

[7.5.1(L) cont'd]

(b)



To tie this exercise in with our discussion in the supplementary notes, let us observe that in this exercise we showed explicitly that $\{x^n\}$ did not converge uniformly on [0,1] (although it does on [0,b] if b < 1). In the notes, we showed that while each x^n was continuous on [0,1], $\lim_{n\to\infty} x^n$ was discontinuous at x = 1. This, as we later proved, could not happen if $\{x^n\}$ converged uniformly on [0,1]. Thus, in the notes we proved indirectly (implicitly) that $\{x^n\}$ did not converge uniformly on [0,1].

7.5.2

It should be clear that for a fixed real number x, $\lim_{n\to\infty} \frac{1}{x+n} = \frac{1}{x+\infty} = \frac{1}{\infty} = 0.$

Thus $\left\{\frac{1}{x+n}\right\}$ converges pointwise to 0 for all real x. This, in turn, means that for a fixed x and a given $\varepsilon > 0$, we can find N such that

$$n > N \to \left|\frac{1}{x+n} - 0\right| < \varepsilon \qquad (1)$$

[7.5.2 cont'd]

If we now work on $\left|\frac{1}{x+n} - 0\right|$ we see

$$\left|\frac{1}{x+n} - 0\right| = \left|\frac{1}{x+n}\right| = \frac{1}{|x+n|} = \frac{1}{x+n}^{*} \tag{2}$$

From (2) we see that

$$\frac{1}{|x+n|} < \varepsilon \leftrightarrow \frac{1}{x+n} < \varepsilon$$

but

$$\frac{1}{x+n} < \varepsilon \leftrightarrow x+n > \frac{1}{\varepsilon}$$

$$\leftrightarrow n > \frac{1}{\varepsilon} - x \tag{3}$$

From (3) we see that if we choose N to be any integer which exceeds $\frac{1}{\varepsilon} - x$** then

$$n > N \rightarrow \left|\frac{1}{x+n} - 0\right| < \varepsilon \tag{4}$$

---

*This is why we let dom $f_n = [0,b]$. Namely, since n is always a positive integer, $|x+n|$ will equal x+n as long as x is also positive. If x is negative we can still invoke such results as $|x+n| \leqslant |x| + |n| = |x| + n$, but the arithmetic becomes a bit more obscure. In essence, the restriction that $x \geqslant 0$ does not alter the general theory, but it does simplify the arithmetic.

**It is assumed that N denotes an integer, but $\frac{1}{\varepsilon} - x$ need not be an integer. A common abbreviation is to let [u] denote the greatest whole number which doesn't exceed u (see discussion of The Greatest Integer Function, Thomas 1.6). We can then let $N = \left[\frac{1}{\varepsilon} - x\right]$ and since $N \leqslant \frac{1}{\varepsilon} - x < N+1$ (since N is the greatest integer which doesn't exceed $\frac{1}{\varepsilon} - x$) it follows that $n > N \rightarrow n \geqslant N+1 > \frac{1}{\varepsilon} - x$.

[7.5.2 cont'd]

Thus we have constructed the N required in (1), but, at the moment, N depends on both x and ε.

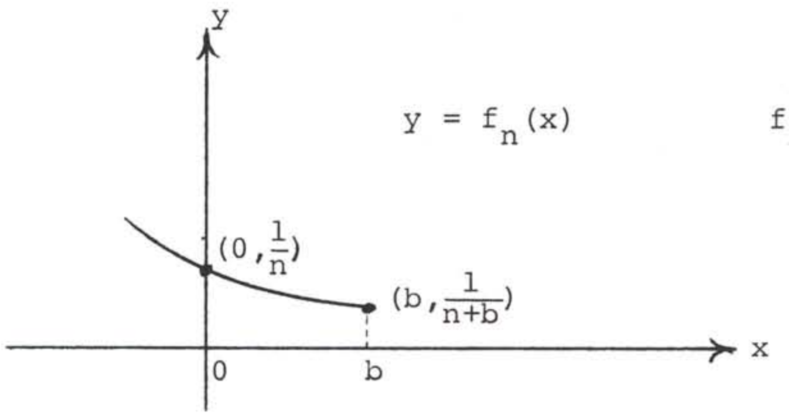
However, the fact that x is positive means that $\frac{1}{\varepsilon} - x < \frac{1}{\varepsilon}$; hence,

$$\text{If } n > \frac{1}{\varepsilon}, \text{ then } n > \frac{1}{\varepsilon} - x, \text{ whereupon } \left|\frac{1}{x+n} - 0\right| < \varepsilon$$

We may now let $N_1 = \left[\frac{1}{\varepsilon}\right]$ (i.e., N_1 is the greatest integer which does not exceed $\frac{1}{\varepsilon}$).

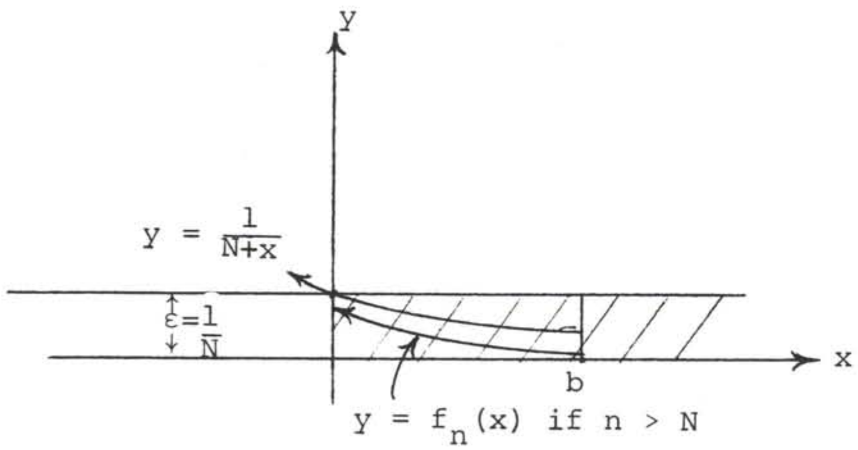
$$\therefore n > N_1 \rightarrow n > N \rightarrow \left|\frac{1}{x+n} - 0\right| < \varepsilon \tag{5}$$

Finally, since $\left[\frac{1}{\varepsilon}\right]$ depends only on ε, not x, (5) tells us that $\left\{\frac{1}{x+n}\right\}$ converges uniformly to 0 on [0,b]. Pictorially,



For simplicity, let $\varepsilon = \frac{1}{N}$. Then, for n > N, we have

[7.5.2 cont'd]



7.5.3

If $f_n(x) = \frac{n}{x+n}$, then $f_n(x) = \frac{1}{\frac{x}{n} + 1}$

$$\therefore \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left[\frac{1}{\frac{x}{n} + 1}\right] = \frac{1}{0 + 1} = \frac{1}{1} = 1$$

Hence, $\left\{\frac{n}{x+n}\right\}$ converges to 1 for all real numbers x. Let us now construct N for a given ε.

We want $\left|\frac{n}{x+n} - 1\right| < \varepsilon$, and $\frac{n}{x+n} - 1 = \frac{-x}{x+n}$

$$\therefore \text{We want } \left|\frac{-x}{x+n}\right| < \varepsilon \qquad (1)$$

Since we are given that x is non-negative, we have that $|-x| = x$ while $|x+n| = x+n$. Hence,

$$\left|\frac{-x}{x+n}\right| = \frac{x}{x+n} \qquad (2)$$

[7.5.3 cont'd]

Substituting (2) into (1), we have

$$\frac{x}{x+n} < \varepsilon$$

$$\therefore \quad \frac{x+n}{x} > \frac{1}{\varepsilon}$$

$$\therefore \quad x+n > \frac{x}{\varepsilon}$$

$$\therefore \quad n > \frac{x}{\varepsilon} - x = x\left(\frac{1}{\varepsilon} - 1\right) \qquad (3)$$

(Again we may assume $0 < \varepsilon < 1$; otherwise, (3) holds trivially. Then, as soon as $0 < \varepsilon < 1$, $\frac{1}{\varepsilon} > 1$ and $\left(\frac{1}{\varepsilon} - 1\right)$ is also positive.)

$\therefore$ If we let $N = \left[x\left(\frac{1}{\varepsilon} - 1\right)\right]$, we see that

$$n > N \rightarrow \left|\frac{n}{x+n} - 1\right| < \varepsilon \qquad (4)$$

To show that $\left\{\frac{n}{x+n}\right\}$ converges uniformly to 1 on [0,b] we must be able to show that we can find N_1 which doesn't depend on x so that (4) holds when N is replaced by N_1.

To this end, we observe that since $0 \leqslant x \leqslant b$, $x\left(\frac{1}{\varepsilon} - 1\right) \leqslant b\left(\frac{1}{\varepsilon} - 1\right)$

$$\left[x\left(\frac{1}{\varepsilon} - 1\right)\right] \leqslant \left[b\left(\frac{1}{\varepsilon} - 1\right)\right]$$

Letting $N_1 = \left[b\left(\frac{1}{\varepsilon} - 1\right)\right]$, we have that $n > N_1 \rightarrow n > N\left(= \left[x\left(\frac{1}{\varepsilon} - 1\right)\right]\right)$. Hence

$$n > N_1 \rightarrow n > N \rightarrow \left|\frac{n}{x+n} - 1\right| < \varepsilon$$

[7.5.3 cont'd]

But N_1 doesn't depend on x and our proof is complete.

(Note a rather important difference in the result of this exercise compared with the previous one. In the previous exercise, we could let $N_1 = \left[\frac{1}{\varepsilon}\right]$ and this not only was independent of x, but it was also independent of x being bounded. In this exercise, however, $N_1 = \left[b\left(\frac{1}{\varepsilon} - 1\right)\right]$, and, clearly, N_1 increases without bound as b increases without bound. In other words, now uniform convergence depends on x being bounded. We shall exploit this in the next exercise.)

7.5.4

$$f_n(x) = \frac{n}{n+x}.$$

$$\therefore \lim_{n\to\infty} f_n(x) = 1$$

$$\therefore \lim_{x\to\infty}\left[\lim_{n\to\infty} f_n(x)\right] = \lim_{x\to\infty} [1] = 1 \qquad (1)$$

On the other hand,

$$\lim_{x\to\infty} f_n(x) = \lim_{x\to\infty}\left[\frac{n}{n+x}\right] = \frac{n}{n+\infty} = \frac{n}{\infty} = 0$$

$$\therefore \lim_{n\to\infty}\left[\lim_{x\to\infty} f_n(x)\right] = \lim_{n\to\infty} [0] = 0 \qquad (2)$$

The result follows from comparing (1) and (2).

(Notice our observation in the previous exercise that the uniform convergence depends on x being bounded. In this exercise we let $x\to\infty$ which meant that x was allowed to increase without bound.)

