Unit 5: The Cross Product

1.5.1(L)

We know that one such vector is, by definition, $\vec{A} \times \vec{B}$. In fact, any other such vector must be a scalar multiple of $\vec{A} \times \vec{B}$.

Now, since $\vec{A} = 3\vec{i} + 4\vec{j} + 5\vec{k}$ and $\vec{B} = 2\vec{i} + 6\vec{j} + 7\vec{k}$, we have that

$$\vec{V} = (3\vec{i} + 4\vec{j} + 5\vec{k}) \times (2\vec{i} + 6\vec{j} + 7\vec{k}) \tag{1}$$

is the vector we seek.

The point is that to compute $\vec{V}$ in Cartesian coordinates we do not need any "formal recipes." Rather, we may proceed structurally by use of the distributive property of cross products.

That is:

$$
\begin{aligned}
(3\vec{i} + 4\vec{j} + 5\vec{k}) \times (2\vec{i} + 6\vec{j} + 7\vec{k}) &= 3\vec{i} \times (2\vec{i} + 6\vec{j} + 7\vec{k}) \\
&\quad + 4\vec{j} \times (2\vec{i} + 6\vec{j} + 7\vec{k}) \\
&\quad + 5\vec{k} \times (2\vec{i} + 6\vec{j} + 7\vec{k}) \\
&= (3\vec{i} \times 2\vec{i}) + (3\vec{i} \times 6\vec{j}) \\
&\quad + (3\vec{i} \times 7\vec{k}) + (4\vec{j} \times 2\vec{i}) \\
&\quad + (4\vec{j} \times 6\vec{j}) + (4\vec{j} \times 7\vec{k}) \\
&\quad + (5\vec{k} \times 2\vec{i}) + (5\vec{k} \times 6\vec{j}) \\
&\quad + (5\vec{k} \times 7\vec{k}) \tag{2}
\end{aligned}
$$

By the associative property that $m\vec{A} \times n\vec{B} = mn\, \vec{A} \times \vec{B}$, equation (2) may be rewritten as

$$6\vec{i} \times \vec{i} + 18\vec{i} \times \vec{j} + 21\vec{i} \times \vec{k} + 8\vec{j} \times \vec{i} + 24\vec{j} \times \vec{j} + 28\vec{j} \times \vec{k}$$

$$+ 10\vec{k} \times \vec{i} + 30\vec{k} \times \vec{j} + 35\vec{k} \times \vec{k} \tag{3}$$

If we next observe that

1.5.1(L) continued

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$$

while $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$ and $\vec{k} \times \vec{i} = \vec{j}$, we see that equation (3) becomes

$$6\vec{0} + 18\vec{k} - 21\vec{j}^* - 8\vec{k} + 24\vec{0} + 28\vec{i} + 10\vec{j} - 30\vec{i} + 35\vec{0} =$$

$$- 2\vec{i} - 11\vec{j} + 10\vec{k} \tag{4}$$

The determinant technique merely allows us to obtain the answer in a more compact form. Namely,

$$\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 5 \\ 2 & 6 & 7 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} - \vec{j} \begin{vmatrix} 3 & 5 \\ 2 & 7 \end{vmatrix} + \vec{k} \begin{vmatrix} 3 & 4 \\ 2 & 6 \end{vmatrix}$$

$$= \vec{i}(28 - 30) - \vec{j}(21 - 10) + \vec{k}(18 - 8)$$

$$= -2\vec{i} - 11\vec{j} + 10\vec{k} \tag{5}$$

The fact that (4) and (5) are identical indicates the convenience of the determinant method in finding the same answer that the direct method yields.

The main point is that by the arithmetic of vectors we can show directly that

$$(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times (b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) = (a_2b_3 - a_3b_2)\vec{i}$$

$$+ (a_3b_1 - a_1b_3)\vec{j}$$

$$+ (a_1b_2 - a_2b_1)\vec{k} \tag{6}$$

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*Note the need for stressing order. Namely, $\vec{i} \times \vec{k} = -(\vec{k} \times \vec{i})$ not $\vec{k} \times \vec{i}$. Consequently, since $\vec{k} \times \vec{i} = \vec{j}$, $\vec{i} \times \vec{k} = -\vec{j}$.

