Unit 5: Determinants

3.5.1(L)

a. $$\begin{vmatrix} 3+2 & 1+5 \\ 4 & 7 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 4 & 7 \end{vmatrix} + \begin{vmatrix} 2 & 5 \\ 4 & 7 \end{vmatrix} \tag{1}$$

$$= (21 - 4) + (14 - 20)$$

$$= 11. \tag{2}$$

<u>Check:</u>

$$\begin{vmatrix} 3+2 & 1+5 \\ 4 & 7 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 7 \end{vmatrix} \tag{3}$$

$$= 35 - 24 = 11,$$

Which agrees with (2).

<u>Note #1</u>

Given specific numbers, as in this exercise, we would most likely use (3) rather than (1). The technique used in (1) is most useful when we are working with "literal" constants. That is, since we cannot simplify $a_{11} + b_{11}$, we would use (1) to obtain

$$\begin{vmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{vmatrix}. \tag{4}$$

[This is analogous to the situation in algebra where one learns that $(a + b)^2 = a^2 + 2ab + b^2$, yet computes $(3 + 4)^2$ as $7^2 = 49$, rather than as $3^2 + 2(3)(4) + 4^2$.]

<u>Note #2</u>

The validity of (4) stems from the fact that

$$\begin{vmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

3.5.1(L) continued

denotes

$$D(v_1 + v_1', v_2) \text{ where } \begin{cases} v_1 = a_{11}u_1 + a_{12}u_2 \\ v_1' = b_{11}u_1 + b_{12}u_2 \\ v_2 = a_{21}u_1 + a_{22}u_2 \end{cases}$$

and one of our axioms for D is that

$$D(v_1 + v_1', v_2) = D(v_1, v_2) + D(v_1', v_2)$$

Note #3

While our demonstration was for $n = 2$, the result is valid for all n. This is due to the fact that D is defined to be linear. That is

$$D(v_1,\ldots,v_i+v_i',\ldots,v_n) = D(v_1,\ldots,v_i,\ldots,v_n) + D(v_1,\ldots,v_i',\ldots,v_n)$$

for any space $V = [u_1,\ldots,u_n]$.

Note #4

While we shall not bother to prove it here, there is an interesting result which relates the determinant of a matrix to the determinant of the transpose of the matrix. Essentially, any theorems referring to the rows of a matrix remains valid for the columns. In particular, the determinant of a matrix equals the determinant of its transpose. We shall illustrate these ideas in part (b).

b. We may use the technique of part (a) successively to obtain

$$\begin{vmatrix} 3+2 & 1+5 \\ 4+6 & 7+9 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 4+6 & 7+9 \end{vmatrix} + \begin{vmatrix} 2 & 5 \\ 4+6 & 7+9 \end{vmatrix} \tag{5}$$

But

$$\begin{vmatrix} 3 & 1 \\ 4+6 & 7+9 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 4 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 6 & 9 \end{vmatrix} \tag{6}$$

and

$$\begin{vmatrix} 2 & 5 \\ 4+6 & 7+9 \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 4 & 7 \end{vmatrix} + \begin{vmatrix} 2 & 5 \\ 6 & 9 \end{vmatrix}. \tag{7}$$

3.5.1(L) continued

Using (6) and (7) in (5) we obtain

$$\begin{vmatrix} 3+2 & 1+5 \\ 4+6 & 7+9 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 4 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 6 & 9 \end{vmatrix} + \begin{vmatrix} 2 & 5 \\ 4 & 7 \end{vmatrix} + \begin{vmatrix} 2 & 5 \\ 6 & 9 \end{vmatrix}$$

$$= (21 - 4) + (27 - 6) + (14 - 20) + (18 - 30)$$

$$= 17 + 21 - 6 - 12$$

$$= 20. \qquad (8)$$

As a check,

$$\begin{vmatrix} 3+2 & 1+5 \\ 4+6 & 7+9 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 10 & 16 \end{vmatrix} = 80 - 60 = 20,$$

which agrees with (8).

<u>Note #5</u>

We could have used columns rather than rows to solve this problem as follows:

$$\begin{vmatrix} 3+2 & 1+5 \\ 4+6 & 7+9 \end{vmatrix} = \begin{vmatrix} 3 & 1+5 \\ 4 & 7+9 \end{vmatrix} + \begin{vmatrix} 2 & 1+5 \\ 6 & 7+9 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 \\ 4 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 5 \\ 4 & 9 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 6 & 7 \end{vmatrix} + \begin{vmatrix} 2 & 5 \\ 6 & 9 \end{vmatrix}$$

$$= (21 - 4) + (27 - 20) + (14 - 6) + (18 - 30)$$

$$= 17 + 7 + 8 - 12$$

$$= 20.$$

More literally

$$\begin{vmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{vmatrix}$$

$$+ \begin{vmatrix} b_{11} & b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{vmatrix}$$

3.5.1(L) continued

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \qquad (9)$$

$$= (a_{11}a_{22} - a_{12}a_{21}) + (a_{11}b_{22} - a_{12}b_{21}) + (b_{11}a_{22} - a_{21}b_{12}) + (b_{11}b_{22} - b_{12}b_{21}).$$

As a check of this last result we may obtain by direct computation:

$$\begin{vmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{vmatrix} = (a_{11} + b_{11})(a_{22} + b_{22}) - (a_{12} + b_{12})(a_{21} + b_{21})$$

$$= a_{11}a_{22} + a_{11}b_{22} + b_{11}a_{22} + b_{11}b_{22} - a_{12}a_{21} - a_{12}b_{21} - a_{21}b_{12} - b_{12}b_{21}$$

$$(a_{11}a_{22} - a_{12}a_{21}) + (a_{11}b_{22} - a_{12}b_{21}) + (b_{11}a_{22} - a_{21}b_{12}) + (b_{11}b_{22} - b_{12}b_{21}).$$

On the other hand, using columns rather than rows, we would obtain

$$\begin{vmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} + b_{12} \\ a_{21} & a_{22} + b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} + b_{12} \\ b_{21} & a_{22} + b_{22} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}. \qquad (10)$$

3.5.1(L) continued

The right sides of (9) and (10) agree since the right side of (10) is $(a_{11}a_{22} - a_{12}a_{21}) + (a_{11}b_{22} - a_{21}b_{12}) + (b_{11}a_{22} - a_{12}b_{21}) + (b_{11}b_{22} - b_{12}b_{21})$.

<u>Note #6</u>

Equation (9) reveals a result that may seem a bit "unpleasant". Namely, it might seem nice to believe that "the determinant of a sum equals the sum of the determinants". But, equation (9) shows that this need not be true. In more detail, suppose we let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Then

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}.$$

Now, letting $|A|$ denote the determinant of A, etc., we have from (8) that

$$|A + B| = |A| + \begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix} + |B| . \qquad (11)$$

Hence,

$$|A + B| = |A| + |B| \leftrightarrow \begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix} = 0.$$

We illustrate this idea in part (c).

c. We have

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 2 & 5 \\ 6 & 9 \end{bmatrix}.$$

Therefore,

3.5.1(L) continued

$$A + B = \begin{bmatrix} 3 + 2 & 1 + 5 \\ 4 + 6 & 7 + 9 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 10 & 16 \end{bmatrix} .$$

Hence,

$$|A + B| = 80 - 60 = 20$$

while

$$|A| + |B| = (21 - 4) + (18 - 30) = 5.$$

This is sufficient to prove that $|A + B|$ <u>need</u> not equal $|A| + |B|$.