7.5.5(L)

(a) Since nx and 1 + nx are both continuous, $\frac{nx}{1 + nx}$ will be continuous provided $1 + nx \neq 0$. But $1 + nx \geq 1$ since both n and x are non-negative (n because it is a whole number and x because it is [0,1]). In particular, for all x under consideration $1 + nx \neq 0$.

$$\therefore \text{ If } f_n(x) = \frac{nx}{1 + nx},\ f_n \text{ is continuous on } [0,1]$$

(b) If $x = 0$, $\frac{nx}{1 + nx} = \frac{0}{1} = 0$

$$\therefore \lim_{n\to\infty} \frac{nx}{1 + nx} = \lim_{n\to\infty} 0 = 0 \text{ if } x = 0 \qquad (1)$$

If $x \neq 0$ then

$$\lim_{n\to\infty} \frac{nx}{1 + nx} = \lim_{n\to\infty} \left[\frac{x}{\frac{1}{n} + x}\right] = \lim_{n\to\infty} \left[\frac{x}{0 + x}\right]$$

$$= \lim_{n\to\infty} \frac{x}{x} = \lim_{n\to\infty} 1^* = 1 \qquad (2)$$

Combining (1) and (2) we see that

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1 + nx} = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } 0 < x \leq 1 \end{cases}$$

$\therefore f(x)$ is not continuous on [0,1]. In particular, f is discontinuous at $x = 0$.

---

*The statement $\frac{x}{x} = 1$ requires that $x \neq 0$, otherwise we have the indeterminate $\frac{0}{0}$ form. This, hopefully, motivates why when we compute $\lim_{n\to\infty} \frac{nx}{1 + nx}$, we look at $x = 0$ as a special case.

[7.5.5(L) cont'd]

(c) We have proven that if $\{f_n\}$ converges uniformly to f on [a,b] and each f_n is continuous on [a,b], then f must also be continuous on [a,b]. In our problem we saw in (a) that each f_n was continuous on [0,1]. We saw in (b), however, that f was not continuous on [0,1]. Hence $\{f_n\}$ does not converge uniformly to f on [0,1] (otherwise f would also have been continuous on [0,1]).

(d) Part (c) offers an indirect proof that convergence was not uniform on [0,1]. In part (d) we would again like to emphasize the more direct computational approach.

Since x = 0 is the "bad" point, we shall, for the moment, disregard it and look at

$$\frac{nx}{1 + nx} \text{ where } 0 < b \leq x \leq 1$$

For any $x \in [b,1]$ $\lim_{n\to\infty} \frac{nx}{1 + nx} = 1$

This means that for any $\varepsilon > 0$ we can find N such that

$n > N \rightarrow \left|\frac{nx}{1 + nx} - 1\right| < \varepsilon$

As in previous problems, we now examine

$$\left|\frac{nx}{1 + nx} - 1\right| < \varepsilon$$

Well

$$\left|\frac{nx}{1 + nx} - 1\right| = \left|\frac{nx-1 - nx}{1 + nx}\right| = \left|\frac{-1}{1 + nx}\right| = \frac{|-1|}{|1 + nx|} = \frac{1}{1 + nx}$$

(since x > 0)

[7.5.5(L) cont'd]

$$\therefore \left|\frac{nx}{1+nx} - 1\right| < \varepsilon \leftrightarrow \frac{1}{1+nx} < \varepsilon$$

$$\leftrightarrow 1 + nx > \frac{1}{\varepsilon}$$

$$\leftrightarrow nx > \frac{1}{\varepsilon} - 1$$

$$\leftrightarrow n > \frac{1}{x}\left(\frac{1}{\varepsilon} - 1\right) \qquad (3)$$

If we examine (3) we see that as long as $0 < b \leqslant x \leqslant 1$, $\frac{1}{x}$ is bounded. Namely $\frac{1}{b} \geqslant \frac{1}{x} \geqslant 1$

$$\therefore n > \frac{1}{b}\left(\frac{1}{\varepsilon} - 1\right) \rightarrow n > \frac{1}{x}\left(\frac{1}{\varepsilon} - 1\right) \rightarrow \left|\frac{nx}{1+nx} - 1\right| < \varepsilon$$

and since $\frac{1}{b}\left(\frac{1}{\varepsilon} - 1\right)$ does not depend on x, the convergence is uniform.

That is, $\left\{\frac{nx}{1+nx}\right\}$ converges uniformly to 1 on [b,1] where $b \gneq 0$. The key comes in when we look at (3) as $x \rightarrow 0^+$. In this case $\frac{1}{x}$ increases without bound which indicates that "close" to 0, N depends on x.

<u>7.5.6(L)</u>

(a) $$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left[\frac{n^2x}{1+n^3x^2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{\frac{x}{n}}{\frac{1}{n^3} + x^2}\right]$$

$$= \frac{0}{0 + x^2}$$

$$= 0 \text{ (unless } x = 0\text{, since then we have } \tfrac{0}{0}\text{)}$$

[7.5.6(L) cont'd]

If $x = 0$, $\dfrac{n^2x}{1 + n^3x^2} = 0$

$$\therefore \lim_{n\to\infty} f_n(x) = 0 \text{ for all real } x$$

$$\therefore \int_0^1 \lim_{n\to\infty} f_n(x)dx = \int_0^1 0\ dx = 0 \qquad (1)$$

On the other hand,

$$\int_0^1 f_n(x)dx = \int_0^1 \frac{n^2x}{1 + n^3x^2}\ dx = \left.\frac{\ln\left(1 + n^3x^2\right)}{2n}\right|_0^1$$

$$= \frac{\ln(1 + n^3)}{2n}$$

$$\therefore \lim_{n\to\infty}\int_0^1 f_n(x)dx = \lim_{n\to\infty}\left[\frac{\ln(1 + n^3)}{2n}\right] \qquad (2)$$

Equation (2) brings us to grips with an important computational issue. Notice that $\lim_{n\to\infty}\left[\frac{\ln(1 + n^3)}{2n}\right]$ is of the form "$\frac{\infty}{\infty}$" if we replace n by ∞. In many respects $\frac{\infty}{\infty}$ is the same as $\frac{0}{0}$, (for example, if one thinks of $\infty = \frac{1}{0}$, then $\frac{\infty}{\infty} = \frac{1}{0} \div \frac{1}{0} = \frac{0}{0}$) and so $\frac{\infty}{\infty}$ is also called an indeterminate form.

To be sure, there are shrewd ways of trying to guess $\lim_{n\to\infty}\left[\frac{\ln(1 + n^3)}{2n}\right]$. One technique might be to say that for large values of n, $\ln(1 + n^3) \approx \ln n^3 = 3\ln n$

[7.5.6(L) cont'd]

$\therefore\ \dfrac{\ln(1 + n^3)}{2n} \approx \dfrac{3\ \ln n}{2n} = \dfrac{3}{2}\left(\dfrac{\ln n}{n}\right)$, but we already know that

$\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$

We guess that $\lim\limits_{n\to\infty} \left[\dfrac{\ln(1 + n^3)}{2n}\right] = 0$

We shall prove this result more rigorously as a note at the end of this exercise. For the time being, assume that it's true. Then, from (2),

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \left[\frac{\ln(1 + n^3)}{2n}\right] = 0 \qquad (3)$$

Comparing (1) and (3), we have

$$\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx$$

(b) Let

$$y = f_n(x) = \frac{n^2 x}{1 + n^3 x^2} \qquad (4)$$

Then

$$y' = f_n'(x) = \frac{\left(1 + n^3x^2\right)n^2 - n^2x(2n^3x)}{\left(1 + n^3x^2\right)^2}$$

$$= \frac{n^2 - n^5x^2}{\left(1 + n^3x^2\right)^2} \qquad (5)$$

[7.5.6(L) cont'd]

From (4) we observe that

(i) $f_n(0) = 0$ for every n

(ii) $f_n(x) \gtrless 0 \leftrightarrow x \gtrless 0$ (i.e., $f_n(x)$ is positive when x is positive, negative when x is negative). In fact

(iii) $f_n(x) = -f_n(x)$ (i.e., f_n is an odd function)

From (5) we see that

(iv) $f_n{}'(0) = n^2$

Combining this with (i) means that each curve $y = f_n(x)$ passes through (0,0) but the slope at (0,0) increases without bound as n does.

(v) $$\begin{aligned} f_n{}'(x) = 0 &\leftrightarrow n^2 - n^5x^2 = 0 \\ &\leftrightarrow x^2 = \frac{1}{n^3} \\ &\leftrightarrow x = \pm \frac{1}{n^{\frac{3}{2}}} = \pm n^{-\frac{3}{2}} \end{aligned}$$

If $x = \pm n^{-\frac{3}{2}}$ then

$$f_n(x) = \frac{n^2\left(\pm n^{-\frac{3}{2}}\right)}{1 + n^3\left(\pm n^{-\frac{3}{2}}\right)^2} = \frac{\pm n^{\frac{1}{2}}}{1 + n^3(n^{-3})} = \frac{\pm\sqrt{n}}{2}$$

More specifically,

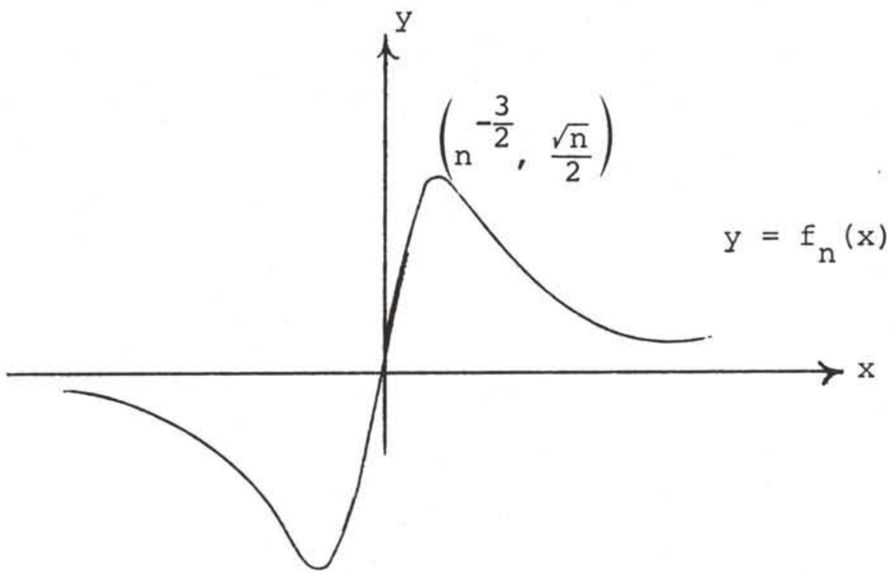
$$f_n\left(n^{-\frac{3}{2}}\right) = \frac{\sqrt{n}}{2} \quad f_n\left(-n^{-\frac{3}{2}}\right) = \frac{-\sqrt{n}}{2}$$

[7.5.6(L) cont'd]

