
CALCULUS REVISITED
PART 2
A Self-Study Course

STUDY GUIDE
Block 3
Partial Derivatives

Herbert I. Gross
Senior Lecturer

Center for Advanced Engineering Study
Massachusetts Institute of
Technology

Copyright © 1972 by
Massachusetts Institute of Technology
Cambridge, Massachusetts

All rights reserved. No part of this book may be reproduced in any form or by any means without permission in writing from the Center for Advanced Engineering Study, M.I.T.

CONTENTS

Study Guide

Block 3: Partial Derivatives

Pretest	3.ii
Unit 1: Functions of More Than One Variable	3.1.1
Unit 2: An Introduction to Partial Derivatives	3.2.1
Unit 3: Differentiability and the Gradient	3.3.1
Unit 4: The Directional Derivative in n-Dimensional Vector Spaces (Optional)	3.4.1
Unit 5: The Chain Rule, Part 1	3.5.1
Unit 6: The Chain Rule, Part 2	3.6.1
Unit 7: More on Derivatives of Integrals	3.7.1
Unit 8: The Total Differential	3.8.1
Quiz	3.Q.1

Solutions

Block 3: Partial Derivatives

Pretest	S.3.ii
Unit 1: Functions of More Than One Variable	S.3.1.1
Unit 2: An Introduction to Partial Derivatives	S.3.2.1
Unit 3: Differentiability and the Gradient	S.3.3.1
Unit 4: The Directional Derivative in n-Dimensional Vector Spaces (Optional)	S.3.4.1
Unit 5: The Chain Rule, Part 1	S.3.5.1
Unit 6: The Chain Rule, Part 2	S.3.6.1
Unit 7: More on Derivatives of Integrals	S.3.7.1
Unit 8: The Total Differential	S.3.8.1
Quiz	S.3.Q.1

Study Guide

BLOCK 3:
PARTIAL DERIVATIVES

Pretest

1. Let $w = f(x,y) = \frac{2xy}{x^2 + y^2}$, $(x,y) \neq (0,0)$. Show that
 $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ depends on the path by which (x,y) approaches $(0,0)$.
2. Find the equation of the plane which is tangent to the surface $x^4 + y^6z + xyz^5 = 3$ at $(1,1,1)$.
3. Suppose w depends on r but not on θ , say $w = h(r)$, and that h is a twice-differentiable function of r . Determine $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$, expressed in terms of r .
4. Find the equation of the curve C if C passes through the origin and has its slope at each point (x,y) given by

$$\frac{dy}{dx} = \frac{-(2xe^y + e^x)}{(x^2 + 1)e^y}.$$

5. Given that $g(y) = \int_0^1 \frac{x^y - x^b}{\ln x} dx$ where $y > b > -1$, determine $g'(y)$.

Unit 1: Functions of More Than One Variable


1. Lecture 3.010

n-dimensional vector spaces

input vector \xrightarrow{f} machine $\xrightarrow{\text{output}}$ scalar

Example:
 $\vec{v} = x\vec{i} + y\vec{j}$
 $f(\vec{v}) = \pi x^2 y$
 $(3\vec{i} + 4\vec{j}) \xrightarrow{f} \pi(3^2)(4) = 36\pi$
 $(4\vec{i} + 3\vec{j}) \xrightarrow{f} \pi(4^2)(3) = 48\pi$
 $f(3,4) = 36\pi$
 $f(4,3) = 48\pi$

Now consider



Volume = $\pi x^2 y$
 $= f(x,y)$
 $f(3,4) = 36\pi$
 $\therefore (x,y)$ need not be viewed as an arrow but as an ordered pair (2-tuple)

n-tuples

$T = f(x,y,z,t)$

input 4-tuple \xrightarrow{f} machine $\xrightarrow{\text{output}}$ scalar

(x_1, x_2, x_3, x_4) is a 4-tuple
 (x_1, \dots, x_n) is an n-tuple
 we let \underline{x} denote (x_1, \dots, x_n) , rather than \vec{x} since "arrows" may be inappropriate

a.