<u>Note #7</u>

In this exercise we saw that $|A + B| - |A| - |B| = 15$. According to (11), then

$$\begin{vmatrix} a_{11} + b_{12} \\ a_{21} + b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix}$$

should equal 15. As a check,

$$\begin{vmatrix} 3 & 5 \\ 4 & 9 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 6 & 7 \end{vmatrix} = (27 - 20) + (14 - 6) = 7 + 8 = 15.$$

<u>Note #8</u>

Do not confuse parts (b) and (c). What (c) showed was that we could not replace $|A + B|$ by $|A| + |B|$. What (b) showed was that we could obtain a "correction factor". In other words, (b) showed that we could expand $|A + B|$ by taking one row (column) at a time.

3.5.2(L)

In the previous exercise we may have been a bit crushed to discover that the determinant of a sum need not be the sum of the determinants. If nothing else, such a result should at least make us a bit wary of trusting our intuition, and accordingly,

3.5.2(L) continued

it should make us seem a bit more tolerant when it comes to giving proofs.

Our point is that one very important property of determinants is that the determinant of a product is the product of the determinants; but we cannot pass this off as being "self-evident" if only because the corresponding statement about sums seems just as self-evident.

What we shall do in this exercise is prove the result for the case n = 2. Our technique shall be the one which generalizes to all dimensions, but by restricting our attention to n = 2 we may avoid part of the mass of computational detail and notation that often obscures the structure of the proof. As we shall see, the proof relies rather strongly on the results of Exercise 3.5.1.

a. Since

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

we can be certain that

$$|AB| = \begin{vmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{vmatrix} . \tag{1}$$

By the linear properties of the determinant, as discussed in Exercise 3.5.1, we see from (1) that $|AB|$ may be written as

$$\begin{vmatrix} a_{11}b_{11} & a_{11}b_{12} \\ a_{21}b_{11} & a_{21}b_{12} \end{vmatrix} + \begin{vmatrix} a_{11}b_{11} & a_{12}b_{22} \\ a_{21}b_{11} & a_{22}b_{22} \end{vmatrix} + \begin{vmatrix} a_{12}b_{21} & a_{11}b_{12} \\ a_{22}b_{21} & a_{21}b_{12} \end{vmatrix} + \begin{vmatrix} a_{12}b_{21} & a_{12}b_{22} \\ a_{22}b_{21} & a_{22}b_{22} \end{vmatrix} . \tag{2}$$

Factoring out the common factors in the rows or columns of the terms in (2), we see that (2) may be rewritten in the form

3.5.2(L) continued

$$a_{11}a_{21}\begin{vmatrix} b_{11} & b_{12} \\ b_{11} & b_{12} \end{vmatrix} + b_{11}b_{22}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + b_{21}b_{12}\begin{vmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{vmatrix} + a_{12}a_{22}\begin{vmatrix} b_{21} & b_{22} \\ b_{21} & b_{22} \end{vmatrix}. \tag{3}$$

But both

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{11} & b_{12} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} b_{21} & b_{22} \\ b_{21} & b_{22} \end{vmatrix}$$

are 0 since each of these determinants has two equal rows [i.e., $D(v_1,v_1) = 0$, etc.].

Moreover,

$$\begin{vmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

since one determinant is obtained from the other merely by interchanging two columns [i.e., $D(v_1,v_2) = -D(v_2,v_1)$,etc.]

With these observations, (3) becomes

$$b_{11}b_{22}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - b_{21}b_{12}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^{*} (b_{11}b_{22} - b_{21}b_{12}). \tag{4}$$

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*Remember

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

is a number not a matrix, and can be factored from the given expression.

3.5.2(L) continued

Finally, since

$$b_{11}b_{22} - b_{21}b_{12} = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix},$$

we see from (1) and (4) that

$$|AB| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

$$= |A||B| \text{ .} \qquad (5)$$

<u>Note #1</u>

In the relatively simple case $n = 2$, we could verify by direct computation that $|AB| = |A||B|$ but notice how cumbersome the direct computation becomes for large values of n. Our technique never involves our having to expand by cofactors, etc., but rather has us repeatedly write a determinant as a simpler sum of determinants. Moreover, our technique does not require that we conjecture the result in advance. That is, we began with $|AB|$ and showed that $|AB| = |A||B|$.

b. Since $D(u_1,\ldots,u_n) = 1$, our coding system says that

$$\begin{vmatrix} 1 & 0 & 0 \ldots & 0 \\ 0 & 1 & 0 \ldots & 0 \\ 0 & 0 & 1 \ldots & 0 \\ \vdots & & & \\ 0 & 0 & 0 \ldots 0 & 1 \end{vmatrix} = 1.$$

In other words, the determinant of I is 1.

Now suppose A is invertible, than A^{-1} exists such that

$$AA^{-1} = I.$$

Accordingly,

$$|AA^{-1}| = |I| \text{ .} \qquad (6)$$

3.5.2(L) continued

By part (a) we know that $|AA^{-1}| = |A||A^{-1}|$ and we have just recalled that $|I| = 1$. Hence (6) becomes

$$|A||A^{-1}| = 1. \tag{7}$$

From (7) we obtain

$$|A| = \frac{1}{|A^{-1}|}.$$

That is

$$|A^{-1}| = |A|^{-1}. \tag{8}$$

Note #2

The beauty of (8) is that it identifies the inverse of a matrix with the inverse of a number. That is, $|A^{-1}|$ refers to finding the determinant of the matrix which is the inverse of A, while $|A|^{-1}$ refers to finding the determinant of A (which is a number) and then taking its reciprocal. We illustrate this in part (c).

c. With

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}$$

we see at once that

$$|A| = 7 - 4 = 3. \tag{9}$$

We may now find A^{-1}, namely:

$$\left[\begin{array}{cccc} 1 & 2 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & 2 & 1 & 0 \\ 0 & 3 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc} 1 & 2 & 1 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} \end{array}\right]$$

3.5.2(L) continued

$$\sim\begin{bmatrix} 1 & 0 & \frac{7}{3} & -\frac{2}{3} \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} \end{bmatrix},$$

hence,

$$A^{-1} = \begin{bmatrix} \frac{7}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}. \tag{10}$$

Accordingly,

$$\begin{aligned} |A^{-1}| &= \frac{7}{3}\left(\frac{1}{3}\right) - \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) \\ &= \frac{7}{9} - \frac{4}{9} = \frac{3}{9} \\ &= \frac{1}{3}. \end{aligned} \tag{11}$$

Here are a few points which occur just within the confines of the present exercise.