$\therefore \left(n^{-\frac{3}{2}}, \frac{\sqrt{n}}{2}\right)$ is a high point

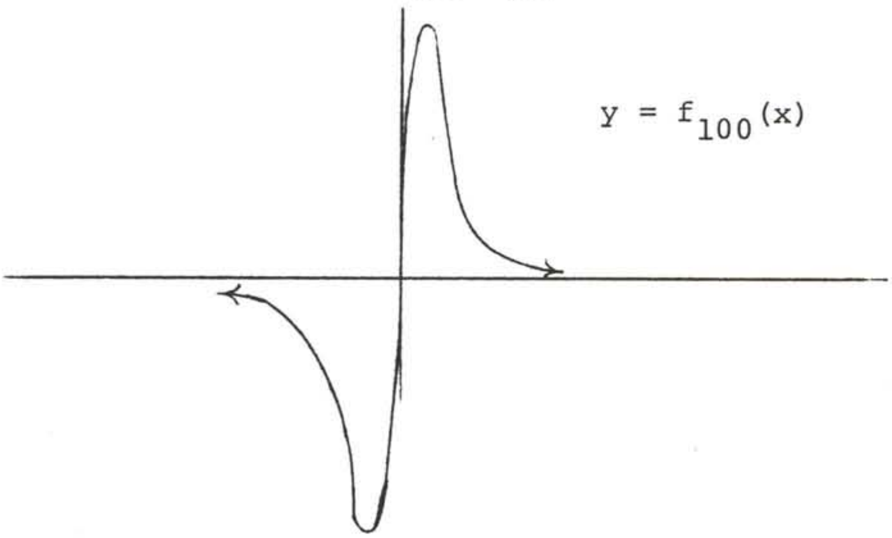
$\therefore \left(-n^{-\frac{3}{2}}, \frac{-\sqrt{n}}{2}\right)$ is a low point

Putting this together we have



$n = 100 \rightarrow n^{-\frac{3}{2}} = \frac{1}{1000}, \frac{\sqrt{n}}{2} = 5$

[7.5.6(L) cont'd]



As n gets large $n^{-\frac{3}{2}}$ gets close to 0 while $\frac{\sqrt{n}}{2}$ gets large. That is, the high point of $y = f_n(x)$ occurs closer to the y-axis and higher and higher up as $n\to\infty$. For example, if $n = 10^{100}$ then $n^{-\frac{3}{2}} = \left[(10)^{100}\right]^{-\frac{3}{2}} = 10^{-150}$ (which is 0 to 149 decimal place accuracy) while $\frac{\sqrt{n}}{2} = \frac{10^{50}}{2} = 5 \times 10^{49}$ (which is rather huge).

This leads us to suspect that $\{f_n\}$ does <u>not</u> converge uniformly to 0 on [0,1].

Moreover, as $f_n(x)$ becomes more complicated algebraically, it becomes more difficult to find N explicitly for a given $\varepsilon > 0$. As a result it is a valuable tool to be able to sketch the curves $y = f_n(x)$ and then get a pictorial insight as to what is actually happening.

(c) In order that we do not have to take too much on faith, we have chosen an example for which it is still possible, without too much difficulty, to find N for a given ε.

[7.5.6(L) cont'd]

Namely we want

$$\left|\frac{n^2x}{1 + n^3x^2} - 0\right| < \varepsilon$$

$$\therefore \frac{n^2x}{1 + n^3x^2} < \varepsilon \quad \text{since x is positive}$$

$$\therefore \frac{1 + n^3x^2}{n^2x} > \frac{1}{\varepsilon} \tag{6}$$

Since $n^2x \geqslant x$ (remember $n^2 = 1,4,9,16,$ and $x > 0$), we deduce from (6) that

$$\frac{1 + n^3x^2}{n^2x} > \frac{1}{\varepsilon} \quad \underline{\text{if}} \quad \frac{1 + n^3x^2}{x} > \frac{1}{\varepsilon}$$

(i.e. $\dfrac{1 + n^3x^2}{x} \geqslant \dfrac{1 + n^3x^2}{n^2x}$ since $n^2 \geqslant 1$)

$$\text{Now } \frac{1 + n^3x^2}{x} > \frac{1}{\varepsilon} \leftrightarrow 1 + n^3x^2 > \frac{x}{\varepsilon}$$

$$\leftrightarrow n^3x^2 > \frac{x}{\varepsilon} - 1$$

$$\leftrightarrow n^3 > \frac{1}{\varepsilon x} - \frac{1}{x^2}$$

$$\leftrightarrow n > \sqrt[3]{\frac{1}{\varepsilon x} - \frac{1}{x^2}}$$

That is: $n > \sqrt[3]{\dfrac{1}{\varepsilon x} - x^2} \rightarrow \left|\dfrac{n^2x}{1 + n^3x^2}\right| < \varepsilon$

[7.5.6(L) cont'd]

We then see that as $x \to 0$, $\sqrt[3]{\frac{1}{\varepsilon x} - x^2} \to \infty$ and thus that uniform convergence is lost as $x \to 0$.

(d) This is a major learning point of this exercise. Namely, we have proven that if $\{f_n\}$ converges uniformly to f on [a,b], then $\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx$. We did not prove the converse.

That is, we did not show that if $\lim_{n\to\infty}\int_a^b f_n(x) = \int_a^b \lim_{n\to\infty} f_n(x)dx$ then $\{f_n\}$ converges uniformly to f on [a,b].

Now, from this exercise, we see why we did not prove the converse. Namely, the converse is false!

In particular, in this exercise we showed explicitly a case wherein

(1) $\lim_{n\to\infty}\int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx$

but

(2) $\{f_n\}$ did not converge uniformly to f on [0,1]

<u>A NOTE ON 0/0-TYPE FORMS</u>

When we dealt with such forms as $\lim_{x\to 0} \frac{\sin x}{x}$ or $\lim_{x\to a} \frac{x^2 - a^2}{x - a}$, we were fortunate in that we had <u>relatively</u> simple devices at our disposal to compute these 0/0-type indeterminate forms.

The main problem is that there are many transcendental functions that take on the 0/0 form but for which there are no obvious straight-forward techniques for computing the limit.

[7.5.6(L) cont'd]

One device that is sometimes used is series. For example, suppose we wish to compute $\lim_{x\to 0} \frac{\sinh x}{x}$. We have already seen that sinh x may be represented by the power series:

$$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

Hence

$$\frac{\sinh x}{x} = 1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots.$$

(To be on safer ground, what we should do is say that

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + R_n(x,0)$$

where $\lim R_n(x,0) = 0$. We would then conclude that

$$\frac{\sinh x}{x} = 1 + \frac{x^2}{3!} + \cdots + \frac{x^{2n}}{(2n+1)!} + \frac{R_n(x,0)}{x}$$

and then show that $\lim_{n\to\infty} \frac{R_n(x,0)}{x}$ is also equal to 0 whereupon $\lim_{x\to 0} \frac{\sinh x}{x} = 1$.)

We may talk more about the series approach in the next unit. At this time, what we would prefer to do is to introduce a rather powerful tool for evaluating 0/0 forms. The technique we have in mind is known as

<u>l'Hopital's Rule</u>

[7.5.6(L) cont'd]

(For a more thorough treatment, read Thomas 18.7.)

What l'Hopital's Rule says is:

Suppose f(a) = g(a) = 0 and that $\lim_{x\to a} \frac{f'(t)}{g'(t)}$ exists. Then $\lim_{x\to a} \frac{f(t)}{g(t)}$ exists, and, in particular,

$$\lim_{x\to a} \frac{f(t)}{g(t)} = \lim_{x\to a} \frac{f'(t)}{g'(t)}^{*}$$

For example, if we were to use this rule to compute $\lim_{t\to 0} \frac{\sinh t}{t}$, we would have f(t) = sinh t, g(t) = t, and a = 0. Then

$$\lim_{t\to 0} \frac{f'(t)}{g'(t)} = \lim_{t\to 0} \frac{\cosh t}{1} = \cosh 0 = 1$$

Therefore,

$$\lim_{t\to 0} \frac{\sinh t}{t} \quad \left(= \lim_{t\to 0} \frac{f(t)}{g(t)}\right) = 1.$$

The proof of this result is a refinement of the mean value theorem. In other words, l'Hopital's Rule could have been introduced as early as Block II, but perhaps it fits better in the context of Block VII.

Section 18.7 in Thomas supplies the details of the proof of l'Hopital's Rule in an elementary manner, and as a result we will not reproduce the proof here.

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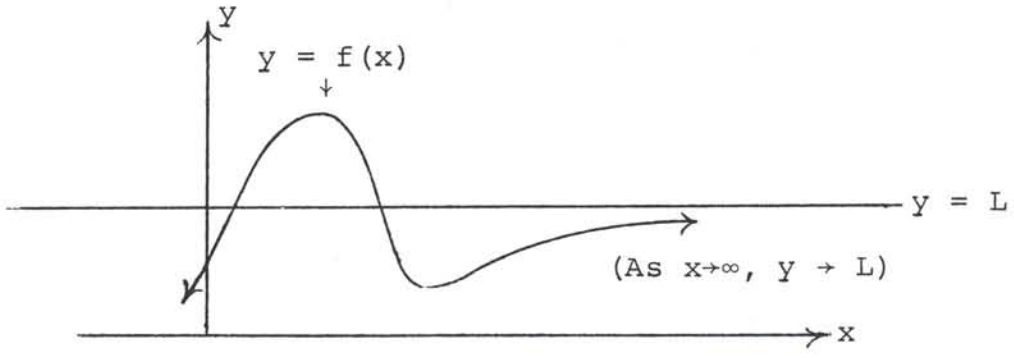
*Note the form $\frac{f'(t)}{g'(t)}$. It is <u>not</u> the derivative of $\frac{f(t)}{g(t)}$ but rather the quotient of the derivatives of f(t) and g(t).

[7.5.6(L) cont'd]

Instead we will talk about a subtlety that is not discussed in as much detail in the text but which is essential in our present consideration. Notice that in our problem we are dealing with $\lim_{n\to\infty}$ rather than $\lim_{x\to\infty}$. This is much more crucial than just a change in symbolism. Namely, n denotes a whole number (i.e., n = 1,2,3,...) while x denotes any real number. In the technical jargon, n is called a discrete variable, while x is a continuous variable.

The key point is that the proof of l'Hopital's Rule uses differential calculus, and differential calculus applies to continuous variables - not to discrete variables. (This will be illustrated by an example at the end of this note.) We "bail out" of this dilemma by observing that if $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} f(n) = L$, also. This means, in essence, that we may view n as if it were a continuous variable when we compute $\lim_{n\to\infty} f(n)$.

To see why, perhaps a picture is most helpful. Suppose $\lim_{x\to\infty} f(x) = L$. Then, graphically, we might have

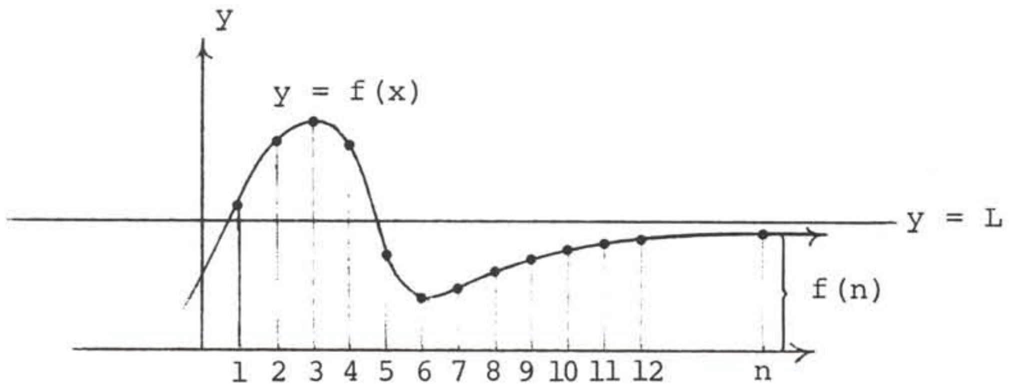


(Figure 1)

[7.5.6(L) cont'd]

Notice, next, that the points $(n, f(n))$ are a subset of the points which make up the curve $y = f(x)$.