Structure of n-dim vector spaces (n-space)

Let $S_n = \{x_1, \dots, x_n\}$ and $\underline{a} = (a_1, \dots, a_n), \underline{b} = (b_1, \dots, b_n)$

Based on $n=1, 2, \text{ or } 3$ we let

- $\underline{a} = \underline{b}$ mean $a_1 = b_1, \dots, a_n = b_n$
- $\underline{a} + \underline{b} = (a_1 + b_1, \dots, a_n + b_n)$
- $c\underline{a} = (ca_1, \dots, ca_n)$ where c is any number

"Non-trivial" Example of a 4-space

Let (a_0, a_1, a_2, a_3) denote $a_0 + a_1 x + a_2 x^2 + a_3 x^3$

n-tuples are not automatically n-spaces

Let (a,b) mean $a+b$

Then $(4,5) = (6,3)$ but $4 \neq 6, 5 \neq 3$

Limits

Let $f(x_1, x_2, x_3, x_4) = x_1^3 + x_2 + x_3^2 + 2x_4$

$\therefore \lim_{(x_1, x_2, x_3, x_4) \rightarrow (1, 3, 1, 2)} f(x_1, x_2, x_3, x_4) = 1^3 + 3 + 1^2 + 2(2) = 9$

$\lim_{x \rightarrow a} f(x) = L$ means given $\epsilon > 0$, can find $\delta > 0$ such that $0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon$

b.

What is $|\underline{x} - \underline{a}|$, if $\underline{x} = (x_1, \dots, x_n), \underline{a} = (a_1, \dots, a_n)$?

1-dim: $|\underline{x} - \underline{a}| = \sqrt{(x_1 - a_1)^2}$

2-dim: $|\underline{x} - \underline{a}| = \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2}$

3-dim: $|\underline{x} - \underline{a}| = \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2}$

Definition

$\|\underline{x} - \underline{a}\| = \sqrt{(x_1 - a_1)^2 + \dots + (x_n - a_n)^2}$

$\|\underline{x} - \underline{a}\| = 0 \iff x_1 = a_1, \dots, x_n = a_n$

$\therefore \underline{x}$ near \underline{a} still means $\|\underline{x} - \underline{a}\|$ is small

$\lim_{(x_1, \dots, x_n) \rightarrow (a_1, \dots, a_n)} f(x_1, \dots, x_n) = L$ means given $\epsilon > 0$, can find $\delta > 0$ such that $0 < \sqrt{(x_1 - a_1)^2 + \dots + (x_n - a_n)^2} < \delta \rightarrow |f(x_1, \dots, x_n) - L| < \epsilon$

c.

2. Read Supplementary Notes, Chapter 4.
3. Read Thomas, Section 15.1.
4. (Optional) Read Thomas, Sections 12.10 and 12.11. (These sections will help you feel more at home with equations of surfaces. The idea is that just as the graphs of functions of a single variable are curves in the xy -plane, the graphs of functions of two real variables are surfaces in space. Except for any peace-of-mind that you get in feeling at home with the various equations, it should be noted that we can survive the remainder of this course without recourse to accurate graphs just as was the case in functions of a single real variable.)

5. Exercises:

3.1.1(L)

Define $\|\underline{x}\|$ as the Minkowski metric. That is, if $\underline{x} = (x_1, \dots, x_n)$, then $\|\underline{x}\| = \max\{|x_1|, \dots, |x_n|\}$.

a. Show that

1. $\|\underline{x}\| \geq 0$ for all \underline{x} and $\|\underline{x}\| = 0$ if and only if $\underline{x} = \underline{0}$.

2. $\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$

3. $\|\underline{ax}\| = |a| \|\underline{x}\|$

b. Compute $\underline{x} \cdot \underline{y}$, $\|\underline{x}\|$, and $\|\underline{y}\|$ (where we are still using the Minkowski metric) if $\underline{x} = (2, 4, 1)$ and $\underline{y} = (4, 4, 5)$. From this conclude that it need not be true that $|\underline{x} \cdot \underline{y}| \leq \|\underline{x}\| \|\underline{y}\|$.