1. The matrix A^{-1} in (10) can be written as

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 7 & -2 \\ -2 & 1 \end{bmatrix}.$$

That is, when we multiply a <u>matrix</u> by a number, <u>each</u> <u>entry</u> of the <u>matrix</u> is multiplied by that number.

On the other hand, when we multiply the <u>determinant</u> of A^{-1}, i.e., $|A^{-1}|$, by a number we multiply <u>any</u> <u>one</u> <u>row</u> (or column, but not both) of A^{-1} by that number. For example, in computing $|A^{-1}|$ from (10), we find that

$$\begin{vmatrix} \frac{7}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}\begin{vmatrix} 7 & -2 \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix}$$

3.5.2(L) continued

$$= \frac{1}{3}\left\{\frac{1}{3}\begin{vmatrix} 7 & -2 \\ -2 & 1 \end{vmatrix}\right\}$$

$$= \frac{1}{9}\begin{vmatrix} 7 & -2 \\ -2 & 1 \end{vmatrix}$$

$$= \frac{1}{9}(7 - 4)$$

That is, we "factored out" 1/3 twice, once for each row (or column) in which it was a common factor.

$$= \frac{1}{3}.$$

Had we simply written that

$$\begin{vmatrix} \frac{7}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}\begin{vmatrix} 7 & -2 \\ -2 & 1 \end{vmatrix}$$

we would have obtained the incorrect answer, 1, as the value of the determinant.

2. If AB = 0 (where now 0 denotes the zero matrix), then it is not true that either A or B must be the zero matrix. For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

but neither

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

nor

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

is the zero matrix.

This should not be confused with the <u>numerical</u> result that if $|AB| = 0$ (where now 0 is the <u>number</u> 0) then $|A| = 0$ or $|B| = 0$. In summary, the product of two matrices can be the zero.

3.5.2(L) continued

Comparing (9) and (11) we conclude that

$$|A| = \frac{1}{|A^{-1}|}\ .$$

Note #3

This exercise proves that if a matrix is invertible, its determinant cannot equal zero. Namely, since

$$|A||A^{-1}| = 1,$$

neither $|A|$ nor $|A^{-1}|$ can be zero.

More generally, since $|AB| = |A|\ |B|$, we see that $|A| \neq 0$ and $|B| \neq 0 \leftrightarrow |AB| \neq 0$; or from a different emphasis, if $|AB| = 0 \leftrightarrow |A| = 0$ or $|B| = 0$.

Note #4

Because of all our previous work with matrices and the relatively little work we've done with determinants, there is a danger that we may confuse matrix properties with determinant properties. Many non-zero matrices (recall that the zero-matrix is the matrix, all of whose entries are zero) have a zero determinant. In fact, this is precisely the definition of a singular matrix. At any rate, notice that the product of two matrices can be the zero-matrix even though neither matrix is the zero-matrix. For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The two matrices on the left side of the equality are singular but non-zero. However, the product of two matrices cannot be singular unless at least one of the two matrices is singular. It is this fact that is stated in terms of determinants by

$$|AB| = 0 \text{ if and only if } |A| = 0 \text{ or } |B| = 0.$$

Stated without the language of determinants, all we are saying is that if both A^{-1} and B^{-1} exist then $(AB)^{-1}$ also exists, and in particular, it is given by $B^{-1}A^{-1}$ [since $(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AA^{-1} = I$].

3.5.2(L) continued

In summary, we must not confuse <u>the</u> zero matrix with a matrix whose determinant is zero (as examples of which are all singular matrices).

3.5.3(L)

If P is an n by n invertible matrix and A is any n by n matrix, then we have:

$$\begin{aligned} |PAP^{-1}| &= |(PA)P^{-1}| \\ &= |PA|\ |P^{-1}| \\ &= (|P||A|)\ |P^{-1}| \\ &= |P||A||P^{-1}|. \qquad (1) \end{aligned}$$

The fact that P is invertible means that $|P| \neq 0$. Hence, $|P^{-1}| = \frac{1}{|P|}$ is well defined. Hence, the (numerical) factors $|P|$ and $|P^{-1}|$ cancel in (1) to yield $|PAP^{-1}| = |A|$.

<u>A NOTE ABOUT LINEAR TRANSFORMATIONS</u>

In the previous unit we discussed how the same linear transformation could be coded by many different matrices, depending on the basis being used to describe the domain of the transformation. We showed that if A and B were any two such matrices, then there existed a non-singular matrix P such that $B = PAP^{-1}$. In fact, we used this fact as the motivation for our definition of what it meant for two matrices to be <u>similar</u>.

During the discussion we also mentioned that the transformation itself did not depend on the basis being used. Only the matrix depended on the choice of basis. Thus, one would like to feel that there should be some invariant about the matrix of the transformation; that is, some fact that would be true for every similar matrix. What we have shown in this exercise is that there is at least one such invariant (there are also others but we shall not pursue this here). Namely, since $|PAP^{-1}| = |A|$ we know that if A and B are similar then these

3.5.3(L) continued

two matrices have the same determinant. In other words, the infinitely many different matrices which code the same linear transformation are characterized (among other ways) by the fact that they all have the same determinant.

3.5.4 (optional)

We have

$$A + I = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} + 1 & a_{12} \\ a_{21} & a_{22} + 1 \end{bmatrix} .$$

Hence,

$$\det (A + I) = \begin{vmatrix} a_{11} + 1 & a_{12} \\ a_{21} & a_{22} + 1 \end{vmatrix}$$

$$= (a_{11} + 1)(a_{22} + 1) - a_{12}a_{21}$$

$$= a_{11}a_{22} + a_{11} + a_{22} + 1 - a_{12}a_{21}$$

$$= (a_{11}a_{22} - a_{12}a_{21}) + 1 + (a_{11} + a_{22}). \tag{1}$$

Now

$$a_{11}a_{22} - a_{12}a_{21} = \det (A)$$

and

$$1 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \det I.$$

Therefore, we may deduce from (1) that

$$\det (A + I) = \det(A) + \det(I) + a_{11} + a_{22}. \tag{2}$$

3.5.4 continued

In itself, (2) might not seem like a very impressive result, but it has several important ramifications. From our immediate point of view, perhaps the most important consequence of (2) is that it shows us that "the determinant of a sum equals the sum of the determinants" is usually a false statement. Indeed, we have shown in (2) that

$$\det(A + I) = \det(A) + \det(I)$$

is true only when $a_{11} + a_{22} = 0$. That is, unless the sum of the diagonal elements of A is zero,

$$\det(A + I) \neq \det(A) + \det(I).$$

It is also interesting to note that the sum of the diagonal elements of a square matrix is given a special name. It is called the <u>trace</u> of the matrix. We shall exhibit an interesting property of the trace of a matrix in part (b).

b. $$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -5 & 2 \\ 0 & -1 & -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{bmatrix}. \qquad (1)$$

Hence, we conclude from (1) that if

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

then

$$B^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}.$$

3.5.4 continued

Therefore,

$$B^{-1}AB = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 13 & 23 \\ 31 & 54 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -7 \\ 8 & 15 \end{bmatrix} . \qquad (2)$$

Thus for (2) we conclude that the trace of $B^{-1}AB$, written $T_r(B^{-1}AB)$, equals $-3 + 15 = 12$.