Namely,



(Figure 2)

From Figure 2 it should be clear that as $n \to \infty$, $f(n) \to L$.

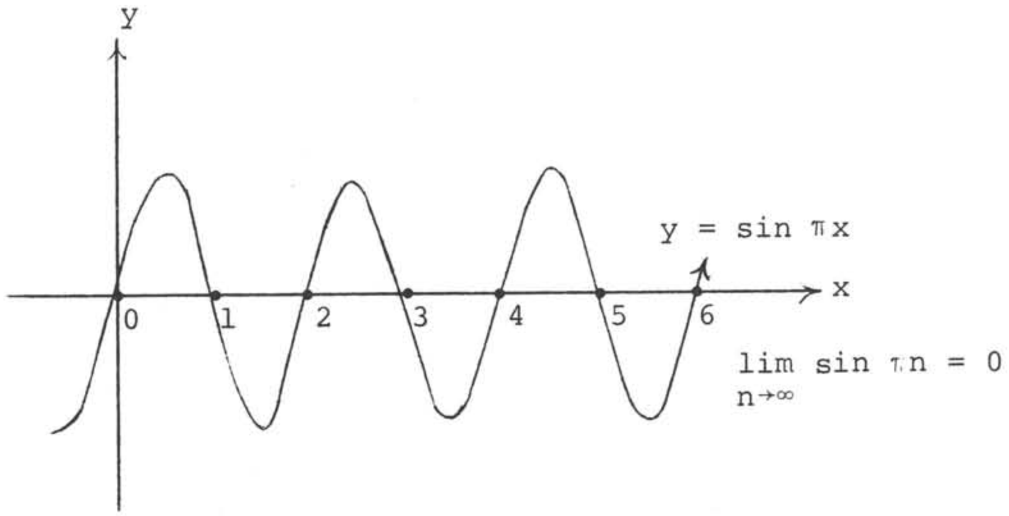
<u>Note</u>

The converse is not true. That is, if $\lim_{n\to\infty} f(n) = L$ it <u>need not</u> be true that $\lim_{x\to\infty} f(x) = L$. As a simple example consider

$$f(x) = \sin \pi x$$

Then $\lim_{x\to\infty} f(x)$ <u>doesn't exist</u> since the values of $f(x)$ continuously oscillate between -1 and 1. On the other hand, for any whole number n, $f(n) - \sin \pi n = 0$. Hence $\lim_{n\to\infty} f(n) = 0$. Pictorially,

[7.5.6(L) cont'd]



At any rate, then, if we treat n as a continuous variable and it turns out that $\lim_{n\to\infty} f(n) = L$, then it will also turn out that $\lim_{n\to\infty} f(n) = L$, if n is viewed as a discrete variable. This is what justifies our use of l'Hopital's Rule in our present exercise.

—— o ——

There is an "interesting" proof that 2 = 1 by means of calculus if we confuse a discrete variable with a continuous variable. Namely,

$1^2 = 1$ (i.e., one 1)

$2^2 = 2 + 2$ (i.e., two 2's)

$3^2 = 3 + 3 + 3$ (i.e., three 3's)

.
.
.

[7.5.6(L) cont'd]

In general, then,

$$x^2 \equiv \underbrace{x + x + \ldots + x}_{x \text{ times}} \qquad (1)$$

If we now differentiate both sides of (1), we obtain

$$2x = \underbrace{1 + 1 + \ldots + 1}_{x \text{ times}}$$

or

$$2x = x \qquad (2)$$

Since x need not be 0, we may cancel x from both sides of equation (2) to obtain

$$2 = 1 \qquad (3)$$

How could we have arrived at such a ridiculous conclusion (unless you believe 2 = 1)? The answer lies in the fact that equation (1) <u>is only true if x is a whole number</u>*. More symbolically, (1) should have been written

$$n^2 = \underbrace{n + \ldots + n}_{n \text{ times}}$$

---

*That is we can add 1,2,3,4,5,... etc. terms but we cannot form, for example, the sum of π terms. That is, to say π^2 means add π to itself π times is a bit absurd. The notion that multiplication is repeated addition applies only to whole numbers.

[7.5.6(L) ccnt'd]

Equation (3) offers us direct proof that the rules of differentiation cannot be applied to discrete variables since our application of differentiation here led to the absurdity that 2 = 1. (More structurally, the concept of derivative required that we could make the difference between x_1 and $x_1 + \Delta x$ as small as we wished. If, however, the domain of our function consists of whole numbers, either $\Delta x = 0$ or else it is at least 1 since this is the minimum difference between two unequal whole numbers.)

7.5.7

Notice that $\lim\limits_{n\to\infty} 2nxe^{-nx^2} = \lim\limits_{n\to\infty} \frac{2nx}{e^{nx^2}}$ and this is an $\frac{\infty}{\infty}$- form, which, hopefully, suggests L'Hopital's Rule. (To use l'Hopital's Rule on $\lim\limits_{n\to\infty} 2nxe^{-nx^2}$ we must have either a $\frac{0}{0}$ form or a $\frac{\infty}{\infty}$ form and that is why we write $2nxe^{-nx^2}$ as $\frac{2nx}{e^{nx^2}}$.)

Remembering that we are treating n as the variable and x as a constant, we obtain

$$\lim_{n\to\infty} 2nxe^{-nx^2} = \lim_{n\to\infty} \frac{2nx}{e^{nx^2}} = \lim_{n\to\infty} \frac{2x}{x^2 e^{nx^2}} = \lim_{n\to\infty} \frac{2}{xe^{nx^2}}^{*} = \frac{1}{x\infty} = \frac{1}{\infty} = 0$$

$$\therefore \lim_{n\to\infty} 2nxe^{-nx^2} = 0 \qquad (1)$$

---

*This result requires that $x \neq 0$ since we cancelled x from numerator and denominator. If, however, x = 0 it is clear that the result still holds since $2nxe^{-nx^2}$ then equals 0.

[7.5.7 cont'd]

Another way of anticipating the result (1), is to observe that e^{-nx^2} goes to 0 "faster than" 2nx goes to infinity. That is, $\lim_{n\to\infty} ne^{-n} = 0$, from which a few computational steps allow us to conclude that $\lim_{n\to\infty} 2nxe^{-nx^2} = 0$.

(b) By (a), $\lim_{n\to\infty} f_n(x) = 0$

$$\therefore \int_0^1 \lim_{n\to\infty} f_n(x)dx = \int_0^1 0\, dx = 0 \qquad (2)$$

(c) $\int_0^1 f_n(x)dx = \int_0^1 2nxe^{-nx^2}dx = -e^{-nx^2}\Big|_0^1$

$$= -e^{-n} - (-e^{-0}) = -e^{-n} + 1 = 1 - \frac{1}{e^n}$$

$$\therefore \lim_{n\to\infty}\int_0^1 f_n(x)dx = \lim_{n\to\infty}\left(1 - \frac{1}{e^n}\right) = 1 - 0 = 1 \qquad (3)$$

(d) Comparing (2) and (3) we see that

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)dx \qquad (4)$$

But (4) cannot happen if $\{f_n\}$ converges uniformly to f on [0,1].

Therefore, $\{f_n\}$ does not converge uniformly to f on [0,1] because (4) <u>does</u> happen! i.e. $\left|2nxe^{-nx^2}\right|$ does not converge uniformly to 0 on [0,1].

7.5.8(L)

(a) Actually, this is a learning exercise at this stage only in the sense that we want to illustrate our concluding remarks in Chapter X, Section G of the supplementary notes concerning the fact that a series is a sequence of partial sums.

We have

$$f_n(x) = \sum_{k=0}^{n} \frac{x^2}{(1 + x^2)^k}$$

$$= x^2\left(1 + \frac{1}{1 + x^2} + \frac{1}{(1 + x^2)^2} + \frac{1}{(1 + x^2)^3} + \ldots + \frac{1}{(1 + x^2)^n}\right) \quad (1)$$

The sum in parentheses in (1) is a geometric progression with ratio $\frac{1}{1 + x^2}$ and since $x^2 \geqslant 0$, $\frac{1}{1 + x^2} \leqslant 1$ with equality holding $\leftrightarrow$ $x = 0$.

If we let, say, $s_n = 1 + \frac{1}{1 + x^2} + \ldots + \frac{1}{(1 + x^2)^n}$ we see that (and this is the same technique we used earlier to evaluate geometric series)

$$\frac{1}{1 + x^2} s_n = \frac{1}{(1 + x^2)} + \frac{1}{(1 + x^2)^2} + \ldots + \frac{1}{(1 + x^2)^{n+1}}$$

$$\therefore \quad s_n - \frac{1}{1 + x^2} s_n = 1 - \frac{1}{(1 + x^2)^{n+1}}$$

$$\therefore \quad \left(1 - \frac{1}{1 + x^2}\right)s_n = 1 - \frac{1}{(1 + x^2)^{n+1}}$$

[7.5.8(L) cont'd]

$$\therefore \quad \frac{x^2}{1 + x^2} s_n = 1 - \frac{1}{(1 + x^2)^{n+1}}$$

$$\therefore \quad s_n = \frac{1 + x^2}{x^2}\left[1 - \frac{1}{(1 + x^2)^{n+1}}\right] \; (x \neq 0) \qquad (2)$$

Putting (2) into (1) we have

$$f_n(x) = x^2 \, s_n(x) = x^2\left[\frac{1 + x^2}{x^2}\left(1 - \frac{1}{(1 + x^2)^{n+1}}\right)\right]$$

$$= (1 + x^2)\left(1 - \frac{1}{(1 + x^2)^{n+1}}\right) \quad x \neq 0$$

$$\therefore \lim_{n\to\infty} f_n(x) = 1 + x^2 \text{ if } x \neq 0$$

while $\lim_{n\to\infty} f_n(0) = \sum_{k=0}^{n} \frac{0^2}{(1 + 0^2)^k} = 0$, or $\lim_{n\to\infty} f_n(x) = 0$, if $x = 0$.