1.5.1(L) continued

Notice that (6) has a rather distinctive, easy-to-remember form
quite apart from the determinant notation. Namely, only the second
and third subscripts appear in the first (i.e., the $\vec{i}$) component,
only the first and third subscripts appear in the second component,
and only the first and second subscripts appear in the third com-
ponent. Moreover, the term is positive if the subscripts appear
in the correct cyclic order and negative otherwise. By cyclic
order, we merely mean: write 1, 2, and 3 in the given order and
imagine them to arranged on a circle rather than on a straight
line. Then a cyclic rearrangement is any one that may be obtained
by starting at any of the three numbers and proceeding in the
direction of 123. That is 231 and 312 are cyclic arrangements of
123. In a similar way the cyclic arrangements of 1234 are 2341,
3412, and 4123. Pictorially,
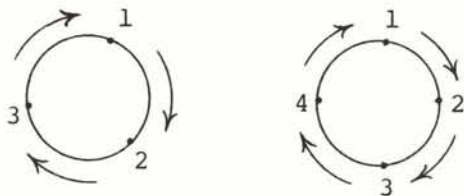
It is simply that equation (6) is easier to remember in the de-
terminant form. We want those who are not too familiar with de-
terminant notation to realize that there is no need to know de-
terminants to find cross products. In still other words, had de-
terminants not already been invented for better reasons, it is un-
likely they would have been invented to compute cross products.

1.5.2(L)

a. From the points A(1,2,3), B(3,3,5) and C(4,8,1), we may form the
   vectors $\vec{AB}$ and $\vec{AC}$. We then know that for a line to be perpendi-
   cular to a plane it must be perpendicular to each line in the
   plane. In particular, the line we seek must be perpendicular to
   both AB and AC. If we vectorize the problem, we seek a vector
   perpendicular to both $\vec{AB}$ and $\vec{AC}$, but from the material of this
   unit, we know that one such vector is $\vec{AB} \times \vec{AC}$. Since $\vec{AB} = (2,1,2)$
   and $\vec{AC} = (3,6,-2)$, we have:

1.5.2(L) continued

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 2 \\ 3 & 6 & -2 \end{vmatrix}$$

$$= (-2 - 12)\vec{i} - \vec{j}(-4 - 6) + \vec{k}(12 - 3)$$

$$= -14\vec{i} + 10\vec{j} + 9\vec{k}$$

Thus, the correct answer to (a) is any scalar multiple of $-14\vec{i} + 10\vec{j} + 9\vec{k}$.

Notice that there was nothing sacred about choosing $\vec{AB}$ and $\vec{AC}$ as our vectors. For example, we might have elected to pick $\vec{AB}$ and $\vec{BC}$ as our pair of vectors. In this case, $\vec{BC} = (1,5,-4)$ and then $\vec{AB} \times \vec{BC}$ would be given by

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 2 \\ 1 & 5 & -4 \end{vmatrix}$$

or

$$(-4 - 10)\vec{i} + (-8 - 2)(-\vec{j}) + (10 - 1)\vec{k}$$

which is again $-14\vec{i} + 10\vec{j} + 9\vec{k}$.

b. In part (a) we found a vector perpendicular to a plane by computing a cross product of two vectors in the plane. In the example we chose, the cross product was the same in both cases. In general, we would not expect two different pairs of vectors in the plane to give the same cross product although we would expect the answers to be scalar multiples of one another. The reason that both answers were the same in (a) is part of our study in (b). The key point is that if the vectors $\vec{A}$ and $\vec{B}$ are placed at a common origin, we may think of the parallelogram determined by $\vec{A}$ and $\vec{B}$ as adjacent sides. In this case, the magnitude of $\vec{A} \times \vec{B}$ denotes the area of the parallelogram.