Note #1:

We have written $B^{-1}AB$ whereas in the part we have talked about BAB^{-1}. Notice that if we let $P = B^{-1}$ then $B^{-1}AB$ takes the PAP^{-1}. In other words, we may write either $B^{-1}AB$ or BAB^{-1} in our definition of similar matrices. In computing the matrix of the transformation, however, it does make a difference whether we write $B^{-1}AB$ or BAB^{-1} but we shall say more about this in the next unit.

Note #2:

In what may seem to be a coincidence, the trace of A is also 12 (namely, the diagonal elements of A are 4 and 8). Thus, at least in this special case, we have that A and $B^{-1}AB$ have the same trace. The interesting point is that this is not a coincidence. While we shall not bother proving this result in our course, the fact is that if B is any non-singular n by n matrix and A is any n by n matrix then A and $B^{-1}AB$ always have the same trace.

Recalling our definition in the previous unit that A is similar to C means that $C = B^{-1}AB$, we see that similar matrices have the same trace. In particular, then,in reference to our coding a linear transformation by a matrix, this shows that the matrix which denotes the transformation, while it varies with the choice of basis, has the same trace, regardless of the basis

3.5.4 continued

being used. In other words, when we talk about the trace of the matrix, it makes no difference which basis is being used. In still other words, the trace of the matrix which describes the linear transformation is also an invariant with respect to the basis.

3.5.5(L)

a. Using the method of cofactors and expanding along the first row, we obtain

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 6 \\ 2 & 5 & 8 & 9 \\ 3 & 4 & 9 & 8 \end{vmatrix}$$

$$= \underset{1}{\begin{vmatrix} 3 & 4 & 6 \\ 5 & 8 & 9 \\ 4 & 9 & 8 \end{vmatrix}} -2 \underset{2}{\begin{vmatrix} 1 & 4 & 6 \\ 2 & 8 & 9 \\ 3 & 9 & 8 \end{vmatrix}} +3 \underset{3}{\begin{vmatrix} 1 & 3 & 6 \\ 2 & 5 & 9 \\ 3 & 4 & 8 \end{vmatrix}} -4 \underset{4}{\begin{vmatrix} 1 & 3 & 4 \\ 2 & 5 & 8 \\ 3 & 4 & 9 \end{vmatrix}}. \qquad (1)$$

We may now expand 1 , 2 , 3 , and 4 by cofactors along the first row to obtain:

$$1 = 3\begin{vmatrix} 8 & 9 \\ 9 & 8 \end{vmatrix} -4\begin{vmatrix} 5 & 9 \\ 4 & 8 \end{vmatrix} +6\begin{vmatrix} 5 & 8 \\ 4 & 9 \end{vmatrix}$$

$$= 3(64 - 81) - 4(40 - 36) + 6(45 - 32)$$

$$= -51 -16 +78$$

$$= 11. \qquad (2)$$

$$2 = \begin{vmatrix} 8 & 9 \\ 9 & 8 \end{vmatrix} -4\begin{vmatrix} 2 & 9 \\ 3 & 8 \end{vmatrix} +6\begin{vmatrix} 2 & 8 \\ 3 & 9 \end{vmatrix}$$

$$= (64 - 81) - 4(16 - 27) + 6(18 - 24)$$

$$= -17 +44 -36$$

$$= -9. \qquad (3)$$

3.5.5(L) continued

$$③ = \begin{vmatrix} 5 & 9 \\ 4 & 8 \end{vmatrix} -3\begin{vmatrix} 2 & 9 \\ 3 & 8 \end{vmatrix} +6\begin{vmatrix} 2 & 5 \\ 3 & 4 \end{vmatrix}$$

$$= (40 - 36) - 3(16 - 27) + 6(8 - 15)$$

$$= 4 + 33 - 42$$

$$= -5.$$

$$④ = \begin{vmatrix} 5 & 8 \\ 4 & 9 \end{vmatrix} -3\begin{vmatrix} 2 & 8 \\ 3 & 9 \end{vmatrix} +4\begin{vmatrix} 2 & 5 \\ 3 & 4 \end{vmatrix}$$

$$= (45 - 32) - 3(18 - 24) + 4(8 - 15)$$

$$= 13 + 18 - 28$$

$$= 3. \tag{5}$$

Using the results of (2),(3),(4), and (5) in (1) we obtain

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 6 \\ 2 & 5 & 8 & 9 \\ 3 & 4 & 9 & 8 \end{vmatrix} = 11 - 2(-9) + 3(-5) - 4(3)$$

$$= 11 + 18 - 15 - 12$$

$$= 2. \tag{6}$$

Note #1

Our main aim in having you do part (a) is so that you can convince yourself that the method of cofactors, while structurally sound, is extremely cumbersome to apply in even simple numerical cases. That is, a 3 by 3 determinant can hardly be called complicated, yet the amount of computation in part (a) already seems to be getting quite heavy.

3.5.5(L) continued

Our point is that the method of cofactors is extremely important when we are dealing with "literal" matrices, but in the case that we are dealing with numerical examples, it is much easier to "row-reduce" the determinant, just as we did with matrices. In this respect, we invoke the axiom that the determinant is unchanged if we replace any row (column) by itself plus a constant multiple of any other row (column). Here, again we see the difference between matrices and determinants; when we row-reduce a matrix we get different though equivalent (i.e., they span the same space) matrices, but when we row-reduce a determinant in the above-mentioned context, we do not change the determinant. With this idea in mind one usually evaluates an n by n determinant by row reducing it to a form in which in one row (or column) all the entries, but one, are zero.

In part (b) we illustrate this technique by getting the determinant into the form in which all but the first entry of the first column are zero.

b.
$$\underset{\text{(i)}}{\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 6 \\ 2 & 5 & 8 & 9 \\ 3 & 4 & 9 & 8 \end{vmatrix}} = \underset{\text{(ii)}}{\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & 0 & -4 \end{vmatrix}} . \qquad (7)$$

We obtained (ii) from (i) by replacing the 2nd row of (i) by the 2nd minus the 1st; the 3rd, by the 3rd minus twice the first; and the 4th, by the 4th minus three times the first.