Letting $f(x) = \lim_{n\to\infty} f_n(x)$, we see that

$$f(x) = \begin{cases} 1 + x^2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$\therefore \lim_{x\to 0} f(x) = \lim_{x\to 0} (1 + x^2) = 1 \neq f(0)$$

$\therefore$ f is not continuous at $x = 0$. (3)

[7.5.8(L) cont'd]

(b) Since $\frac{x^2}{(1 + x^2)^k}$ is the quotient of two continuous functions and $1 + x^2 \neq 0$ for every x, $\frac{x^2}{(1 + x^2)^k}$ is also continuous. Then, since any finite sum of continuous functions is continuous, it follows that $f_n(x) = \sum_{k=0}^{n} \frac{x^2}{(1 + x^2)^k}$ is continuous.

Hence $\{f_n\}$ cannot converge uniformly to f on any interval which includes x = 0, for if it did f would also have to be continuous at x = 0, but from (3) we see that this isn't true! In other words, an infinite sum of continuous functions need not be continuous.

7.5.9(L)

(a) For a given x, we may compute $\lim_{n\to\infty} \frac{\sin nx}{\sqrt{n}}$ to be 0 since $-1 \leqslant \sin nx \leqslant 1 \rightarrow \left|\frac{\sin nx}{\sqrt{n}}\right| \leqslant \frac{1}{\sqrt{n}} \rightarrow 0$.

To show uniform convergence we must exhibit for $\varepsilon > 0$, a number N, which depends on ε but not x, such that
$n > N \rightarrow \left|\frac{\sin nx}{\sqrt{n}}\right| < \varepsilon$.

Now, since $\frac{|\sin nx|}{\sqrt{n}} \leqslant \frac{1}{\sqrt{n}}$ it follows that $\left|\frac{\sin nx}{\sqrt{n}}\right| < \varepsilon$ if $\frac{1}{\sqrt{n}} < \varepsilon$. But $\frac{1}{\sqrt{n}} < \varepsilon \leftrightarrow \sqrt{n} > \frac{1}{\varepsilon}$ or $n > \frac{1}{\varepsilon^2}$.

Letting $N = \left[\frac{1}{\varepsilon^2}\right]$, we see that

$$n > N \rightarrow n > \frac{1}{\varepsilon^2} \rightarrow \frac{1}{n} < \varepsilon \rightarrow \left|\frac{\sin nx}{\sqrt{n}}\right| < \varepsilon$$

[7.5.9(L) cont'd]

Since $\left[\frac{1}{\varepsilon^2}\right]$ doesn't depend on x,

$\left\{\frac{\sin nx}{\sqrt{n}}\right\}$ converges uniformly to 0

(b) $\lim_{n\to\infty} f_n(x) = 0$

$$\therefore \left[\lim_{n\to\infty} f_n(x)\right]' = \frac{d\left[\lim_{n\to\infty} f_n(x)\right]}{dx} = \frac{d0}{dx} = 0 \qquad (1)$$

On the other hand, $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ implies that

$$f_n'(x) = \frac{n \cos nx}{\sqrt{n}} = \sqrt{n} \cos nx \qquad (2)$$

$\therefore \lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \sqrt{n} \cos nx = \infty$ (since $\cos nx = 1$ when $x = 0$, 2π, 4π.... we see that $\sqrt{n} \cos(n\, 2k\, \pi) = \sqrt{n} \to \infty$ as $n\to\infty$)

$$\therefore \lim_{n\to\infty} f_n'(x) \neq \left[\lim_{n\to\infty} f_n(x)\right]' \qquad (3)$$

since, in any event, $\lim_{n\to\infty} f_n'(x)$ is not identically 0.

Thus, we have an example wherein

(1) $\{f_n\}$ converges uniformly to f

(2) f_n' and f' exist

but

(3) $\lim_{n\to\infty} f_n' \neq f'$

[7.5.9(L) cont'd]

This does not contradict our theorem about differentiability since $\lim_{n\to\infty} f_n{}' = f'$ was only guaranteed if $\{f_n{}'\}$ itself is a uniformly convergent sequence. Clearly, however, $\{\sqrt{n}\cos nx\}$ is not uniformly convergent since it isn't even pointwise convergent; for example, $\lim_{n\to\infty} \sqrt{n}\cos 2\pi n = \lim_{n\to\infty} \sqrt{n} = \infty$.

UNIT 6: Uniform Convergence Applied to Power Series

7.6.1 (L)

Recall that a fundamental result is that if the interval (radius) of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is R, then

(1) $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely if $|x| < R$

(2) $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly if $|x| \leqslant x_1 < R$

(3) $\sum_{n=0}^{\infty} a_n x^n$ diverges if $|x| > R$.

Thus, from a purely mechanical point of view, we need only determine R to solve this problem. To this end, the ratio test serves nicely.

Let $a_n = |5x|^n$, $a_{n+1} = |5x|^{n+1}$.

Then $\frac{a_{n+1}}{a_n} = |5x| = 5\ |x|$

$$\therefore \rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 5\ |x|$$

$$\therefore \rho < 1 \longleftrightarrow 5\ |x| < 1$$

$$\longleftrightarrow |x| < \frac{1}{5} \quad . \qquad (1)$$

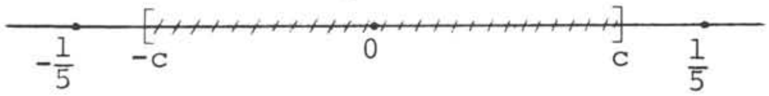
From (1), we have that $R = \frac{1}{5}$. (Notice that up to this point we are reviewing the procedure used in Unit 4 of this Block.)

[7.6.1 (L) cont'd]

Thus, from (2) above $\sum_{n=0}^{\infty} (5x)^n$ converges uniformly if $|x| \leqslant c < \frac{1}{5}$.
(2)

Diagramatically:

(1) choose $c \; \varepsilon (0,\frac{1}{5})$.



(2) Then $\sum_{n=0}^{\infty} (5x)^n$ converges uniformly if $x \; \varepsilon [-c,c]$.

Notice, however, that we specifically requested use of the Weierstass M-test to find the answer. One reason for this, is that the proof of the fundamental result makes use of this test. (In other words, by using the M-test, we reinforce the proof of the fundamental result.) Another reason is that it is not always convenient to depend on the ratio test to find R; hence, the M-test gives us an alternative approach.

At any rate, to use the M-test here we observe that $\sum_{n=0}^{\infty} b^n$ converges if $|b| < 1$ and diverges if $|b| \geqslant 1$.

Hence $\sum_{n=0}^{\infty} (5x)^n$ converges uniformly, by the M-test, if and only if:

$$|5x|^n \leqslant |b|^n \text{ where } b < 1 \quad . \tag{3}$$

But (3) holds $\longleftrightarrow |5x| \leqslant b,\ b < 1$

$\longleftrightarrow 5|x| \leqslant b,\ b < 1$

[7.6.1 (L) cont'd]

$$\longleftrightarrow |x| \leq \frac{b}{5}\ ,\ b < 1$$

$$\longleftrightarrow |x| \leq c\ ,\ c < \frac{1}{5} \qquad (c = \frac{b}{5}) \quad . \tag{4}$$

A comparison of (2) and (4) show that the two methods yield the same result.

7.6.2

$$\frac{1}{1 + x^{2n}} < \frac{1}{x^{2n}} = \left(\frac{1}{x^2}\right)^n$$

$$\sum\left(\frac{1}{x^2}\right)^n \text{ converges} \longleftrightarrow \frac{1}{x^2} < 1$$

$$\longleftrightarrow x^2 > 1$$

$$\longleftrightarrow |x| > 1 \quad . \tag{1}$$

Thus, if b is any number such that $|b| > 1$, we see that $\sum_{n=0}^{\infty} \frac{1}{1 + b^{2n}} < \sum_{n=0}^{\infty} \left(\frac{1}{b^2}\right)^n$ where $\sum_{n=0}^{\infty} \left(\frac{1}{b^2}\right)^n$ is a positive convergent series.

Therefore, by the M-test:

$$\sum_{n=0}^{\infty} \frac{1}{1 + x^{2n}} \text{ is uniformly convergent if } 1 < |b| \leq |x| \quad .$$

[7.6.2 cont'd]

(By the ratio test techniques we would have

$$a_n = \frac{1}{|1 + x^{2n}|} = \frac{1}{1 + x^{2n}} \qquad (\text{since } x^{2n} \geq 0)$$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{1 + x^{2n}}{1 + x^{2n+2}} = \frac{\frac{1}{x^{2n+2}} + \frac{1}{x^2}}{\frac{1}{x^{2n+2}} + 1} .$$

$$\text{Now } \lim_{n\to\infty} \frac{1}{x^{2n+2}} = 0 \longleftrightarrow |x| > 1$$

$$\therefore |x| > 1, \longrightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{0 + \frac{1}{x^2}}{0 + 1} = \frac{1}{x^2}$$

$$\therefore \rho = \frac{1}{x^2}$$

and $\rho < 1 \longleftrightarrow x^2 > 1$

$$\longleftrightarrow |x| > 1 \qquad (2)$$

and clearly (1) and (2) agree .)

7.6.3 (L)

In this exercise, we wish to emphasize the power of series in handling certain problems. We shall both review certain basic principles and apply them is rather simple situations (we use simple cases so as not to obscure the theory; in later exercises we gradually let the cases become computationally more complex).

[7.6.3 (L) cont'd]

(a) Now in part (a) we are emphasizing that portion of our discussion in the notes in which we showed that if a series converged to a given function uniformly, then its coefficients had to behave in a rather special way. Recall that we showed that if $\sum_{n=0}^{\infty} a_n x^n$ converged uniformly to f(x), then

$$a_n = \frac{f^{(n)}(0)}{n!} \quad . \qquad (1)$$

Hence, if both $\sum_{n=0}^{\infty} b_n x^n$ and $\sum_{n=0}^{\infty} c_n x^n$ converge to the same function, (1) allows us to conclude that for each n, $b_n = c_n$ since, in particular, each is equal to $\frac{f^{(n)}(0)}{n!}$.

In other words, if $\sum a_n x^n$ converges uniformly to f(x), then it is the only series which does.

(b) We could let f(x) = x sin x and then compute $a_n \frac{f^{(n)}(0)}{n!}$ to obtain the desired series. If we do this, we obtain

$$\left.\begin{aligned}
f(x) &= x \sin x \\
f'(x) &= x \cos x + \sin x \\
f''(x) &= -x \sin x + \cos x + \cos x = 2 \cos x - x \sin x \\
f'''(x)\ & -2 \sin x - \sin x - x \cos x = -3 \sin x - x \cos x \\
f^{(4)}(x) &= -3 \cos x + x \sin x - \cos x = -4 \cos x + x \sin x
\end{aligned}\right\} \quad (1)$$

[7.6.3 (L) cont'd]

$$\left.\begin{aligned} \therefore f(0) &= 0 & \therefore a_0 &= 0 \\ f'(0) &= 0 & a_1 &= 0 \\ f''(0) &= 2 & a_2 &= \frac{f''(0)}{2!} = 1 \\ f'''(0) &= 0 & a_3 &= 0 \\ f^{(4)}(0) &= -4 & a_4 &= \frac{f^{(4)}(0)}{4!} = \frac{-4}{4!} = -\frac{1}{3!} \end{aligned}\right\} \quad (2)$$

$$\therefore x \sin x = x^2 - \frac{x^4}{3!} + \ldots. \quad (3)$$

The trouble is deriving (3) is that the formula for obtaining (1) (hence, also (2)) may seem non-simple.

