Unit 4: Matrices as Linear Functions

4.4.1

a. Since

$u = 6x + 5y$

and

$v = x + y$

our matrix of coefficients A is given by $\begin{bmatrix} 6 & 5 \\ 1 & 1 \end{bmatrix}$.

Therefore,

$\det A = 6 - 5 = 1 \neq 0$

therefore,

A^{-1} exists

therefore,

$\underline{f}^{-1}$ exists.

b. If we compute A^{-1} directly, we have:

$$\left[\begin{array}{cc|cc} 6 & 5 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 6 & 5 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -6 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & -5 \\ 0 & 1 & -1 & 6 \end{array}\right]$$

Therefore, if $A = \begin{bmatrix} 6 & 5 \\ 1 & 1 \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} 1 & -5 \\ -1 & 6 \end{bmatrix}$.

Translated into the language of systems of equations, this tells us that if

$$\left.\begin{aligned} u &= 6x + 5y \\ v &= x + y \end{aligned}\right\} \qquad (1)$$

4.4.1 continued

then

$$\left.\begin{aligned} x &= u - 5v \\ y &= -u + 6v \end{aligned}\right\} \qquad (2)$$

[Notice, of course, that we could have obtained (2) from (1) without a knowledge of matrices, but the matrix coding technique is very desirable for higher dimensional linear systems.]

From (2), we see that

$$\underline{f}^{-1}(u,v) = (x,y)$$

where

$$\begin{cases} x = u - 5v \\ y = -u + 6v \end{cases}$$

c. In particular,

$$\underline{f}^{-1}(16,3) = [16 - 5(3), -16 + 6(3)] = (1,2).$$

As a check, by (1),

$$\underline{f}(1,2) = [6(1) + 5(2), 1 + 2] = (16,3).$$

d. The line in question has as its equation

$$y = 2x.$$

With this substitution, equation (2) becomes

$$-u + 6v = 2u - 10v$$

or

$$3u = 16v. \qquad (3)$$

4.4.1 continued

This checks with (c) since (1,2) is on $y = 2x$ and (16,3) is on $3u = 16v$.

Note

Had we let $y = 2x$ in equation (1), we would have obtained

$$\begin{cases} u = 6x + 10x = 16x \\ v = x + 2x = 3x \end{cases} \tag{4}$$

Equation (4) not only tells us that $\underline{f}$ maps $y = 2x$ onto $3u = 16v$, but it tells us the point-by-point images. For example, if we pick a general point $(x_0,2x_0)$ on $y = 2x$, then by (4), letting $x = x_0$, we have

$u = 16x_0$

$v = 3x_0$

so that

$\underline{f}(x_0,2x_0) = (16x_0,3x_0)$.

In other words, equation (3) yields the range of $\underline{f}$ without reference to individual images of elements in the xy-plane, while equation (4) allows us not only to find the range of $\underline{f}$, but the individual images as well.

4.4.2

a. We have

$$\left.\begin{aligned} u &= x + 4y \\ v &= 3x + 12y \end{aligned}\right\} \tag{1}$$

so our matrix of coefficients, A, is $\begin{bmatrix} 1 & 4 \\ 3 & 12 \end{bmatrix}$, whereupon det A = $12 - 12 = 0$. Hence, A^{-1} fails to exist. Therefore, $\underline{f}^{-1}$ doesn't exist.

4.4.2 continued

b. Dividing the second equation in (1) by the first, we obtain

$$\frac{v}{u} = \frac{3x + 12y}{x + 4y} = \frac{3(x + 4y)}{(x + 4y)}$$

Therefore, $v = 3u$, except possibly when $x + 4y = 0$.

However, when $x + 4y = 0$, equation (1) tells us that $u = 0$ and $v = 0$, and this also satisfies $v = 3u$.

Therefore, the range of $\underline{f}$ is given by $\{(u,v):v = 3u\}$ and pictorially, this is the straight line of slope 3 passing through the origin in the uv-plane.