The first column of (7) has 0's everywhere except for a 1 in the first row. Hence, if we expand (ii) by cofactors along the first column we obtain

$$1\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ -2 & 0 & -4 \end{vmatrix} \underbrace{\left\{ -0\begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 1 \\ -2 & 0 & 4 \end{vmatrix} +0\begin{vmatrix} 2 & 3 & 4 \\ 1 & 1 & 2 \\ -2 & 0 & -4 \end{vmatrix} -0\begin{vmatrix} 2 & 3 & 4 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} \right\}}_{=\,0} \qquad (8)$$

3.5.5(L) continued

Comparing (7) and (8) we see that the 4 by 4 determinant (i) is equal to the 3 by 3 determinant (iii) where

$$\text{(iii)} = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ -2 & 0 & 4 \end{vmatrix}. \tag{9}$$

We may again row-reduce (iii) to obtain

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ -2 & 0 & -4 \end{vmatrix} = \underset{\text{(iv)}}{\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \end{vmatrix}} \tag{10}$$

and expanding (iv) by cofactors along the first column, we obtain

$$\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = 0 - (-2) = 2. \tag{11}$$

Combining steps (7) through (11) [and we reproduce these steps so that you see the "big picture"] we obtain

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 6 \\ 2 & 5 & 8 & 9 \\ 3 & 4 & 9 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & 0 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ -2 & 0 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \end{vmatrix}$$

3.5.5(L) continued

$$= \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}$$

$$= 2. \tag{12}$$

Clearly (12) agrees with (6), but certainly there is no doubt that the method of part (b) is more "sane" than that of part (a).

Note #2:

As a compromise between the techniques of (a) and (b), we should observe that we may continue the row reduction technique until we have a diagonal 4 by 4 matrix. The idea is that for a diagonal matrix the determinant is simply the product of the diagonal elements. The key point here is that if we use this method, no reference at all is made to cofactors. In other words, we may evaluate the determinant without recourse to anything but the axioms under which the determinant function was derived.

More specifically, we have:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 6 \\ 2 & 5 & 8 & 9 \\ 3 & 4 & 9 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & 0 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 0 \end{vmatrix}. \tag{13}$$

We may now factor a 2 out of the bottom row of the last determinant in (13) to obtain

$$2\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= 2.$$

3.5.5(L) continued

Observations

1. The reason that we know that the determinant of a diagonal matrix is the product of the diagonal entries is as follows (we use the 4 by 4 case, but the generalization should be clear):

Given:

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{23} & 0 \\ 0 & 0 & 0 & a_{44} \end{vmatrix}$$

we may "factor out" a_{11} from the first row; a_{22}, from the second row; a_{33}, from the third row; and a_{44}, from the fourth row to obtain

$$a_{11}a_{22}a_{33}a_{44} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

and this, in turn, equals $a_{11}a_{22}a_{33}a_{44}$ since $|I| = 1$.

2. Actually if the entires of A are all zero below the main diagonal (i.e., $a_{ij} = 0$ if $i > j$) then the determinant of A is still the product of the diagonal entries. Namely, given

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{vmatrix}$$

we may expand by cofactors along the first column to obtain successively

$$a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ 0 & a_{33} & a_{34} \\ 0 & 0 & a_{44} \end{vmatrix} = a_{11}\left\{a_{22}\begin{vmatrix} a_{33} & a_{34} \\ 0 & a_{44} \end{vmatrix}\right\} = a_{11}a_{22}a_{33}a_{44}.$$

3.5.6

$$\begin{vmatrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 3 & 2 & 4 & 5 \\ 2 & 5 & 3 & 4 & 7 \\ 3 & 4 & 4 & 4 & 8 \\ 4 & 9 & 2 & 3 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 3 & 1 & 0 & 1 \\ 0 & 1 & 1 & -2 & -1 \\ 0 & 5 & -2 & -5 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & -2 & -5 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & -5 & 10 \end{vmatrix} \qquad (1)$$

and factoring out 2 from the 4th row of (1) and -5 from the 5th row, we obtain

$$-10 \begin{vmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & -2 \end{vmatrix} = -10 \underbrace{\begin{vmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & -4 \end{vmatrix}}_{(i)} \qquad (2)$$

Since (i) is a triangular matrix (i.e., all entries below the main diagonal are zero), its determinant is the product of diagonal entries, i.e., $(1)(1)(1)(-1)(-4) = 4$. Combining this result with (1) and (2) we have that

$$\begin{vmatrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 3 & 2 & 4 & 5 \\ 2 & 5 & 3 & 4 & 7 \\ 3 & 4 & 4 & 4 & 8 \\ 4 & 9 & 2 & 3 & 9 \end{vmatrix} = -40.$$

3.5.7 (optional)

Our main aim in this exercise is to try to explain how the formula of expanding by the use of cofactors evolves from our three basic axioms used to define D. The key step, of course, is the linear property of D. For example, we know by linearity that

$$D(a_{11}u_1 + a_{12}u_2,\ a_{21}u_1 + a_{22}u_2)$$

$$= D(a_{11}u_1,\ a_{21}u_1 + a_{22}u_2) + D(a_{12}u_2,\ a_{21}u_1 + a_{22}u_2).$$

If we now write this same result in the more conventional notion for determinants, we obtain that:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{vmatrix}. \tag{1}$$

An interesting question is that of determining how (1) could be derived directly from our previous recipes without reference to the D-notation. This is answered in part (a) of this exercise.

a. Using as hindsight the facts that

$$\begin{vmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{vmatrix}$$

"codes" $D(a_{11}u_1,\ a_{21}u_1 + a_{22}u_2)$, and $a_{11}u_1 = a_{11}u_1 + 0u_2$, we rewrite

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \tag{2}$$

as

$$\begin{vmatrix} a_{11} + 0 & 0 + a_{12} \\ a_{21} & a_{22} \end{vmatrix}. \tag{3}$$

3.5.7 continued

Notice the position of 0 in the first row of (3). In the first column it <u>follows</u> a_{11}, but in the second column, it <u>precedes</u> a_{12}. The reason for this is that a_{11} is the coefficient of u_1 while a_{12} is the coefficient of u_2; so that we may think of a_{11} in (2) as denoting $a_{11}u_1 + 0u_2$; while a_{12} denotes $0u_1 + a_{12}u_2$. Be this as it may, however, we now use the result discussed in exercise 3.5.1 and rewrite (3) as

$$\begin{vmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{vmatrix} . \tag{4}$$

We may now rewrite the bottom row of each determinant of (4) by replacing a_{21} by $a_{21} + 0$ and a_{22} by $0 + a_{22}$. We then obtain

$$\begin{vmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 \\ a_{21} + 0 & 0 + a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 \\ a_{21} & 0 \end{vmatrix} + \begin{vmatrix} a_{11} & 0 \\ 0 & a_{22} \end{vmatrix} \tag{5}$$

and

$$\begin{vmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 0 & a_{12} \\ a_{21} + 0 & 0 + a_{22} \end{vmatrix} = \begin{vmatrix} 0 & a_{12} \\ a_{21} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ 0 & a_{22} \end{vmatrix} . \tag{6}$$

If we now put the results of (2) through (6) together we have:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \underbrace{\begin{vmatrix} a_{11} & 0 \\ a_{21} & 0 \end{vmatrix}}_{1} + \underbrace{\begin{vmatrix} a_{11} & 0 \\ 0 & a_{22} \end{vmatrix}}_{2} + \underbrace{\begin{vmatrix} 0 & a_{12} \\ a_{21} & 0 \end{vmatrix}}_{3} + \underbrace{\begin{vmatrix} 0 & a_{12} \\ 0 & a_{22} \end{vmatrix}}_{4} \tag{7}$$