3.1.2

Mimic the proof of the corresponding 1-dimensional case to prove that if \underline{x} and \underline{a} belong to E^n and $\lim_{\underline{x} \rightarrow \underline{a}} f(\underline{x}) = L_1$ while $\lim_{\underline{x} \rightarrow \underline{a}} g(\underline{x}) = L_2$, then

$$\lim_{\underline{x} \rightarrow \underline{a}} [f(\underline{x}) + g(\underline{x})] = L_1 + L_2$$

3.1.3(L)

- a. Using the Minkowski metric, suppose $\epsilon > 0$ is given; find δ such that for this choice of δ

$$0 < \|(x,y) - (2,3)\| < \delta \rightarrow |x^2 + y^3 - 31| < \epsilon .$$

- b. Interpret the answer in (a) geometrically and explain why the same value of δ as in (a) would have sufficed had we used the Euclidean metric rather than the Minkowski metric.

3.1.4(L)

Let $\underline{x} = (x_1, x_2, x_3, x_4)$ and let $\underline{1} = (1, 1, 1, 1)$. Define f by

$$f(\underline{x}) = x_1^2 + 2x_2 + x_3^3 + x_4^2. \text{ Prove that } f \text{ is continuous at } \underline{x} = \underline{1}.$$

3.1.5

Let f , \underline{x} and $\underline{1}$ be as in Exercise 3.1.4. For a given $\epsilon > 0$, find δ such that

$$0 < \|\underline{x} - \underline{1}\| < \delta \rightarrow |f(\underline{x}) - 5| < \epsilon .$$

3.1.6(L)

Let f be defined by

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2} .$$

- a. Is f continuous at $(0,0)$?
b. Compute both $\lim_{y \rightarrow 0} [\lim_{x \rightarrow 0} f(x,y)]$ and $\lim_{x \rightarrow 0} [\lim_{y \rightarrow 0} f(x,y)]$.
c. Investigate the behaviour of

$$\lim_{(x,y) \rightarrow (0,0)} f(x,y)$$

in more detail by introducing polar coordinates.

3.1.7

Let f be defined by

$$\frac{2xy}{x^2 + y^2}$$

- Show that $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ depends on the path by which (x,y) approaches $(0,0)$.
- Compute $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ if (x,y) approaches $(0,0)$ along the ray $\theta = \frac{\pi}{4}$.
- Show that if (x,y) approaches $(0,0)$ either along the x -axis or the y -axis then $\lim f(x,y) = 0$.

3.1.8(L)

Define g by

$$g(x,y) = \begin{cases} \frac{2xy}{x^2 + y^2} & , \text{ if } (x,y) \neq (0,0) \\ 0 & , \text{ if } (x,y) = (0,0) \end{cases}$$

- Show that g is not continuous at $(0,0)$.
- Show that $\lim_{(x,y) \rightarrow (0,0)} g(x,y) = g(0,0)$ if (x,y) is allowed to approach $(0,0)$ along either axis.

3.1.9(L)

Let the function $f: E^2 \rightarrow E$ be continuous. Prove that f cannot be 1-1.

Comment

The following two exercises are optional. They may be omitted without loss of continuity to our present discussion. Their main purpose is to supply the interested reader with a few clues as to how analytic proofs are carried out in n -dimensional vector spaces (with n greater than three) using the ordinary properties of real number arithmetic.

3.1.10

Let \underline{a} and \underline{b} belong to E^4 . Prove that our definition of $\underline{a} = \underline{b}$ is an equivalence relation because of the fact that "ordinary" equality is an equivalence relation on the set of real numbers.

3.1.11

Let \underline{a} , \underline{b} and \underline{c} be elements of E^4 . With the dot product as defined in our supplementary notes, prove that

$$\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$$

MIT OpenCourseWare
<http://ocw.mit.edu>

Resource: Calculus Revisited: Multivariable Calculus
Prof. Herbert Gross

The following may not correspond to a particular course on MIT OpenCourseWare, but has been provided by the author as an individual learning resource.

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.