<u>Note #1</u>

Had we so desired, we could have concluded that $v = 3u$ by our matrix coding technique. Namely,

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 3 & 12 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right]$$

which tells us that the constraint is $0 = -3u + v$, or $v = 3u$. It also tells us that once the constraint is met, $x + 4y = u$. In summary, this information tells us that

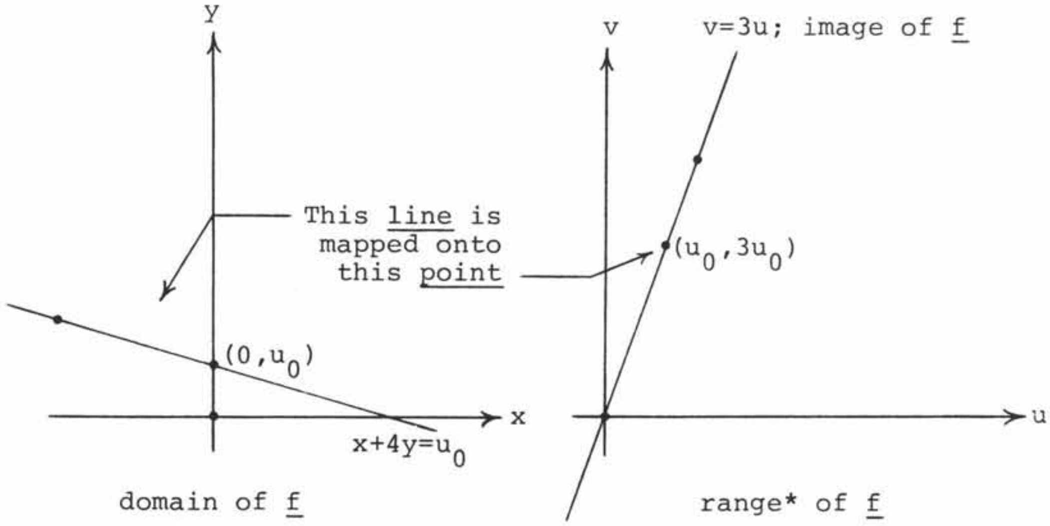
(1) $\underline{f}$ maps no element (x,y) into (u,v) unless $v = 3u$.

(2) If $v = 3u$, then $\underline{f}(x,y) = (u,v) \leftrightarrow x + 4y = u$.

<u>Note #2</u>

Our discussion in Note #1 does not require a geometric interpretation, but such an interpretation does exist. Namely,

4.4.2 continued



c. This occurs when u = 8, so that the set we seek is x + 4y = 8, or $y = -\frac{1}{4}x + 2$, which is a line of slope $-\frac{1}{4}$ and y-intercept equal to 2.

d. We simply find where this line is intersected by the line $y = -\frac{1}{4}x + 2$ since in (c), we showed that every point on $y = -\frac{1}{4}x + 2$ is mapped onto (8,24).

[As an algebraic check, we solve

$$\begin{cases} x + 4y = 8 \\ 2x + 9y = 17 \end{cases}$$

to obtain x = 4 and y = 1, and we then observe that $\underline{f}(4,1) = [4 + 4(1), 3(4) + 12(1)] = (8,24)$.]

e. No other point (x,y) of 2x + 9y = 17 lies on x + 4y = 8. Given any such point, however, it does lie on some line of the parallel family $x + 4y = u_0$, where $u_0 \neq 8$. Since all points on $x + 4y = u_0$ map into $(u_0,3u_0)$, this point also maps into $(u_0,3u_0)$, and since $u_0 \neq 8$, $(u_0,3u_0) \neq (8,24)$. Therefore,

$\underline{f}(x,y) \neq (8,24)$.

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*Recall that when f is onto, then the range of f and the image of f are the same. Otherwise, as in this case, they are different.

4.4.3

a. We have

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 3 & -2 & 1 \end{array}\right] \quad (1)$$

From the rows of (1), we have:

$$\left.\begin{aligned} x + z &= 3u - v \\ y &= -2u + v \end{aligned}\right\} \quad (2)$$

and the constraint that

$$3u - 2v + w = 0. \quad (3)$$

From (3), we see that the range of f is the set

$\{(u,v,w): 3u - 2v + w = 0\}$.

b. The geometric interpretation is that we may view f as mapping xyz-space into uvw-space. (This is an extension of the 2-dimensional case in which f mapped the xy-plane into the uv-plane.) Then f maps the <u>entire</u> xyz-space into (actually, onto) <u>the plane</u> $3u - 2v + w = 0$ (or more commonly, $w = -3u + 2v$) in uvw-space.