Determinants 1 and 4 are 0. We may see this by observing that each has a column of 0's or by observing that in each determinant one row is a scalar multiple of the other. In any event, we are left with

3.5.7 continued

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 \\ 0 & a_{22} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ a_{21} & 0 \end{vmatrix}$$

$$+ \; a_{11}a_{22}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + a_{12}a_{21}\underset{5}{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}} \tag{8}$$

Determinant 5 has an interesting form. It is a <u>permutation</u> of the identity determinant. That is, we can obtain the identity from 5 by interchanging the two rows. Now, we know that we change the sign of the determinant whenever we interchange two rows. Hence,

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix},$$

and since

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

we conclude from (8) that

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \tag{9}$$

b. In all candor, it should appear that our derivation of (9) was liking snatching defeat from the jaws of victory! After all, we derived this result in the lecture. Indeed, the technique used in this exercise was precisely the same as that used in the lecture except for the symbolism used to denote the determinant function.

The important point is that in the format used in part (a) of this exercise we begin to see the structure which justifies the sign convention of the method of cofactors. The key idea

3.5.7 continued

is that whenever we interchange two rows (columns) of a square matrix we change the sign of the determinant of the matrix. Thus, given a permutation of the identity matrix I, we can count how many times we must interchange two rows to obtain the identity matrix. If the number of times is even, the determinant is 1 (since every interchange changes the sign) while if the number of interchanges is odd the determinant is -1.

Before pursuing this in more detail, let us reinforce the technique by verifying the formula for computing

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad (10)$$

We have that (10) is equal to

$$\begin{vmatrix} a_{11} + 0 + 0 & 0 + a_{12} + 0 & 0 + 0 + a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

which in turn is equal to

$$\underset{6}{\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} + \underset{7}{\begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} + \underset{8}{\begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \qquad (11)$$

We may now rewrite the second row of each determinant in (7) as

$$a_{21} + 0 + 0 \quad 0 + a_{22} + 0 \quad 0 + 0 + a_{23}$$

in which case we obtain:

$$6 = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} + 0 + 0 & 0 + a_{22} + 0 & 0 + 0 + a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

3.5.7 continued

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \qquad (12)$$

Since the first two rows of (9) are scalar multiples of one another, we have that the value of (9) is zero (more formally, we factor out a_{11} from the first row of (9) and a_{21} from the second row, so that the determinant has the form:

$$a_{11}a_{21}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

in which case two rows of the new determinant are equal).

To compute (10) or (11) we rewrite the third row of each as

$$a_{31} + 0 + 0 \quad 0 + a_{32} + 0 \quad 0 + 0 + a_{33}.$$

For example, (11) becomes

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} + 0 + 0 & 0 + a_{32} + 0 & 0 + 0 + a_{33} \end{vmatrix}$$

$$= \underset{(12)}{\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix}} + \underset{(13)}{\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix}} + \underset{(14)}{\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix}}. \qquad (13)$$

Now (12) and (14) are each zero. Namely in (12) we see that the first and third rows are scalar multiples of one another (or in terms of the transpose matrix, the second column of (12) represents the zero vector); while in (14) the second and third rows are scalar multiples of one another.

It is (13) which is of interest to us. That is, we may factor a_{11} from the first row; a_{23} from the second row, and a_{32} from the third row to obtain:

3.5.7 continued

$$a_{11}a_{23}a_{32}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

15

Notice that 15 is a permutation of the identity. In particular, 15 may be converted into

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

simply by interchanging the second and third rows. More specifically

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1. \tag{15}$$

Combining the results of (13), (14), and (15), we see that

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - a_{11}a_{23}a_{32}. \tag{16}$$

<u>Note:</u>

We could have said that

$$\begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{\ 1} & a_{32} & a_{33} \end{vmatrix} \end{array} = a_{11}\begin{vmatrix} 0 & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{11}(0 - a_{23}a_{32})$$

$$= - a_{11}a_{23}a_{32}$$

and this seems much simpler than our method of deriving (16). Keep in mind, however, that the purpose of this exercise is to <u>justify</u> the method of expansion by cofactors. In other words, the "short cut" is a form of circular reasoning in that it

3.5.7 continued

presupposes the validity of the method of cofactors.

In any event we can perform a similar analysis on determinants 7 and 8. Sparing you the details, the overall technique takes on the form:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underset{6}{\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} + \underset{7}{\begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} + \underset{8}{\begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \tag{17}$$

[You may want to observe that the right side of (17) is equivalent to expanding the left scale by cofactors with respect to the first row.]

As we saw, 6 may be rewritten as

$$\underbrace{\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}_{=0} + \underbrace{\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} + \underbrace{\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & \underset{\substack{\uparrow \\ 0}}{0} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & \underset{\substack{\uparrow \\ 0}}{0} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{23} \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{21} & \underset{\substack{\uparrow \\ 0}}{0} & 0 \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & \underset{\substack{\uparrow \\ 0}}{0} & a_{33} \end{vmatrix}$$

3.5.7 continued

$$= a_{11}a_{22}a_{33}\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix} + a_{21}a_{23}a_{32}\begin{vmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}. \qquad (18)$$

Similarly 7 becomes

$$\begin{vmatrix}0 & a_{12} & 0\\ a_{21} & 0 & 0\\ a_{31} & a_{32} & a_{33}\end{vmatrix} + \underbrace{\begin{vmatrix}0 & a_{12} & 0\\ 0 & a_{22} & 0\\ a_{31} & a_{32} & a_{33}\end{vmatrix}}_{\substack{= 0 \text{ since first two rows are scalar multi-}\\ \text{ples of each other.}}} + \begin{vmatrix}0 & a_{12} & 0\\ 0 & 0 & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$

$$= \underbrace{\begin{vmatrix}0 & a_{12} & 0\\ a_{21} & 0 & 0\\ a_{31} & 0 & 0\end{vmatrix}}_{0} + \underbrace{\begin{vmatrix}0 & a_{12} & 0\\ a_{21} & 0 & 0\\ 0 & a_{32} & 0\end{vmatrix}}_{0} + \begin{vmatrix}0 & a_{12} & 0\\ a_{21} & 0 & 0\\ 0 & 0 & a_{33}\end{vmatrix}$$

$$+ \begin{vmatrix}0 & a_{12} & 0\\ 0 & 0 & a_{23}\\ a_{31} & 0 & 0\end{vmatrix} + \underbrace{\begin{vmatrix}0 & a_{12} & 0\\ 0 & 0 & a_{23}\\ 0 & a_{32} & 0\end{vmatrix}}_{0}$$

$$+ \underbrace{\begin{vmatrix}0 & a_{12} & 0\\ 0 & 0 & a_{23}\\ 0 & 0 & a_{33}\end{vmatrix}}_{0}$$

$$= a_{12}a_{21}a_{33}\underbrace{\begin{vmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{vmatrix}}_{16} + a_{12}a_{23}a_{31}\underbrace{\begin{vmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{vmatrix}}_{17}. \qquad (19)$$