In particular, the parallelogram which has $\vec{AB}$ and $\vec{AC}$ as adjacent sides has area equal to $|\vec{AB} \times \vec{AC}|$. Of course, this is the same parallelogram as the one which has $\vec{AB}$ and $\vec{BC}$ as adjacent sides.

1.5.2(L) continued

Hence, the area of this parallelogram is also given by $\vec{AB} \times \vec{BC}$. This will explain geometrically why $\vec{AB} \times \vec{BC}$ and $\vec{AB} \times \vec{AC}$ could at "worst" be the negatives of one another. Namely, they have the same direction (both are perpendicular to the same plane) and they have the same magnitude (the area of the parallelogram).

c. The point now is that triangle ABC has half the area of the parallelogram formed by $\vec{AB}$ and $\vec{AC}$, and as seen in our discussion of (b) the area of the parallelogram is given by $|\vec{AB} \times \vec{AC}|$.

Accordingly, the area of the triangle is given by

$|\vec{AB} \times \vec{AC}|/2$

or

$\sqrt{(-14)^2 + (10)^2 + (9)^2}/2$

or

$\sqrt{377}/2 \approx 19.4/2 = 9.7$

1.5.3(L)

a. As an introductory aside, let us first observe that the concept of skew lines exists in three-dimensional geometry but not in two-dimensional geometry. That is, when we say that parallel lines are lines which do not intersect, we are assuming that the lines being considered all lie in the same plane. Clearly, however, given two non-intersecting lines in three-dimensional space it might well happen that there is no one plane which contains both lines.

When two lines in different planes do not intersect we call these lines a pair of skew lines. It seems natural (hopefully) to define the distance between skew lines as follows. There are infinitely many planes that can pass through a given line. Essentially, once we have one plane that passes through the line we may rotate that
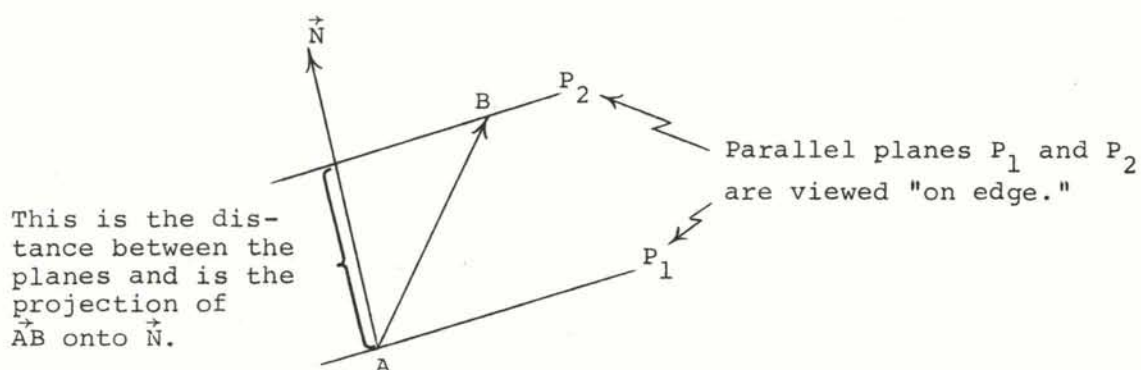
1.5.3(L) continued

plane about the line, and in this way we change the position of
the plane (hence we change the plane) but it still passes through
the given line.

We pass planes through each of the lines and we then "pivot" the
planes until they are parallel. Stated more concisely, we imbed
the lines in a pair of parallel planes. We then define the dis-
tance between the skew lines to be the (perpendicular) distance
between the two parallel planes.

While the study of planes and lines is the subject of the next
unit, most of the basic ideas are already available to us.