c. From (2), if $u = v = 0$, then $x + z = 0$ and $y = 0$. Therefore,

$$\underline{f}(x,y,z) = (0,0,0) \leftrightarrow x + z = 0 \text{ and } y = 0$$

$$\leftrightarrow z = -x,\ y = 0$$

Therefore,

$$S = \{(x,0,-x)\} \to \underline{f}(S) = \{(0,0,0)\}.$$

4.4.3 continued

d. Using the equation of a line in 3-space, we see that L is given by

$u = v = -w \quad (= t)$

$\left[\text{i.e.; } \frac{u - 0}{1} = \frac{v - 0}{1} = \frac{w - 0}{-1} \; (= t)\right].$

Putting this into (2) yields

$$\left.\begin{aligned} x + z &= 3t - t = 2t \\ y &= -2t + t = -t \end{aligned}\right\} \qquad (4)$$

whereupon

$x + z = -2y$

i.e. $\underline{f}(x,y,z) \in L \leftrightarrow (x,y,z)$ belongs to the plane $x + z = -2y$.

[As a check, we may replace z by $-x - 2y$ in the equations

$x + y + z = u$

$2x + 3y + 2z = v$

$x + 3y + z = w$

to obtain $-y = u = v$; $y = w$; or $u = v = -w$.]

4.4.4

Here we no longer have a graphical interpretation, but the arithmetic ideas remain the same.

4.4.4 continued

a.
$$\begin{array}{cccc|cccc} \underline{x_1} & \underline{x_2} & \underline{x_3} & \underline{x_4} & \underline{b_1} & \underline{b_2} & \underline{b_3} & \underline{b_4} \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 5 & 3 & 4 & 0 & 1 & 0 & 0 \\ 3 & 5 & 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 4 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & -3 & 0 & 1 & 0 \\ 0 & -2 & -2 & -4 & -3 & 0 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 5 & -2 & 0 & 0 \\ 0 & 1 & 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & -2 & 1 \end{bmatrix} \qquad (1)$$

The last two rows of (1) give us the constraints

$$\left.\begin{array}{l} -5b_1 + b_2 + b_3 = 0 \\ \text{and} \\ 3b_1 - 2b_3 + b_4 = 0 \end{array}\right\} \qquad (2)$$

or, in terms of b_1 and b_2,

$$\left.\begin{array}{l} b_3 = 5b_1 - b_2 \\ b_4 = 2b_3 - 3b_1 = 2(5b_1 - b_2) - 3b_1 \\ \quad = 7b_1 - 2b_2 \end{array}\right\} \qquad (3)$$

Equations (3) tell us that the 3rd row (equation) must be five times the first minus the second [check: $5(x_1 + 2x_2 + x_3 + x_4) - (2x_1 + 5x_2 + 3x_3 + 4x_4) = 3x_1 + 5x_2 + 2x_3 + x_4$], while the 4th equation is seven times the first minus two times the second.

4.4.4 continued

In any event, combining (3) [or (2)] with (1), we have that the given system is equivalent to

$$\left.\begin{aligned} x_1 - x_3 - 3x_4 &= 5b_1 - 2b_2 \\ x_2 + x_3 + 2x_4 &= -2b_1 + b_2 \end{aligned}\right\} \qquad (4)$$

subject to the constraints

$$b_3 = 5b_1 - b_2$$

$$b_4 = 7b_1 - 2b_2$$

In other words, if either $b_3 \neq 5b_1 - b_2$ or $b_4 \neq 7b_1 - 2b_2$, then the given system has no solutions; while if $b_3 = 5b_1 - 2b_2$ <u>and</u> $b_4 = 7b_1 - 2b_2$, then the given system has infinitely many solutions as indicated by (4). [Namely, we may choose, for example, x_3 and x_4 at random and then solve uniquely for x_1 and x_2.]

b. In particular, if the constraints are met, pick x_3 and x_4 at random whereupon (4) yields

$$\left.\begin{aligned} x_1 &= x_3 + 3x_4 + 5b_1 - 2b_2 \\ x_2 &= -x_3 - 2x_4 - 2b_1 + b_2 \end{aligned}\right\} \qquad (5)$$

The absence of b_3 and b_4 in (5) merely reflects our constraints. That is, for (5) to hold, we must have that $b_3 = 5b_1 - b_2$ and $b_4 = 7b_1 - 2b_2$.

c. Here, again, we have the same problem, only reworded in terms of $\underline{f}:E^4 \to E^4$.