A powerful alternative way is to recall that within its interval of uniform convergence a power series behaves like a polynomial. Among other things:

If $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly for $|x| < R_1$ and $\sum_{n=0}^{\infty} b_n x^n$ converges uniformly for $|x| < R_2$. Then $\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right)$ converges uniformly for $|x| < \min\{R_1, R_2\}$; where

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n,\ c_n = (a_0 b_0 + a_1 b_{n-1} + \ldots + a_n b_0).$$

Applying this idea to the present exercise; x is already a uniformly convergent series which represents x for all real x. (That is, $x \equiv 0 + 1x + 0x^2 + 0x^3 \ldots$, and from (a) the series representation is unique.)

[7.6.3 (L) cont'd]

On the other hand, $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ converges uniformly to sin x for all real x .

$$\therefore x \sin x \equiv x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots\right)$$

$$= x^2 - \frac{x^4}{4!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \ldots + \frac{(-1)^n x^{2n+2}}{(2n+1)!} + \ldots \qquad (4)$$

Notice that (3) agrees with (4), as far as we developed (3), but (4) was much easier to obtain. Also observe that (4) is valid for all real x since it is the product of two uniformly convergent series each of which is valid for all real x .

(c) $$\int_0^{1/2} x \sin x \, dx = \int_0^{1/2} \left\{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}\right\} dx$$

$$= \int_0^{1/2} \left\{x^2 - \frac{x^4}{3!} + \ldots + \frac{(-1)^n x^{2n+2}}{(2n+1)!} + \ldots\right\} dx \quad . \qquad (5)$$

Now the crucial point is that for uniformly convergent series we may interchange the order of integration and summation. Hence, (5) may be written as:

$$\left.\begin{aligned} \int_0^{1/2} x \sin x \, dx &= \sum_{n=0}^{\infty} \int_0^{1/2} \frac{(-1)^n x^{2n+2}}{(2n+1)!} dx \\ &= \sum_{n=0}^{\infty} \left\{\frac{(-1)^n x^{2n+3}}{(2n+3)(2n+1)!}\right|_0^{1/2} \\ &= \left(\frac{1}{3} x^3 - \frac{x^5}{5(3!)} + \frac{x^7}{7(5!)} - \frac{x^9}{9(7!)} + \ldots\right)\Bigg|_0^{1/2} \end{aligned}\right\} (6)$$

[7.6.3 (L) cont'd]

Therefore, from (6),

$$\frac{1}{3}\left(\frac{1}{2}\right)^3 - \frac{1}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{3!}\right)$$

underestimates $\int_0^{1/2} x \sin x \, dx$ by no more than

$$\frac{1}{7}\left(\frac{1}{2}\right)^7\left(\frac{1}{5!}\right) .$$

Now,

$$\frac{1}{3}\left(\frac{1}{2}\right)^3 - \frac{1}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{3!}\right) = \frac{1}{24} - \frac{1}{960} = \frac{39}{960} = \frac{13}{320}$$

while $\frac{1}{7}\left(\frac{1}{2}\right)^7\left(\frac{1}{5!}\right) = \frac{1}{107,520}$.

Hence,

$$\frac{13}{320} \leqslant \int_0^{1/2} x \sin x \, dx \leqslant \frac{13}{320} + \frac{1}{107,520} . \qquad (7)$$

Since $\frac{1}{107,520} = 0.000009 < 0.00001$ and $\frac{13}{320} = 0.04062$, (7) becomes:

$$0.04062 \leqslant \int_0^{1/2} x \sin x \, dx \leqslant 0.04063 . \qquad (8)$$

From (8)

$$\int_0^{1/2} x \sin dx = 0.0406 \text{ correct to 4 decimal place}$$

accuracy. (9)

[7.6.3 (L) cont'd]

(d) It happens that by integration by parts we can evaluate $\int_0^{1/2} x \sin x \, dx$ directly.

By way of review:

$$u = x \qquad dv = \sin x \, dx$$

$$du = dx \qquad v = -\cos x$$

$$\therefore \int_0^{1/2} x \sin x \, dx = uv \Big|_{x=0}^{1/2} - \int_0^{1/2} v \, du$$

$$= -x \cos x \Big|_{x=0}^{1/2} + \sin x \Big|_0^{1/2}$$

$$= -\tfrac{1}{2} \cos \tfrac{1}{2} + \sin \tfrac{1}{2} \quad .$$

From the tables (recalling again that $\sin \frac{1}{2}$ means $\sin \frac{1}{2}$ radian, etc.)

$$\sin \tfrac{1}{2} = .4794$$

$$\cos \tfrac{1}{2} = .8776$$

$$\therefore \tfrac{1}{2} \cos \tfrac{1}{2} = .4388$$

$$\therefore \sin \tfrac{1}{2} - \tfrac{1}{2} \cos \tfrac{1}{2} = .0406 \quad . \qquad (10)$$

[7.6.3 (L) cont'd]

A comparison of (10) and (9) vividly illustrates the power of our series techniques. To be sure, (10) was easier to obtain than (9) (that is, integration by parts was easier then series), but in many cases (see, for example, the next exercise) we have no alternative for the series approach. The reason we chose $\int_0^{1/2} x \sin x \, dx$ was so that we could check the accuracy of our result by another method.

7.6.4

The basic difference between this exercise and the previous one is that in this example there is no convenient way of determining $\int x e^{-x^3} dx$ by our usual techniques of integration.

What we do is begin with:

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots + \frac{u^n}{n!} + \dots \qquad , |u| < \infty$$

$$\therefore e^{-x^3} = 1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \dots + (-1)^n \frac{x^{3n}}{n!} + \dots \qquad , |x| < \infty$$

$$\therefore x e^{-x^3} = x - x^4 + \frac{x^7}{2!} - \frac{x^{10}}{3!} + \dots + \frac{(-1)^n x^{3n+1}}{n!} + \dots \qquad , |x| < \infty$$

$$\therefore \int_0^{1/2} x e^{-x^3} dx = \int_0^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{n!} dx$$

$$= \sum_{n=0}^{\infty} \int_0^{1/2} \frac{(-1)^n x^{3n+1}}{n!} dx \qquad \underline{\text{by uniform convergence}}$$

[7.6.4 cont'd]

$$= \sum_{n=0}^{\infty} \int_0^{1/2} \frac{(-1)^n x^{3n+2}}{(3n+2)(n!)} \Bigg|_0^{1/2}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{3n+2}(3n+2)(n!)}$$

$$= \frac{1}{8} - \frac{1}{2^5(5)(1)} + \frac{1}{2^8(8)2!} - \frac{1}{2^{11}(11)3!}$$

$$= \frac{1}{8} - \frac{1}{160} + \frac{1}{4096} - \frac{1}{135,168} \quad . \qquad (1)$$

$$\frac{1}{135,168} = 0.000007^+$$

∴ From (1)

$\frac{1}{8} - \frac{1}{160} + \frac{1}{4096}$ is greater than $\int_0^{1/2} x\, e^{-x^3}\, dx$ by no more than $\frac{1}{135,168}$.

Now

$$\frac{1}{8} - \frac{1}{160} + \frac{1}{4096} = \frac{1}{2^3} - \frac{1}{2^5 \cdot 5} + \frac{1}{2^{12}} = \frac{5(2)^9 - 2^7 + 5}{5(2)^{12}}$$

$$= \frac{5(512) - 128 + 5}{5(4096)} = \frac{2437}{20,480}$$

∴

$$\frac{2437}{20,480} - \frac{1}{135,168} \leq \int_0^{1/2} x\, e^{-x^3}\, dx \leq \frac{2437}{20,480} \quad . \qquad (2)$$

[7.6.4 cont'd]

$$\frac{2437}{20,480} = 0.11890^{+}$$

$$\frac{1}{135,168} = 0.000007^{+} < 0.00001 \quad .$$

$\therefore$ From (2) we see that

$$\int_0^{1/2} x\, e^{-x^3}\, dx = 0.1189$$

is correct to at least the given number of decimal places.

7.6.5

(a) $\dfrac{1}{(x-2)(x-3)} = \dfrac{1}{x-3} - \dfrac{1}{x-2}$ (by partial fractions) (1)

Now, one way of getting a power series for $\frac{1}{x-a}$ is by letting

$f(x) = \dfrac{1}{(x-a)} = (x-a)^{-1}$ $\qquad f(0) = -\dfrac{1}{a}$

$\therefore f'(x) = -(x-a)^{-2}$ $\qquad f'(0) = -\dfrac{1}{a^2}$

$f''(x) = 2(x-a)^{-3}$ $\qquad f''(0) = -2\,\dfrac{1}{a^3}$

$f'''(x) = -3(!(x-a)^{-4}$ $\qquad f'''(0) = -3!\,\dfrac{1}{a^4}$

$f^{(n)}(x) = (-1)^n\, n!(x-a)^{-(n+1)}$ $\qquad f^{(n)}(0) = -n!\,\dfrac{1}{a^{n+1}}$

$$\therefore a_n = \frac{f^{(n)}(0)}{n!} = \frac{-n!}{n!\ a^{n+1}}$$

$$\therefore \frac{1}{x-a} = \sum_{n=0}^{\infty} \frac{-x^n}{a^{n+1}} = -\left(\frac{1}{a} + \frac{x}{a^2} + \frac{x^2}{a^3} + \dots\right), \quad |x| < |a| \quad \text{(by ratio test)}$$

[7.6.5 cont'd]

$$\therefore \frac{1}{x-3} = \sum_{n=0}^{\infty} \frac{-x^n}{3^{n+1}} \qquad \frac{1}{x-2} = \sum_{n=0}^{\infty} \frac{-x^n}{2^{n+1}}$$

$$\therefore \frac{1}{x-3} - \frac{1}{x-2} = \sum_{n=0}^{\infty} \frac{-x^n}{3^{n+1}} - \sum_{n=0}^{\infty} \frac{-x^n}{2^{n+1}} \tag{2}$$

Now $\sum_{n=0}^{\infty} \frac{-x^n}{3^{n+1}}$ converges absolutely for $|x| < 3$ (ratio test) while $\sum_{n=0}^{\infty} \frac{-x^n}{2^{n+1}}$ converges absolutely for $|x| < 2$. Hence, for $|x| < 2$, both series converge absolutely. Thus, for $|x| < 2$ the series behave like polynomials.

That is, from (2):

$$\frac{1}{x-3} - \frac{1}{x-2} = \sum_{n=0}^{\infty} \frac{-x^n}{3^{n+1}} + \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}}\right) x^n \qquad |x| < 2 \quad . (3)$$

Combining (3) and (1):

$$\frac{1}{(x-2)(x-3)} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}}\right) x^n \qquad |x| < 2 \tag{4}$$

$$\left.\begin{aligned} &= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{9}\right) x + \left(\frac{1}{8} - \frac{1}{27}\right) x^2 \\ &\qquad + \dots + \left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}}\right) x^n + \dots \\ &= \frac{1}{6} + \frac{5}{36}x + \frac{19}{216}x^2 + \dots + \left(\frac{3^{n+1} - 2^{n+1}}{6^{n+1}}\right) x^n + \dots \end{aligned}\right\} \quad |x| < 2 .$$

[7.6.5 cont'd]

(Again notice that it is easier to obtain (4) as we did than to have to compute $\frac{f^{(n)}(0)}{n!}$ where $f(x) = \frac{1}{(x-2)(x-3)}$.)