We should point out that there are other ways of defining the dis-
tance. For example, we could pick a point on one line and drop a
perpendicular from that point to the other line. That would be
the distance from <u>that particular point</u> on one line to the other.
We could then repeat this procedure (sort of like a max-min calculus
problem) for each point on the first line, and the <u>minimum</u> distance
from a point on the first line to the second line could be defined
as the distance between the two lines. It should not be too hard
to check, however, that the method we outlined earlier is equiva-
lent to this method but a bit easier to handle computationally.

b. A rather convenient way of finding the distance between two par-
    allel planes is to form a vector $\vec{AB}$ from the point A in one of the
    planes to a point B in the other, and then project this vector
    onto a vector $\vec{N}$ which is perpendicular to both planes. Pictorially,

Parallel planes P_1 and P_2
are viewed "on edge."

This is the dis-
tance between the
planes and is the
projection of
$\vec{AB}$ onto $\vec{N}$.

1.5.3(L) continued

It is the dot product that allows us, in terms of vectors, to project one line onto another. In the present case, the distance we seek is the magnitude of $\vec{AB}\cdot\vec{u}_N$ where $\vec{u}_N$ is a unit vector in the direction of N. (See Exercises 1.4.6 and 1.4.7 as a review if necessary.)

All that we need now is a vector perpendicular to the parallel planes which contain our given skew lines. Here we employ the properties of the cross product. Namely, let us vectorize the given skew lines and call the resulting vectors $\vec{v}$ and $\vec{w}$. Clearly a vector perpendicular to a plane is perpendicular to every line in the plane. Hence, the vector we seek must, in particular, be perpendicular to both $\vec{v}$ and $\vec{w}$. This means that $\vec{v} \times \vec{w}$ will be one such vector.

Putting our discussion all together, we now have a rather straight-forward recipe. Specifically,

Step 1. Vectorize each of the skew lines, calling the vectors, say, $\vec{v}$ and $\vec{w}$.

Step 2. Form $\vec{v} \times \vec{w}$. (Summarizing our discussion above, this is a vector perpendicular to a pair of parallel planes which "house" the lines.)

Step 3. Let $\vec{c}$ denote a vector that originates at the first line and terminates at the second line.

Step 4. Dot $\vec{c}$ with the unit vector in the direction of $\vec{v} \times \vec{w}$ (that is, project $\vec{c}$ onto $\vec{v} \times \vec{w}$).
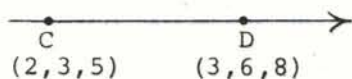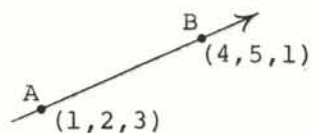
The magnitude of the number obtained in Step 4 is the desired distance.

c. This is just an application of parts (a) and (b). In this case we have

1.5.3(L) continued

Therefore,

$$\vec{AB} = (4-1, 5-2, 1-3) = (3, 3, -2)$$

$$\vec{CD} = (3-2, 6-3, 8-5) = (1, 3, 3)$$

Hence, a vector $\vec{N}$ perpendicular to a pair of parallel planes which house our two skew lines is given by

$$\vec{N} = \vec{AB} \times \vec{CD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 3 & -2 \\ 1 & 3 & 3 \end{vmatrix} = \vec{i}(9+6) - \vec{j}(9+2) + \vec{k}(9-3)$$

$$= 15\vec{i} - 11\vec{j} + 6\vec{k}$$

Therefore,

$$\vec{u}_N = \frac{15\vec{i} - 11\vec{j} + 6\vec{k}}{|15\vec{i} - 11\vec{j} + 6\vec{k}|} = \frac{15\vec{i} - 11\vec{j} + 6\vec{k}}{\sqrt{15^2 + (-11)^2 + 6^2}} = \frac{15\vec{i} - 11\vec{j} + 6\vec{k}}{\sqrt{382}}$$