(i) The fact that $\underline{f}(\underline{x}) = (1,1,1,1)$ means by the definition of $\underline{f}$ that

4.4.4 continued

$$\left.\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 1 \\ 2x_1 + 5x_2 + 3x_3 + 4x_4 &= 1 \\ 3x_1 + 5x_2 + 2x_3 + x_4 &= 1 \\ 3x_1 + 4x_2 + x_3 - x_4 &= 1 \end{aligned}\right\} \qquad (6)$$

This, in turn, is the previous part of the exercise with $b_1 = b_2 = b_3 = b_4 = 1$. In this case, $b_3 \neq 5b_1 - b_2$ (since $1 \neq 5 - 1$). Hence, the constraints are not obeyed and this means that the system (6) has no solutions.

In other words, there exists <u>no</u> $(x_1,x_2,x_3,x_4) \varepsilon E^4$ such that $f(x_1,x_2,x_3,x_4) = (1,1,1,1)$.

More generally, in terms of $\underline{f}:E^4 \rightarrow E^4$, we have shown that the existence of an $\underline{x}$ in E^4 such that $\underline{f}(\underline{x}) = \underline{b}$, where $\underline{b} = (b_1,b_2,b_3,b_4)$, requires that $b_3 = 5b_1 - b_2$ and $b_4 = 7b_1 - 2b_2$. This, in turn, says that unless $\underline{b}$ has the form $(b_1,b_2,5b_1-b_2,7b_1-2b_2)$, there is no $\underline{x}$ in E^4 such that $\underline{f}(\underline{x}) = \underline{b}$. In particular, $\underline{f}$ is not an onto map, since its image contains only those elements of E^4 which are of the form $(b_1,b_2,5b_1-b_2,7b_1-2b_2)$.

(ii) In this part of the exercise, we see that $b_1 = 1$, $b_2 = 1$, $b_3 = 4$, and $b_4 = 5$.

Therefore, $b_3 = 5b_1 - b_2$ is obeyed since $4 = 5(1) - (1)$ and $b_4 = 7b_1 - 2b_2$, since $5 = 7(1) - 2(1)$.

Consequently, our constraints are obeyed and as a result there exist elements $\underline{x}$ in E^4 such that $\underline{f}(\underline{x}) = (1,1,4,5)$.

In fact, if we return to the system (5), we have with $b_1 = b_2 = 1$.

$$\left.\begin{aligned} x_1 &= x_3 + 3x_4 + 3 \\ x_2 &= -x_3 - 2x_4 - 1 \end{aligned}\right\} \qquad (7)$$

In turn, (7) tells us that we may pick x_3 and x_4 at random, whereupon

4.4.4 continued

$$\underline{f}(x_3 + 3x_4 + 3, -x_3 - 2x_4 - 1, x_3, x_4) = (1,1,4,5). \tag{8}$$

For example, letting $x_3 = x_4 = 0$, we have

$$\underline{f}(3,-1,0,0) = (1,1,4,5)$$

while letting $x_3 = x_4 = 1$, we have

$$\underline{f}(7,-4,1,1) = (1,1,4,5).$$

In other words, $\underline{f}$ is not 1-1 since infinitely many $\underline{x} \in E^4$ have the property that $\underline{f}(\underline{x}) = (1,1,4,5)$.

As a check of equation (8), we have

$$\begin{aligned}\underline{f}(x_3 + 3x_4 + 3, -x_3 - 2x_4 - 1, x_3, x_4) &= [(x_3 + 3x_4 + 3) + \\ &+ 2(-x_3 - 2x_4 - 1) + x_3 + x_4, 2(x_3 + 3x_4 + 3) + 5(-x_3 - 2x_4 - \\ &- 1) + 3x_3 + 4x_4, 3(x_3 + 3x_4 + 3) + 5(-x_3 - 2x_4 - 1) + \\ &+ 2x_3 + x_4, 3(x_3 + 3x_4 + 3) + 4(-x_3 - 2x_4 - 1) + x_3 - x_4] = \\ &= (1,1,4,5).\end{aligned}$$

(There is nothing sacred about solving for x_1 and x_2 in terms of x_3 and x_4. The point is that we may solve the original system for any two of the unknowns in terms of the other two, but we shall not pursue this point further here.)