3.5.7 continued

Both (16) and (17) are permutations of the identity matrix. How can we convert (16) to the identity. Well, we would want 1 0 0 to be our first row, but it is now our second row. Hence, we would first interchange our first and second rows to obtain

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1.$$

On the other hand, to convert (17) , we notice that 1 0 0 is the third row, so we interchange it with the present first row to obtain

$$\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = - \underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}}_{(18)} . \tag{20}$$

Looking at (18) we would like 0 1 0 to be the second row rather than the third, so we interchange the second and third rows of (18) to obtain

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1, \tag{21}$$

and combining (20) and (21) establishes that

$$(17) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = - \left\{ - \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \right\}$$

$$= -(-1) = 1.$$

3.5.7 continued

Thus, (19) becomes

$$-a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31}. \tag{22}$$

Finally, leaving the details to you, 8 reduces to

$$a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}. \tag{23}$$

A quick "unethical" way of verifying (23) is to expand 8 by cofactors along the first row. Again, this is circular reasoning since our aim here is to establish the validity of the procedure.

If we now combine (18), (22), and (23), we obtain the well known result:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}. \tag{24}$$

Note:
The major impact of what looks like a "shaggy dog" story here in part (b) is that there was a systematic way of predicting what (24) was going to be, and that this systematic way extends at once to any n by n determinant. Using the 3 by 3 case as an illustration, we observe that the given determinant may be written as the sum of 27 very special determinants. Namely, we get one determinant for each way there is of picking exactly one element from each row. We then factor out the various coefficients and we find that each resulting determinant involves a matrix which either has at least one row (column) of 0's or else the matrix is a permutation of the identity matrix.

The permutation of the identity matrix occurs if and only if we had one of the 27 special matrices in which the selected entries from each row also were from distinct columns. What this means is the following:

3.5.7 continued

If we look at any term which consists of three factors of the a_{ij}'s, this term will be absent from (24) unless both the set of first subscripts and second subscripts are permutations of the numbers 1, 2, 3. For example, notice that the term $a_{11}a_{12}a_{23}$ does not appear in (24) since the first subscript of the a's form the set $\{1,1,2\}$ which is not a permutation of $\{1,2,3\}$. Recall that a permutation of a set S is a 1-1, onto function $f:S \to S$, hence at most a re-arrangement of the order of $\{1,2,3\}$.

Thus, we see that there will be as many distinct terms in (24) as there are ways of obtaining a permutation of the identity matrix. To this end, notice that in the 3 by 3 case there exactly $3! = 6$ such ways. Namely, the row (1 0 0) may be placed in any one of the three rows; once this is done, the row (0 1 0) may be placed in any one of the two remaining rows; and once this is done (0 0 1) must be placed in the last position. Notice that (24) bears out this remark since the determinant in (24) involves six terms.

All that remains to account for in (24) is how to determine whether one of the terms occurs with the plus sign or with the negative sign. Conceptually this is easy to do. We look at the permutation determinant from which this term results and then see how many interchanges are necessary to convert it to the identity. If it takes an odd number then the sign is negative (since each interchange changes the sign of the determinant) while if it takes an even number the sign is positive. (It is an interesting aside that the number of interchanges depends on how we elect to make the changes, but whether the number of interchanges is even or odd <u>does not</u> depend on how we choose to make the interchanges.)

The question is how can we tell the sign without having to reproduce the matrix. This, too, is not very difficult - once you get the central idea.

All we do is order the factors so that either the first subscripts or the second subscripts occur in the "natural" order; i.e., 1, 2, 3. We then look at the other subscript and see how many interchanges are necessary to get them into the natural order. One relatively quick way of doing this is to <u>count the</u>

3.5.7 continued

number of inversions (which means to count the number of times a lesser number comes after a greater number).

Since the idea of inversions may seem new to you, let us pause to illustrate this idea with a specific example. Let us consider the permutation on the set 1,2,3,4,5 which leads to the re-arrangement

$$4 \quad 5 \quad 2 \quad 1 \quad 3. \tag{25}$$

We notice that:

1. Starting with our first member 4, three lesser numbers follow it (namely 2, 1, and 3). So this gives us 3 inversions so far.
2. Our second number 5 is also followed by three lesser ones. So this gives us three more inversions, for a total of six, so far.
3. Our next entry 2 is followed by only one lesser number (namely 1). This yields one more inversion, for a total of 7.
4. Our next number 1 is followed by no lesser numbers. (This is the nice thing about 1. It can't be followed by a lesser number. In a similar way if n is the greatest number in our set, everything which follows it is a lesser number.) So our total number of inversions is still 7.
5. Our last number is 3 and since it is the last number listed, nothing follows it. In particular, no lesser number follows it. Thus, our total number of inversions is 7 which is odd.

Obviously we could have re-arranged (25) into the natural order in a less mystical way. For example, we might have noticed that we wanted 5 to come last but since it was listed second in (25), we would have to interchange the second and fifth entries of (25). That is, our first interchange would yield

$$4 \quad 3 \quad 2 \quad 1 \quad 5. \tag{26}$$

Looking at (26) we know that 4 should occupy the fourth position, but it is presently in the first position, so we

3.5.7 continued

realize that we must interchange the first and fourth terms of (26), and this yields

$$1 \quad 3 \quad 2 \quad 4 \quad 5. \tag{27}$$

We then observe that 2 and 3 should be interchanged in (27) for us to obtain the natural order

$$1 \quad 2 \quad 3 \quad 4 \quad 5,$$

so that altogether we have had to make <u>three</u> interchanges to get (25) into the natural order. This checks with the fact that we had an odd (7) number of inversions.

Keep in mind that we could have put (25) into natural order in other systematic ways. For example we might have decided that we first wanted 1 to appear first, in which case we would have interchanged the 1 and 4 to obtain

$$1 \quad 5 \quad 2 \quad 4 \quad 3.$$

We might then have interchanged 5 and 2 to get 2 in the right place, thus yielding

$$1 \quad 2 \quad 5 \quad 4 \quad 3.$$

We might then have interchanged 3 and 5 to put 3 in the right place, and this would have yielded

$$1 \quad 2 \quad 3 \quad 4 \quad 5,$$

so that again three interchanges brings about the natural order.

In fact, the method of inversion is a special way of counting interchanges. Namely, to get 5 into its proper place, we must have it "jump over" every lesser number and this is equivalent to interchanging it with each lesser number. That is the number of inversions (7) corresponds to the following succession of interchanges:

3.5.7 continued

4	5	2	1	3
4	2	5	1	3
4	2	1	5	3
4	2	1	3	5
2	4	1	3	5
2	1	4	3	5
2	1	3	4	5
1	2	3	4	5.

In terms of our determinant idea, if we were expanding the 5 by 5 determinant $|a_{ij}|$ then the sign of

$$a_{14}a_{25}a_{32}a_{41}a_{53}$$

would be negative because with the first subscripts in the natural order the second ones are the permutation 4 5 2 1 3 which is odd.