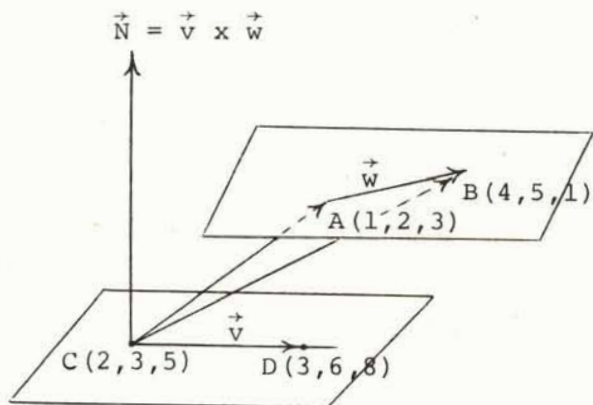Therefore, the desired distance is given by

$$|\vec{CB} \cdot \vec{u}_N| = (4-2, 5-3, 1-5) \cdot \frac{(15, -11, 6)}{\sqrt{382}} = \frac{|(2, 2, -4) \cdot (15, -11, 6)|}{\sqrt{382}}$$

$$= \frac{|30 - 22 - 24|}{\sqrt{382}} = \frac{16}{\sqrt{382}} = \frac{16\sqrt{382}}{382} = \frac{8\sqrt{382}}{191}$$
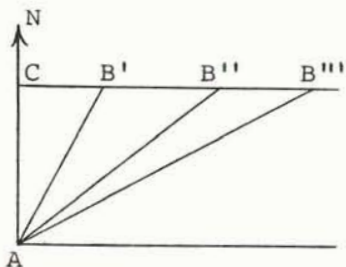
1.5.3(L) continued

Pictorial Summary

Note: Had we used $\vec{CA}$, say, instead of $\vec{CB}$, the distance would have
been given by

$$|\vec{CA}\cdot\vec{u}_N| = \left| (-1,-1,-2) \cdot \frac{(15,-11,6)}{\sqrt{382}} \right|$$

$$= \frac{|-15 + 11 - 12|}{\sqrt{382}}$$

$$= \frac{16}{\sqrt{382}}$$

which agrees with our previous answer. In other words, as one
would probably expect, the distance between the lines is indepen-
dent of the point of reference we choose on each line. Pictorially,
in terms of a "side view"

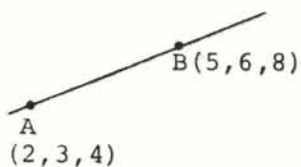$\vec{AC}$ is the vector projection of $\vec{AB'}$, $\vec{AB''}$, $\vec{AB'''}$ onto N.

1.5.4

C(4,5,9)   D(6,11,14)

B(5,6,8)

A
(2,3,4)

$$\vec{AB} = (3,3,4)$$

$$\vec{CD} = (2,6,5)$$

$$\vec{N} = \vec{AB} \times \vec{CD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 3 & 4 \\ 2 & 6 & 5 \end{vmatrix} = \vec{i}(15 - 24) - \vec{j}(15 - 8) + \vec{k}(18 - 6)$$

$$= -9\vec{i} - 7\vec{j} + 12\vec{k}$$

$$|\vec{N}| = \sqrt{(-9)^2 + (-7)^2 + (12)^2} = \sqrt{274}$$

Therefore,

$$\vec{u}_N = \frac{(-9,-7,12)}{\sqrt{274}}$$

Distance between lines $= |\vec{AC} \cdot \vec{u}_N|$

$$= \left| (2,2,5) \cdot \frac{(-9,-7,12)}{\sqrt{274}} \right|$$

$$= \frac{|-18 - 14 + 60|}{\sqrt{274}} = \frac{28}{\sqrt{274}} = \frac{28\sqrt{274}}{274} = \frac{14\sqrt{274}}{137}$$

$$\approx 1.7$$

1.5.5(L)