(b) We have

$$\int_0^1 \frac{dx}{(x-2)(x-3)} = \int_0^1 \left\{ \sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}}\right) x^n \right\} dx \quad . \qquad (5)$$

But the radius of convergence for $\sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}}\right) x^n$ is at least 2. In particular, then, the series converges uniformly on [0,1] . Since the convergence is uniform we may interchange the order of integration and summation in (5) to obtain

$$\int_0^1 \frac{dx}{(x-2)(x-3)} = \sum_{n=0}^{\infty} \left\{ \int_0^1 \left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}}\right) x^n \, dx \right\} \quad . \qquad (6)$$

$$\therefore \int_0^1 \frac{dx}{(x-2)(x-3)} = \sum_{n=0}^{\infty} \left\{ \frac{\left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}}\right) x^{n+1}}{n+1} \Bigg|_0^1 \right\}$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}}\right)}{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{3^{n+1} - 2^{n+1}}{(n+1)\,6^{n+1}}$$

$$= \frac{1}{6} + \frac{5}{72} + \frac{19}{648} + \frac{65}{5184} + \frac{211}{38,880} \cdots + \frac{3^{n+1} - 2^{n+1}}{(n+1)\,6^{n+1}} + \cdots$$

[7.6.5 cont'd]

$$\begin{aligned} \approx\ & 0.1667 \\ & 0.0694 \\ & 0.0293 \\ & 0.0125 \\ & \underline{0.0054} \\ & 0.2833 \quad \approx 0.28 \end{aligned} \tag{7}$$

(c)

$$\begin{aligned} \int_0^1 \frac{dx}{(x-2)(x-3)} &= \int_0^1 \left(\frac{1}{x-3} - \frac{1}{x-2}\right) dx \\ &= \ln|x-3| - \ln|x-2| \Big|_0^1 \\ &= (\ln 2 - \ln 1) - (\ln 3 - \ln 2) \\ &= \ln 2 - 0 - \ln 3 + \ln 2 \\ &= 2 \ln 2 - \ln 3 = \ln 2^2 - \ln 3 \\ &= \ln 4 - \ln 3 \\ &= 1.3863 - 1.0986 \quad \text{(from the tables)} \\ &= 0.2877 \quad . \end{aligned} \tag{8}$$

(7) and (8) are sufficiently compatable to see that the series technique works. We should notice, however, that since all terms in our series are positive, we cannot bound our error by the magnitude of the next term, as we can with alternating series.

7.6.6

Since

$$\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots, \qquad |x| < \infty,$$

it follows that

$$\cos t^2 = 1 - \frac{t^4}{2!} + \frac{t^8}{4!} - \frac{t^{12}}{6!} + \ldots + \frac{(-1)^n t^{4n}}{(2n)!} + \ldots \qquad |t| < \infty$$

$$\therefore \int_0^x \cos t^2\,dt = \int_0^x \left(1 - \frac{t^4}{2!} + \frac{t^8}{4!} - \frac{t^{12}}{6!} + \ldots + \frac{(-1)^n t^{4n}}{(2n)!} + \ldots\right) dt$$

and since the series converges uniformly,

$$\int_0^x \cos t^2\,dt = \left(t - \frac{t^5}{5(2!)} + \frac{t^9}{9(4!)} - \frac{t^{13}}{13(6!)} + \ldots + \frac{(-1)^n t^{4n+1}}{(4n+1)(2n)!} + \ldots\right)\Bigg|_0^x$$

$$= x - \frac{x^5}{10} + \frac{x^9}{216} - \frac{x^{13}}{9360} + \ldots + \frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!} \quad . \qquad (1)$$

(Aside: $\int_0^x \cos t^2\,dt$ is called the Fresnel cosine integral and occurs prominently in the theory of diffraction. The point is that there is no elementary function (or finite combination of such functions), $f(t)$ for which $f'(t) = \cos t^2$. Equation (1) holds for all real x and thus gives us a satisfactory way of computing the integral.)

7.6.7

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then $f(-x) = \sum_{n=0}^{\infty} a_n (-x)^n = \sum_{n=0}^{\infty} (-1)^n a_n x^n$.

$\therefore f(x) = f(-x) \longleftrightarrow$

$$\sum_{n=0}^{\infty} a_n x^n \equiv \sum_{n=0}^{\infty} (-1)^n a_n x^n \quad . \tag{1}$$

$\therefore$By the result of Exercise 7.6.3 (a), equation (1) implies that:

$$a_n = (-1)^n a_n \quad . \tag{2}$$

Equation (2) is a truism when n is even since then all it says is $a_n = (-1)^n a_n = a_n$.

When n is odd, however, we have

$$a_n = -a_n \text{ or } 2a_n = 0$$

$\therefore a_n = 0$ when n is odd.

In other words

$$f(x)=a_0 + a_2x^2 + a_4x^4 + a_6x^6 \ldots.$$

Thus, if f(x) is an even function, the power series which represents it contains only even powers of x. This is another reason for the choice of the name "even" function.

(A similar type of result holds for odd functions of courses; namely, $\sum a_n x^n$ represents an odd function, i.e., $f(x) = -f(-x)$, if and only if $a_n = 0$ when n is even.)

7.6.8 (L)

If at first glance this exercise seems as if it belongs in Unit 3 of this Block, it is because in a certain sense it does belong there. At least, that much is true about part (a).

The main ideas are these:

(1) The product of two convergent series is convergent if at least one of the series is absolutely convergent.

(2) The product is uniformly convergent if each of the factors is.

Now, to compute the product, we don't need (2). If, however, we wish to integrate the product term by term (as we do in part (b)) then (2) is important. That is, unless the series is uniformly convergent, we are on dangerous ground if we reverse the order of summation and integration.

The point is that part (a) is a pre-computation for part (b). In terms of a real-life approach the idea is that there is no elementary function f for which $f'(x) = \frac{\sin x}{1 - x}$. Thus, to compute $\int_0^{0.01} \frac{\sin x}{1 - x}\,dx$ we observe that $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)}$, $|x| < \infty$; while $\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$

$$\therefore \frac{\sin x}{1 - x} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)!}\right)\left(\sum_{n=0}^{\infty} x^n\right)$$

$$\therefore \int_0^{0.01} \frac{\sin x}{1 - x}\,dx = \int_0^{0.01} \left\{\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)!}\right)\left(\sum_{n=0}^{\infty} x^n\right)\right\} dx \quad . \qquad (1)$$

[7.6.8 (L) cont'd]

Part (a) allows us to simplify the integrand on the right hand side of (1) into a single series, whereupon, by uniform convergence we may integrate term-by-term.

In more detail:

(a) We have:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$ *

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*If this recipe still seems "unnatural," notice that this is indeed the way we "pick off" coefficients in the usual product of two polynomials. For example, if we have:

$$(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4)(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4)$$

and we wish to compute the coefficients of x^6 in the product we argue something like this:

Since we add exponents when we multiply, an x^6 term occurs if and only if we multiply (1) an x^0 by an x^6, or (2) an x^1 by an x^5, or (3) an x^2 by an x^4, or (4) an x^3 by an x^3 . (That is, if we want the sum of two whole numbers to equal 6, the pairs must be 0 and 6, 1 and 5, 2 and 4, or 3 and 3). In our problem, there happen to be no x^5 or x^6 terms (i.e., their coefficients are 0), but quick inspection shows us that the required terms are $(a_2x^2)(b_4x^4)$, $(a_4x^4)(b_2x^2)$, $(a_3x^3)(b_2x^3)$ and their sum yields $(a_2b_4 + a_3b_3 + a_4b_2)x^6$.

Generalized, then, the x^n term comes from

$[(a_0)(b_nx^n) + (b_0)(a_nx^n)] + [(a_1x)(b_{n-1}\,x^{n-1}) + (a_{n-1}\,x^{n-1})(b_1x)] + \ldots$ etc.

[7.6.8 (L) cont'd]

and in our case:

$$a_{2n} = 0,\ a_{2n+1} = \frac{(-1)^n}{(2n+1)!},\ b_n = 1$$

$$\therefore c_n = (a_0b_n + a_1b_{n-1} + \dots + a_nb_0) \rightarrow$$

$$c_n = a_0 + a_1 + \dots + a_n \qquad (\text{since } b_0 = b_1 = b_2 = \dots = 1)$$

$$= a_1 + a_3 + a_5 + \dots \qquad (\text{since } a_0 = a_2 = a_4 = \dots = 0)$$

$$c_{2n} = a_1 + a_3 + a_5 + \dots + a_{2n-1}$$

$$c_{2n+1} = a_1 + a_3 + a_5 + \dots + a_{2n+1}$$

$$\therefore c_1 = a_1 = 1$$

$$c_2 = a_1 = 1$$

$$c_3 = a_1 + a_3 = 1 - \frac{1}{3!} = \frac{5}{6}$$

$$c_4 = a_1 + a_3 = \frac{5}{6}$$

$$c_5 = a_1 + a_3 + a_5 = 1 - \frac{1}{3!} + \frac{1}{5!} = 1 - \frac{1}{6} + \frac{1}{120} = \frac{101}{120}\ ^*$$

$$\therefore \sin x \left(\frac{1}{1-x}\right) = x + x^2 + \frac{5}{6}x^3 + \frac{5}{6}x^4 + \frac{101}{120}x^5 + \dots,\ |x| < 1 \qquad (2)$$

---

*Notice that $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \dots = \sin 1 \therefore \lim_{n\to\infty} c_n \gtrneqq 0$.
This, among other things, should remind us why the restriction $|x| < 1$ must apply. Namely, if $x > 1$, $\sum_{n=0}^{\infty} c_nx^n$ clearly diverges.

[7.6.8 (L) cont'd]

(b) Now using (2), we obtain:

$$\int_0^{0.01} \frac{\sin x}{1 - x}\,dx = \int_0^{0.01} \left(x + x^2 + \frac{5}{6}x^3 + \frac{5}{6}x^4 + \frac{101}{120}x^5 + \ldots\right) dx$$

$$= \int_0^{0.01} x\,dx + \int_0^{0.01} x^2\,dx + \frac{5}{6}\int_0^{0.01} x^3\,dx + \frac{5}{6}\int_0^{0.01} x^4\,dx$$

$$+ \ldots \text{ (by uniform convergence)}$$

$$= \frac{1}{2}x^2\Big|_0^{0.01} + \frac{1}{3}x^3\Big|_0^{0.01} + \frac{5}{24}x^4\Big|_0^{0.01} + \frac{1}{6}x^5\Big|_0^{0.01} + \ldots$$

$$= \frac{1}{2}(0.0001) + \frac{1}{3}(0.000001) + \frac{5}{24}(0.00000001) + \frac{1}{6}(0.0000000001)$$

$$+ \ldots$$

$$= 0.00005$$
$$+ 0.00000033\ldots$$
$$+ 0.000000002\ldots$$
$$+ 0.000000000016\ldots$$
$$= \underline{0.0000503}\quad.$$

QUIZ

1. (a) $\lim_{n\to\infty} \frac{5n+7}{8n-3} = \frac{5}{8} \neq 0$. Therefore the series diverges because its nth term does not approach 0.