4.4.5

The augmented matrix technique yields:

4.4.5 continued

$$\begin{array}{c}
\begin{array}{ccccc|ccccc}
\underline{x_1} & \underline{x_2} & \underline{x_3} & \underline{x_4} & \underline{x_5} & \underline{b_1} & \underline{b_2} & \underline{b_3} & \underline{b_4} & \underline{b_5}
\end{array} \\
\left[\begin{array}{ccccc|ccccc}
1 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\
2 & 3 & 2 & 3 & 3 & 0 & 1 & 0 & 0 & 0 \\
3 & 3 & 4 & 5 & 2 & 0 & 0 & 1 & 0 & 0 \\
1 & 3 & -1 & 2 & 5 & 0 & 0 & 0 & 1 & 0 \\
-2 & 1 & -6 & -3 & 5 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array} \sim$$

$$\begin{bmatrix}
1 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & -1 & -3 & 0 & 1 & 0 & 0 \\
0 & 2 & -2 & 0 & 4 & -1 & 0 & 0 & 1 & 0 \\
0 & 3 & -4 & 1 & 7 & 2 & 0 & 0 & 0 & 1
\end{bmatrix} \sim$$

$$\begin{bmatrix}
1 & 0 & 1 & 3 & 0 & 3 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & -1 & -3 & 0 & 1 & 0 & 0 \\
0 & 0 & -2 & 2 & 2 & 3 & -2 & 0 & 1 & 0 \\
0 & 0 & -4 & 4 & 4 & 8 & -3 & 0 & 0 & 1
\end{bmatrix} \sim$$

$$\begin{bmatrix}
1 & 0 & 0 & 4 & 1 & 6 & -1 & -1 & 0 & 0 \\
0 & 1 & 0 & -1 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & -1 & -3 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3 & -2 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -4 & -3 & 4 & 0 & 1
\end{bmatrix} \qquad (1)$$

From the last two rows of (1), we see that

$$-3b_1 - 2b_2 + 2b_3 + b_4 = 0 \qquad (2)$$

and

$$-4b_1 - 3b_2 + 4b_3 + b_5 = 0 \qquad (3)$$

are our constraints.

In terms of b_1, b_2, and b_3, we may rewrite (2) and (3) as

4.4.5 continued

$$\left.\begin{aligned} b_4 &= 3b_1 + 2b_2 - 2b_3 \\ b_5 &= 4b_1 + 3b_2 - 4b_3 \end{aligned}\right\} \qquad (4)$$

and equations (4) are the constraints.

Again, referring to (1), we see that once the constraints, given by (4), are obeyed our system of equations is equivalent to

$$\left.\begin{aligned} x_1 + 4x_4 + x_5 &= 6b_1 - b_2 - b_3 \\ x_2 - x_4 + x_5 &= -2b_1 + b_2 \\ x_3 - x_4 - x_5 &= -3b_1 + b_3 \end{aligned}\right\} \qquad (5)$$

In summary, our system has no solutions if the constraints defined by (4) are not obeyed, but if the constraints are obeyed, the system (5) shows us that there are infinitely many solutions. In particular, we may pick x_4 and x_5 at random, whereupon

$$\left.\begin{aligned} x_1 &= -4x_4 - x_5 + 6b_1 - b_2 - b_3 \\ x_2 &= x_4 - x_5 - 2b_1 + b_2 \\ x_3 &= x_4 + x_5 - 3b_1 + b_3 \end{aligned}\right\} \qquad (6)$$

Hence, the function $\underline{f}:E^5 \to E^5$ is neither 1-1 nor onto. It is not onto since the only images of $\underline{f}$ must have the special form

$$(b_1, b_2, b_3, 3b_1 + 2b_2 - 2b_3, 4b_1 + 3b_2 - 4b_3)$$

since these are the constraints.

Moreover, once $\underline{b}$ has the form

$$(b_1, b_2, b_3, 3b_1 + 2b_2 - 2b_3, 4b_1 + 3b_2 - 4b_3)$$

there are infinitely many $\underline{x} \in E^5$ for which $\underline{f}(\underline{x}) = \underline{b}$. Namely, from (6), we see that x_4 and x_5 may be chosen at random, whereupon

4.4.5 continued

$$\underline{x} = (-4x_4 - x_5 + 6b_1 - b_2 - b_3,\ x_4 - x_5 - 2b_1 + b_2,\ x_4 + x_5 - $$
$$- 3b_1 + b_3,\ x_4,\ x_5).$$

In particular, if $b_1 = b_2 = b_3 = 0$

$$\underline{x} = (-4x_4 - x_5,\ x_4 - x_5,\ x_4 + x_5,\ x_4,\ x_5). \tag{7}$$

Equation (7) shows us that the image of every element of the form $(-4x_4 - x_5, x_4 - x_5, x_4 + x_5, x_4, x_5)$ is $\underline{0}$ where x_4 and x_5 may be chosen at random.