a. By the definition of cross product, the vector $(\vec{A} \times \vec{B}) \times \vec{C}$ is
    perpendicular to both $(\vec{A} \times \vec{B})$ and $\vec{C}$. We first observe that any
    vector perpendicular to $\vec{A} \times \vec{B}$ must itself be parallel to the plane
    determined by $\vec{A}$ and $\vec{B}$. That is, if, for example, we let $\vec{D} = \vec{A} \times \vec{B}$,
    the locus of all vectors perpendicular to $\vec{D}$ is a plane to which $\vec{D}$
    is perpendicular. (We say "a plane" but it might just as well be
    "the plane" if we utilize the fact that we may move any vector to
    any starting place. In other words, we may assume that $\vec{A}$ and $\vec{B}$
    have a common origin and that all other vectors under consideration
    share this origin.) But, since $\vec{D}$ is perpendicular to both $\vec{A}$ and $\vec{B}$
    it is perpendicular to the plane determined by $\vec{A}$ and $\vec{B}$. Thus, the
    locus of all vectors perpendicular to $\vec{D}$ (= $\vec{A} \times \vec{B}$) is the plane
    determined by $\vec{A}$ and $\vec{B}$. Pictorially,
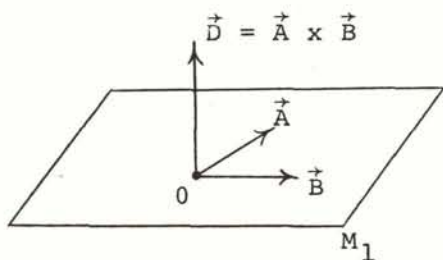
Any vector through 0 perpendicu-
lar to $\vec{D}$ is in the plane M_1
determined by $\vec{A}$ and $\vec{B}$.

In other words, if $\vec{A}$ and $\vec{B}$ are non-parallel, the vector $(\vec{A} \times \vec{B}) \times \vec{C}$
lies in the plane determined by $\vec{A}$ and $\vec{B}$ (if they are parallel then
they do not determine a plane) and at the same time is perpendicu-
lar to $\vec{C}$.

We may rephrase our answer more geometrically as follows. We look
at the plane determined by $\vec{A}$ and $\vec{B}$ (all that is required for this
plane to exist is that $\vec{A}$ and $\vec{B}$ not be parallel*). Call this plane
M_1. We then take the plane which has $\vec{C}$ as its perpendicular

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*If $\vec{A}$ and $\vec{B}$ are parallel then $\vec{A} \times \vec{B} = \vec{0}$ whence $(\vec{A} \times \vec{B}) \times \vec{C} = \vec{0}$ and
the discussion is then "trivial." Similarly, if $\vec{A}$ and $\vec{B}$ are not
parallel, $(\vec{A} \times \vec{B}) \times \vec{C}$ will still equal $\vec{0}$ if $\vec{C}$ is parallel to
$(\vec{A} \times \vec{B})$ since the cross product of parallel vectors always yields
the zero vector. In other words, if either $\vec{A}$ and $\vec{B}$ are scalar
multiples of one another or if $\vec{C}$ is normal to the plane determined
by $\vec{A}$ and $\vec{B}$, the vector $(\vec{A} \times \vec{B}) \times \vec{C}$ is the zero vector.

1.5.5(L) continued

(normal). (Keep in mind that we are assuming that all vectors originate at the same point in our discussion.) Call this plane M_2. Now, unless M_2 and M_1 are parallel they intersect in a line. This line determines the direction of $(\vec{A} \times \vec{B}) \times \vec{C}$.

b. The same argument as in (a) indicates that the vector $\vec{A} \times (\vec{B} \times \vec{C})$ is perpendicular to $\vec{A}$ and lies in the plane determined by $\vec{B}$ and $\vec{C}$. Among other things, then:

$(\vec{A} \times \vec{B}) \times \vec{C}$ lies in the $\vec{A}$-$\vec{B}$ plane

while

$\vec{A} \times (\vec{B} \times \vec{C})$ lies in the $\vec{B}$-$\vec{C}$ plane.

Clearly the $\vec{A}$-$\vec{B}$ plane and the $\vec{B}$-$\vec{C}$ plane need not be the same, which proves the assertion in (b).

In other words, the cross product of three vectors is not an associative operation. This means, in particular, that we must be careful in dealing with any arithmetic involving the cross product. It is such a common thing for us, based on our past experience with multiplication, to assume that $\vec{A} \times \vec{B} \times \vec{C}$ is unambiguous that we can make serious errors if we arbitrarily shift the parentheses in the expression $\vec{A} \times (\vec{B} \times \vec{C})$ (for example, we just showed that it can change the plane in which the vector lies).

