CALCULUS REVISITED
PART 2
A Self-Study Course

STUDY GUIDE
Block 3
Partial Derivatives

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CONTENTS

Study Guide

Block 3: Partial Derivatives

Solutions

Block 3: Partial Derivatives

BLOCK 3:
PARTIAL DERIVATIVES

Pretest

1. Let $w = f(x,y) = \dfrac{2xy}{x^2 + y^2}$, $(x,y) \neq (0,0)$. Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ depends on the path by which (x,y) approaches $(0,0)$.

2. Find the equation of the plane which is tangent to the surface $x^4 + y^6 z + xyz^5 = 3$ at $(1,1,1)$.

3. Suppose w depends on r but not on θ, say $w = h(r)$, and that h is a twice-differentiable function of r. Determine $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}$, expressed in terms of r.

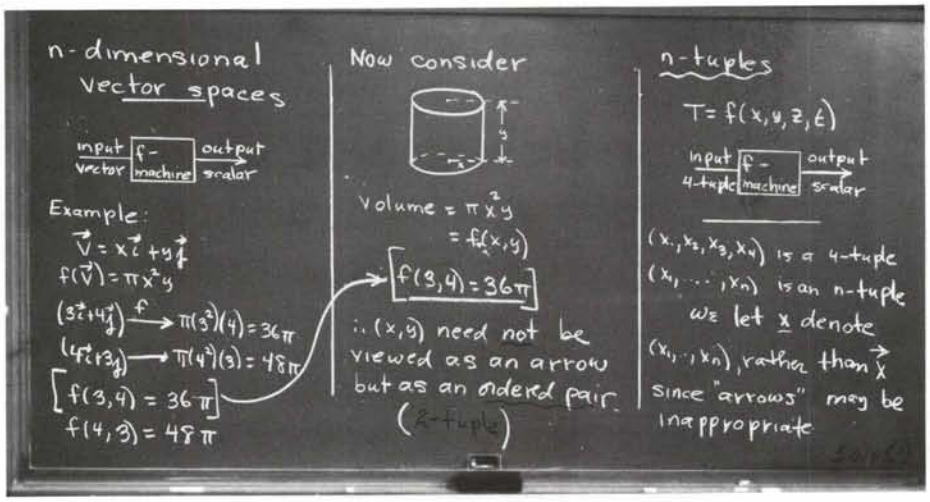
4. Find the equation of the curve C if C passes through the origin and has its slope at each point (x,y) given by

$$\frac{dy}{dx} = \frac{-(2xe^y + e^x)}{(x^2 + 1)e^y}.$$

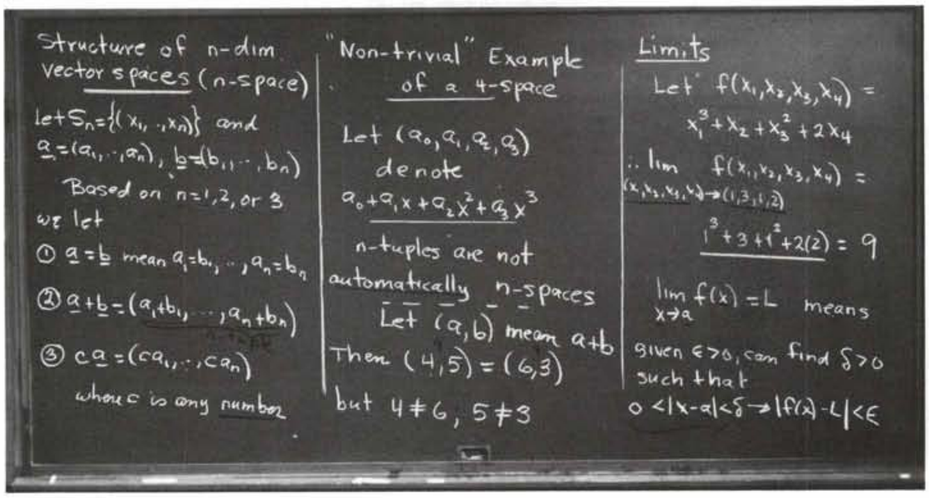
5. Given that $g(y) = \displaystyle\int_0^1 \frac{x^y - x^b}{\ln x}\,dx$ where $y > b > -1$, determine $g'(y)$.

Unit 1: Functions of More Than One Variable

1. Lecture 3.010



a.



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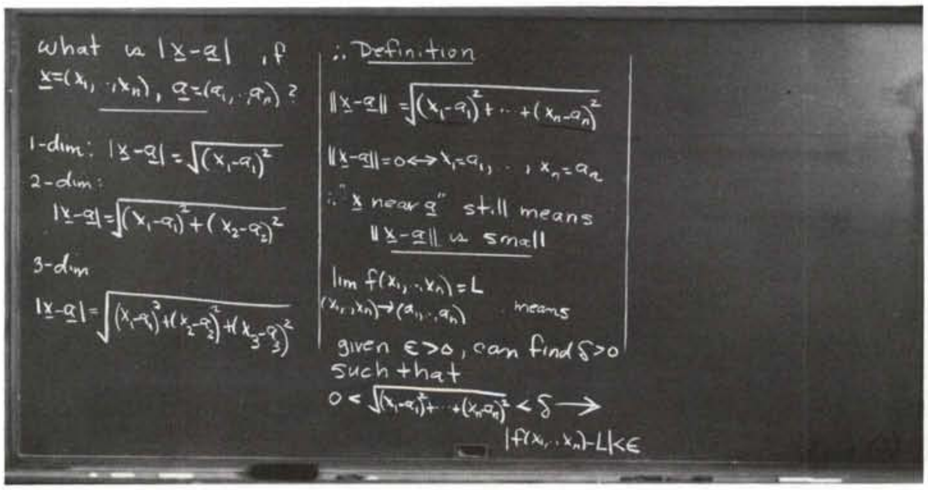

c.