(b) By the integral test $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\leftrightarrow$ $p > 1$. In this case, $p = \frac{1}{2}$. Hence, the series diverges.

(c) Here, we have an alternating series in which the terms monotonically decrease in magnitude and approach 0 as a limit. This is enough to guarantee that the series converges. (From (b), however, we see that it does not converge absolutely; hence, to be precise, the series is conditionally convergent.)

(d) Using the ratio test with $a_n = \frac{10^{6n}}{n!}$, we have $a_{n+1} = \frac{10^{6(n+1)}}{(n+1)!}$. Therefore, $\frac{a_{n+1}}{a_n} = \frac{10^{6(n+1)}}{(n+1)!}\left(\frac{n!}{10^{6n}}\right) = \frac{10^{6n+6}n!}{(n+1)!10^{6n}} = \frac{10^6}{n+1}$

$$\therefore\ \rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{10^6}{n+1} = \frac{10^6}{\infty} = 0$$

$$\therefore\ \rho < 1$$

$\therefore$ The series converges by the ratio test.

(e) By the integral test we compare the series with

$$\int_1^{\infty} \frac{2x}{1+x^2}\,dx. \quad \text{But} \int_1^{\infty} \frac{2x\,dx}{1+x^2} = \ell n(1+x^2)\Big|_1^{\infty} = \ell n\infty - \ell n 2 = \infty$$

$\therefore$ The series diverges by the integral test.

2. (a) We use the ratio test with $a_n = \frac{|x|^n}{n}$

$$\therefore a_{n+1} = \frac{|x|^{n+1}}{n+1}$$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{|x|^{n+1}}{n+1}\left(\frac{n}{|x|^n}\right) = \left(\frac{n}{n+1}\right)|x|$$

$$\therefore \rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)|x| = |x|$$

$$\therefore \rho < 1 \leftrightarrow |x| < 1$$

$\therefore \sum_{n=1}^{\infty} \frac{x^n}{n}$ converges absolutely if $|x| < 1$ and diverges if $|x| > 1$.

Now, we need only test the case $|x| = 1$ - i.e., $x = 1$ and $x = -1$. For $x = 1$, we get $\sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges $\left(\sum \frac{1}{n^p} \text{ with } p = 1\right)$. But if $x = -1$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges since it meets the requirements for an alternating convergent series.

$$\therefore \sum_{n=1}^{\infty} \frac{x^n}{n} \text{ converges} \leftrightarrow -1 \leqslant x < 1$$

(b) Let $a_n = \frac{n!\,|x|^n}{n^n}$. Then $a_{n+1} = \frac{(n+1)!\,|x|^{n+1}}{(n+1)^{n+1}}$

[2. cont'd]

$$\therefore \frac{a_{n+1}}{a_n} = \left[\frac{(n+1)!|x|^{n+1}}{(n+1)^{n+1}}\right] \frac{n^n}{n!|x|^n} = \frac{(n+1)|x|n^n}{(n+1)^{n+1}}$$

$$= \left(\frac{n}{n+1}\right)^n |x| = \left[\frac{1}{\left(1 + \frac{1}{n}\right)^n}\right]|x|$$

$$\therefore \rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{e}|x|$$

$$\therefore \rho < 1 \leftrightarrow \frac{1}{e}|x| < 1 \leftrightarrow |x| < e$$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$ converges absolutely if $|x| < e$ and diverges if $|x| > e$

$\therefore \sum \frac{n!x^n}{n^n}$ converges absolutely if $-e < x < e$

(c) $a_n = \dfrac{n^2|x|^n}{3^n}$, $a_{n+1} = \dfrac{(n+1)^2|x|^{n+1}}{3^{n+1}}$

$$\therefore \frac{a_{n+1}}{a_n} = \left[\frac{(n+1)^2|x|^{n+1}}{3^{n+1}}\right]\left[\frac{3^n}{n^2|x|^n}\right] = \left(\frac{n+1}{n}\right)^2 \frac{1}{3}|x|$$

$$\therefore \rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{3}|x|$$

$$\therefore \rho < 1 \leftrightarrow \frac{1}{3}|x| < 1 \leftrightarrow |x| < 3$$

[2. cont'd]

$\therefore \sum_{n=1}^{\infty} \frac{n^2x^n}{3^n}$ converges absolutely if $|x| < 3$ and diverges if $|x| > 3$

We next test $x = 3$ and $x = -3$

When $x = 3$: $\sum_{n=1}^{\infty} \frac{n^2x^n}{3^n} = \sum_{n=1}^{\infty} \frac{n^2 3^n}{3^n} = \sum_{n=1}^{\infty} n^2$ which diverges to ∞.

When $x = -3$: $\sum_{n=1}^{\infty} \frac{n^2x^n}{3^n} = \sum_{n=1}^{\infty} \frac{n^2(-3)^n}{3^n} = \sum_{n=1}^{\infty} (-1)^n n^2$ which diverges to $\pm\infty$.

$$\therefore \sum_{n=1}^{\infty} \frac{n^2x^n}{3^n} \text{ converges } \leftrightarrow -3 < x < 3$$

3. (a) We have, since $|\sin u| \leqslant 1$ for all u,

$$\left|\frac{\sin nx}{n^2}\right| \leqslant \frac{1}{n^2} \qquad (1)$$

But $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

$\therefore$ By applying the M-test to (1), we have

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2} \text{ converges uniformly}$$

[3. cont'd]

(b) (This is a generalization of part (a). In particular, part (a) was (b) with $a_n = \frac{1}{n^2}$.)

$$|a_n \sin nx| = |a_n|\,|\sin nx| \leqslant |a_n| \qquad (2)$$

But we are given that $\sum_{n=1}^{\infty} |a_n|$ converges. Hence, we may apply the M-test to (2), and obtain the desired result.

4. (a) Since we have a convergent alternating series, the error cannot exceed the magnitude of the next term. Thus we are interested in determining n such that $\frac{1}{\sqrt{n}} < 0.01$.

Now,

$$\frac{1}{\sqrt{n}} < 0.01 \leftrightarrow \sqrt{n} > \frac{1}{0.01} = 100 \leftrightarrow n > (100)^2 = 10{,}000$$

$\therefore \sum_{n=1}^{9,999} \frac{(-1)^n}{\sqrt{n}}$ gives the sum of the series to the desired accuracy. Notice that it takes many terms since the terms are not getting small very rapidly.

(b) The theory is the same as in (a), but now we want $\frac{1}{n^2} < 0.01$. But

$$\frac{1}{n^2} < 0.01 \leftrightarrow n^2 > 100 \leftrightarrow n > 10$$

[4. cont'd]

$\therefore \quad \sum_{n=1}^{9} \frac{(-1)^n}{n^2}$ gives us the sum to the desired degree of accuracy. Notice now that not many terms are needed, since the terms are decreasing quite rapidly.

In other words, to within 0.01, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is given by

$$-1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \frac{1}{36} - \frac{1}{49} + \frac{1}{64} - \frac{1}{81}$$

5. $\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \ldots + \frac{(-1)^n u^{2n+1}}{(2n+1)!} + \ldots \quad |u| < \infty$

$$\therefore \sin t^2 = t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \frac{t^{14}}{7!} + \ldots + \frac{(-1)^n t^{4n+2}}{(2n+1)!} + \ldots \quad |t| < \infty$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+2}}{(2n+1)!}, \quad |t| < \infty$$

$$\therefore \int_0^{\frac{1}{2}} \sin t^2 \, dt = \int_0^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+2}}{(2n+1)!} \, dt$$

$$= \sum_{n=0}^{\infty} \left[\int_0^{\frac{1}{2}} \frac{(-1)^n t^{4n+2}}{(2n+1)!} dt \right] \quad \text{(by uniform convergence)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+3}}{(4n+3)(2n+1)!} \Bigg|_0^{\frac{1}{2}}$$

[5. cont'd]

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n+3}(4n+3)(2n+1)!}$$

$$= \frac{1}{8(3)} - \frac{1}{2^7(7)(3!)} + \frac{1}{2^{11}(11)5!}$$

$$= \frac{1}{24} - \frac{1}{5376} + \frac{1}{2,703,360}$$

$$\frac{1}{24} - \frac{1}{5376} \leq \int_0^{\frac{1}{2}} \sin(t^2)dt \leq \frac{1}{24} - \frac{1}{5376} + \frac{1}{2,703,360}$$

$$\frac{223}{5376} \leq \int_0^{\frac{1}{2}} \sin(t^2)dt \leq \frac{223}{5376} + \frac{1}{2,703,360}$$

$$\therefore \int_0^{\frac{1}{2}} \sin(t^2)dt = 0.04148$$

6. $\frac{1}{e^x(1-x)} = e^{-x}\left(\frac{1}{1-x}\right)$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \quad |x| < \infty$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad |x| < 1$$

[6. cont'd]

$$\therefore\ e^{-x}\,\frac{1}{1-x}\sum_{n=0}^{\infty} c_n x^n \text{ where } c_n = (a_0 b_n + \dots + a_n b_0), \text{ and}$$

$$a_k = \frac{(-1)^k}{k!},\ b_k = 1$$

$$\therefore c_n = (a_0 + \dots + a_n) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \dots$$

i.e.

$$c_0 = a_0 b_0 = 1$$

$$c_1 = a_0 b_1 + a_1 b_0 = a_0 + a_1 = 1 - 1 = 0$$

$$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0 = a_0 + a_1 + a_2 = \frac{1}{2!} = \frac{1}{2}$$

$$c_3 = a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 = a_0 + a_1 + a_2 + a_3 = \frac{1}{2!} - \frac{1}{3!} = -\frac{1}{3}$$

$$c_4 = \dots = c_3 + a_4 = -\frac{1}{3} + \frac{1}{24} = -\frac{7}{24}$$

$$\therefore\ e^{-x}\left(\frac{1}{1-x}\right) = \frac{1}{e^x(1-x)} = 1 + \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{7}{24}x^4 + \dots |x| < 1$$

$$\therefore\ \int_0^{\frac{1}{10}} \frac{dx}{e^x(1-x)} = x + \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{7}{120}x^5 + \dots\Big|_0^{\frac{1}{10}}$$

$$= \frac{1}{10} + \frac{1}{6000} - \frac{1}{120{,}000} - \frac{7}{12{,}000{,}000}$$

$$\begin{array}{rl} = & 0.10000 \\ + & 0.000167 \\ - & \underline{0.000008} \\ & 0.100159 \approx 0.1002 \end{array}$$