1.5.6

A vector which lies in the plane of $\vec{A}$ and $\vec{B}$ and which is perpendicular to $\vec{C}$, as we saw in the previous exercise, is $(\vec{A} \times \vec{B}) \times \vec{C}$. Since $\vec{A} = (1,2,3)$, $\vec{B} = (3,5,4)$, and $\vec{C} = (6,8,9)$ we have:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 3 & 5 & 4 \end{vmatrix} = \vec{i}(8 - 15) - \vec{j}(4 - 9) + \vec{k}(5 - 6)$$

$$= -7\vec{i} + 5\vec{j} - \vec{k}$$

1.5.6 continued

Therefore,

$$(\vec{A} \times \vec{B}) \times \vec{C} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -7 & 5 & -1 \\ 6 & 8 & 9 \end{vmatrix} = \vec{i}(45 + 8) - \vec{j}(-63 + 6) + \vec{k}(-56 - 30)$$

$$= 53\vec{i} + 57\vec{j} - 86\vec{k}$$

(We could also have used the formula $(\vec{A} \times \vec{B}) \times \vec{C} = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{B} \cdot \vec{C})\vec{A}$
where $\vec{A} \cdot \vec{C} = (1,2,3) \cdot (6,8,9) = 6 + 16 + 27 = 49$ and
$\vec{B} \cdot \vec{C} = (3,5,4) \cdot (6,8,9) = 18 + 40 + 36 = 94.$ Then

$$(\vec{A} \cdot \vec{C})\vec{B} - (\vec{B} \cdot \vec{C})\vec{A} = 49(3,5,4) - 94(1,2,3)$$

$$= (147,245,196) + (-94,-188,-282)$$

$$= (53,57,-86)$$

which agrees with our earlier answer.)

1.5.7

In the expression $(\vec{A} \times \vec{B}) \times \vec{C} = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{B} \cdot \vec{C})\vec{A}$ it is more impor-
tant to note that $\vec{A}$ denotes the first vector, $\vec{B}$ the second, and $\vec{C}$
the third than to note the names $\vec{A}$, $\vec{B}$, $\vec{C}$ themselves.

More symbolically,

$$(\vec{1} \times \vec{2}) \times \vec{3} = (\vec{1} \cdot \vec{3})\vec{2} - (\vec{2} \cdot \vec{3})\vec{1}$$

With this in mind, we have

1.5.7 continued

$$\vec{A} \times (\vec{B} \times \vec{C}) = -[(\vec{B} \times \vec{C}) \times \vec{A}]*$$

$$= -[(\vec{B} \cdot \vec{A})\vec{C} - (\vec{C} \cdot \vec{A})\vec{B}]$$

$$= -(\vec{B} \cdot \vec{A})\vec{C} + (\vec{C} \cdot \vec{A})\vec{B}$$

$$= (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

1.5.8

a. $$(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = (\vec{A} \times \vec{B}) \times \vec{E}$$

$$= (\vec{A} \cdot \vec{E})\vec{B} - (\vec{B} \cdot \vec{E})\vec{A}$$

$$= [\vec{A} \cdot (\vec{C} \times \vec{D})]\vec{B} - [\vec{B} \cdot (\vec{C} \times \vec{D})]\vec{A}$$

b. The vector must be in both the $\vec{A}$-$\vec{B}$ plane and the $\vec{C}$-$\vec{D}$ plane. Hence, it is in the intersection of the two planes.

In other words, except in degenerate cases, $(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D})$ vectorizes the line of intersection between the plane determined by $\vec{A}$ and $\vec{B}$ and the plane determined by $\vec{C}$ and $\vec{D}$.