2. Read Supplementary Notes, Chapter 4.

3. Read Thomas, Section 15.1.

4. (Optional) Read Thomas, Sections 12.10 and 12.11. (These sections will help you feel more at home with equations of surfaces. The idea is that just as the graphs of functions of a single variable are curves in the xy-plane, the graphs of functions of two real variables are surfaces in space. Except for any peace-of-mind that you get in feeling at home with the various equations, it should be noted that we can survive the remainder of this course without recourse to accurate graphs just as was the case in functions of a single real variable.)

5. Exercises:

3.1.1(L)

Define $\|\underline{x}\|$ as the Minkowski metric. That is, if $\underline{x} = (x_1,\ldots,x_n)$, then $\|\underline{x}\| = \max\{|x_1|,\ldots,|x_n|\}$.

a. Show that

1. $\|\underline{x}\| \geqslant 0$ for all $\underline{x}$ and $\|\underline{x}\| = 0$ if and only if $\underline{x} = \underline{0}$.

2. $\|\underline{x} + \underline{y}\| \leqslant \|\underline{x}\| + \|\underline{y}\|$

3. $\|a\underline{x}\| = |a|\,\|\underline{x}\|$

b. Compute $\underline{x}\cdot\underline{y}$, $\|\underline{x}\|$, and $\|\underline{y}\|$ (where we are still using the Minkowski metric) if $\underline{x} = (2,4,1)$ and $\underline{y} = (4,4,5)$. From this conclude that it need not be true that $|\underline{x}\cdot\underline{y}| \leqslant \|\underline{x}\|\ \|\underline{y}\|$.

3.1.2

Mimic the proof of the corresponding 1-dimensional case to prove that if $\underline{x}$ and $\underline{a}$ belong to E^n and $\lim_{\underline{x}\to\underline{a}} f(\underline{x}) = L_1$ while $\lim_{\underline{x}\to\underline{a}} g(\underline{x}) = L_2$, then

$$\lim_{\underline{x}\to\underline{a}} [f(\underline{x}) + g(\underline{x})] = L_1 + L_2$$

3.1.3(L)

a. Using the Minkowski metric, suppose $\varepsilon>0$ is given; find δ such that for this choice of δ

$$0 < \|(x,y) - (2,3)\| < \delta \rightarrow |x^2 + y^3 - 31| < \varepsilon .$$

b. Interpret the answer in (a) geometrically and explain why the same value of δ as in (a) would have sufficed had we used the Euclidean metric rather than the Minkowski metric.

3.1.4(L)

Let $\underline{x} = (x_1,x_2,x_3,x_4)$ and let $\underline{1} = (1,1,1,1)$. Define f by

$f(\underline{x}) = x_1{}^2 + 2x_2 + x_3{}^3 + x_4{}^2$. Prove that f is continuous at $\underline{x} = \underline{1}$.

3.1.5

Let f, $\underline{x}$ and $\underline{1}$ be as in Exercise 3.1.4. For a given $\varepsilon>0$, find δ such that

$$0 < \|\underline{x} - \underline{1}\| < \delta \rightarrow |f(\underline{x}) - 5| < \varepsilon \quad .$$

3.1.6(L)

Let f be defined by

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2} \quad .$$

a. Is f continuous at (0,0)?

b. Compute both $\lim_{y\to 0} [\lim_{x\to 0} f(x,y)]$ and $\lim_{x\to 0} [\lim_{y\to 0} f(x,y)]$.

c. Investigate the behaviour of

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

in more detail by introducing polar coordinates.

3.1.7

Let f be defined by

$$\frac{2xy}{x^2 + y^2}$$

a. Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ depends on the path by which (x,y) approaches (0,0).

b. Compute $\lim_{(x,y)\to(0,0)} f(x,y)$ if (x,y) approaches (0,0) along the ray $\theta = \frac{\pi}{4}$.

c. Show that if (x,y) approaches (0,0) either along the x-axis or the y-axis then lim f(x,y) = 0.

3.1.8(L)

Define g by

$$g(x,y) = \begin{cases} \dfrac{2xy}{x^2 + y^2} & \text{, if } (x,y) \neq (0,0) \\ 0 & \text{, if } (x,y) = (0,0) \end{cases}$$

a. Show that g is not continuous at (0,0).

b. Show that $\lim_{(x,y)\to(0,0)} g(x,y) = g(0,0)$ if (x,y) is allowed to approach (0,0) along either axis.

3.1.9(L)

Let the function $f: E^2 \to E$ be continuous. Prove that f cannot be 1-1.

Comment

The following two exercises are optional. They may be omitted without loss of continuity to our present discussion. Their main purpose is to supply the interested reader with a few clues as to how analytic proofs are carried out in n-dimensional vector spaces (with n greater than three) using the ordinary properties of real number arithmetic.

3.1.10

Let $\underline{a}$ and $\underline{b}$ belong to E^4. Prove that our definition of $\underline{a} = \underline{b}$ is an equivalence relation because of the fact that "ordinary" equality is an equivalence relation on the set of real numbers.

3.1.11

Let $\underline{a}$, $\underline{b}$ and $\underline{c}$ be elements of E^4. With the dot product as defined in our supplementary notes, prove that

$$\underline{a} \cdot (\underline{b}+\underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$$