1.5.9

In general, we may think of three vectors emanating from a common point as determining a parallelepiped (which is the three-dimensional analog of a parallelogram, namely, it is a 6-sided figure whose opposite faces are congruent parallelograms). The volume of any parallelepiped is the product of the area of its base and its height. If we think of $\vec{B}$ and $\vec{C}$ as forming the base, then from what we have already seen it is clear that $|\vec{B} \times \vec{C}|$ denotes the area of the parallelogram which is the base of our solid. The height of the solid is the perpendicular distance from $\vec{A}$ onto the base, and this is precisely $|\vec{A} \cos \theta|$, where θ is the angle between $\vec{A}$ and $\vec{B} \times \vec{C}$ (see Figure 1).

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*Note that our established recipe requires the form $(\vec{1} \times \vec{2}) \times \vec{3}$ not $\vec{1} \times (\vec{2} \times \vec{3})$. We obtained the desired form by recalling that $\vec{V} \times \vec{W} = -[\vec{W} \times \vec{V}]$.
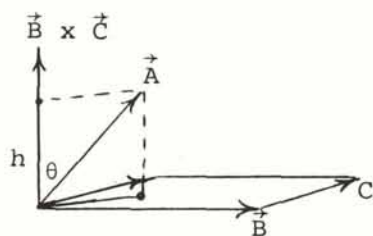

1.5.9 continued

Figure 1

Putting this all together, we see that the volume of the parallel-epiped is given by

$$|\vec{A}| \ |\vec{B} \times \vec{C}| \ |\cos \theta|$$

(where θ is the angle between $\vec{A}$ and $\vec{B} \times \vec{C}$) and by definition of the dot product this is

$$|\vec{A} \cdot (\vec{B} \times \vec{C})| \tag{1}$$

This is a good structural example of where both types of vector products are used in the same formula. Notice also that, except for emphasis, the parentheses in (1) are redundant since $(\vec{A} \cdot \vec{B}) \times \vec{C}$ would be meaningless because $\vec{A} \cdot \vec{B}$ is a scalar, and we only "cross" vectors.

Formula (1) gives us a very convenient test for determining whether three vectors are in the same plane, as well as for finding the volume of parallelepipeds. Namely, if our three vectors lie in the same plane, the volume of the parallelepiped they generate is zero (since in this case all three vectors lie, say, in the base).

If our vectors are given in Cartesian form, equation (1) takes on a particularly convenient computational form. Namely, let $\vec{A} = (a_1, a_2, a_3)$, $\vec{B} = (b_1, b_2, b_3)$, and $\vec{C} = (c_1, c_2, c_3)$. Then $\vec{B} \times \vec{C} = (b_2 c_3 - b_3 c_2, b_3 c_1 - b_1 c_3, b_1 c_2 - b_2 c_1)$, whereupon $\vec{A} \cdot (\vec{B} \times \vec{C})$ becomes

$$a_1(b_2 c_3 - b_3 c_2) + a_2(b_3 c_1 - b_1 c_3) + a_3(b_1 c_2 - b_2 c_1)$$

1.5.9 continued

and it is then easily verified that the above form is equivalent
to the 3 by 3 determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

With these remarks in mind, we have:

a. The vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ are in the same plane if and only if
    $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$ (where, of course, 0 refers to the number zero since
    a dot product is a scalar).

b. In this case,

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} -1 \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} +1 \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}$$

$$= (15 - 16) - (10 - 12) + (8 - 9)$$

$$= -1 + 2 - 1 = 0,$$

which is the required condition that $\vec{A}$, $\vec{B}$, and $\vec{C}$ all lie in one
plane.

c. In this case, we would use the vectors $\vec{OA}$, $\vec{OB}$, and $\vec{OC}$, in which
    case the determinant would be

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 3 & 4 & 5 \end{vmatrix} = (20 - 12) - (10 - 9) + (8 - 12)$$

$$= 3^*, \text{ which is the required volume}$$

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*It is possible that the determinant could turn out to be negative,
in which case we use the absolute value. This is because the
determinant names $\vec{OA} \cdot (\vec{OB} \times \vec{OC})$ while the volume we seek is
$|\vec{OA} \cdot (\vec{OB} \times \vec{OC})|$.

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