In more general terms, the expansion of an n by n matrix will consist of n! terms since this is the number of permutations of I_n. Half of these n! terms will be positive and the other half negative. To see which sign a term has, we line up the factors in the order in which, say, the first subscripts form the natural order 1,...,n (if this cannot be done, the term is not one of the n! being evaluated*). We then look at the permutation formed by the second subscripts and count the number of inversions. If the number of inversions is even, we use the plus sign, otherwise, the negative sign.

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* In the method of part (b), generalized to the n by n case, we observe that we get one special determinant for each way that we can form a matrix using one entry from each row. Since there are n rows and n entries from each row, there are n^n such special determinants (note that this accounts for the 27 in our example wherein n = 3). Of these n^n special determinants, all will equal 0 except for the n! which are permutations of $|I_n|$.

3.5.7 continued

c. Given

$$\begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix}$$

we may think of r_1 coding the row (1 0 0 0 0); r_2, the row (0 1 0 0 0), etc. In this way, I is represented by the sequence of rows

$$r_1 \quad r_2 \quad r_3 \quad r_4 \quad r_5. \tag{28}$$

Our determinant has the rows of I in the order

$$r_3 \quad r_4 \quad r_5 \quad r_2 \quad r_1. \tag{29}$$

As a short cut, we may use just the subscripts in (28) and (29), and we see that we must only count the inversion in

3 4 5 2 1

which is

2 + 2 + 2 + 1 = 7

(i.e., 2 and 1 follows 3; they follow 4; they follow 5; and 1 follows 2).

Hence, the given determinant is -1. More concretely:

$$\begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix}$$

3.5.7 continued

$$= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$= -1.$

<u>Summary</u>:

What we have tried to give you a feeling for in this exercise is where the formula for evaluating determinants comes from, and how the method of expansion by cofactors is a convenient way of keeping track of these results without consciously having to determine whether a given permutation involves an odd or an even number of interchanges to convert it to natural order.

Keep in mind, the fact that we've stressed before. Namely, we worry about cofactors, etc., usually when we are working with literal expressions. In the case of concrete numbers, we often use row-reduction methods, etc. since this often yields the correct answer with much less labor.

3.5.8 (optional)

In our study of investigating the linear independence of the set

$\{e^{r_1 x}, \ldots, e^{r_n x}\}$

we had to evaluate the Wronskian determinant

3.5.8 continued

$$\begin{vmatrix} e^{r_1x} & e^{r_2x} \ldots\ldots & e^{r_nx} \\ r_1e^{r_1x} & r_2e^{r_2x} \ldots\ldots & r_ne^{r_nx} \\ \vdots & & \\ r_1^{n-1}e^{r_1x} & r_2^{n-1}e^{r_2x} \ldots\ldots & r_n^{n-1}e^{r_nx} \end{vmatrix} . \qquad (1)$$

We could then factor e^{r_1x} from the first column of (1), e^{r_2x} from the second column, etc. to obtain

$$e^{(r_1 + \ldots + r_n)x} \begin{vmatrix} 1 & 1 & \ldots\ldots & 1 \\ r_1 & r_2 & \ldots\ldots & r_n \\ \vdots & & & \\ r_1^{n-1} & r_2^{n-1} & \ldots\ldots & r_n^{n-1} \end{vmatrix} \qquad (2)$$

The determinant in (2) is called the Vandemonde determinant and its value consists of the product of all terms of the form $r_i - r_j$ where $i > j$.

For example,

1. $\begin{vmatrix} 1 & 1 \\ r_1 & r_2 \end{vmatrix} = r_2 - r_1.$

2. $\begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{vmatrix} = (r_3 - r_1)(r_3 - r_2)(r_2 - r_1).$

3. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ r_1 & r_2 & r_3 & r_4 \\ r_1^2 & r_2^2 & r_3^2 & r_4^2 \\ r_1^3 & r_2^3 & r_3^3 & r_4^3 \end{vmatrix} = (r_4 - r_1)(r_4 - r_2)(r_4 - r_3)(r_3 - r_1)(r_3 - r_2)(r_2 - r_1),$

etc.

This result shows that the Wronskian of $\{e^{r_1x}, \ldots, e^{r_nx}\}$ equals

3.5.8 continued

$0 \leftrightarrow r_i = r_j$ since $i \neq j$. In other words, if $r_i \neq r_j$ for $i \neq j$, then the determinant in (1) is unequal to 0.

Of course, one does not need the study of linear independence to motivate the Vandemonde determinant. This determinant is valuable in its own right and the purpose of this exercise is to verify its value, at least in the cases (2) and (3) mentioned above [Case (1) is,hopefully, completely trivial].

a. Since we are more used to row-reduction than to column reduction we may use the fact that a matrix and its transpose have the same determinant (otherwise we do use column reduction, which is equivalent to the row-reduction of the transpose). In any event,

$$\begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{vmatrix} = \begin{vmatrix} 1 & r_1 & r_1^2 \\ 1 & r_2 & r_2^2 \\ 1 & r_3 & r_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & r_1 & r_1^2 \\ 0 & r_2 - r_1 & r_2^2 - r_1^2 \\ 0 & r_3 - r_1 & r_3^2 - r_1^2 \end{vmatrix} . \qquad (3)$$

We may now factor $r_2 - r_1$ from the second row of (3) and $r_3 - r_1$ from the third row to obtain

$$\begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^3 \end{vmatrix} = (r_2 - r_1)(r_3 - r_1) \begin{vmatrix} 1 & r_1 & r_1^2 \\ 0 & 1 & r_2 + r_1 \\ 0 & 1 & r_3 + r_1 \end{vmatrix}$$

$$= (r_2 - r_1)(r_3 - r_1) \begin{vmatrix} 1 & r_2 + r_1 \\ 1 & r_3 + r_1 \end{vmatrix}$$

$$= (r_2 - r_1)(r_3 - r_1) \left[\begin{vmatrix} 1 & r_2 \\ 1 & r_3 \end{vmatrix} + \underbrace{\begin{vmatrix} 1 & r_1 \\ 1 & r_2 \end{vmatrix}}_{= 0} \right]$$

(since the two rows are equal) $\longrightarrow$

3.5.8 continued

$$= (r_2 - r_1)(r_3 - r_1)\{(r_3 - r_2)\}$$

$$= (r_3 - r_1)(r_3 - r_2)(r_2 - r_1).$$

b. $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ r_1 & r_2 & r_3 & r_4 \\ r_1^2 & r_2^2 & r_3^2 & r_4^2 \\ r_1^3 & r_2^3 & r_3^3 & r_4^3 \end{vmatrix} = \begin{vmatrix} 1 & r_1 & r_1^2 & r_1^3 \\ 1 & r_2 & r_2^2 & r_2^3 \\ 1 & r_3 & r_3^2 & r_3^3 \\ 1 & r_4 & r_4^2 & r_4^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & r_1 & r_1^2 & r_1^3 \\ 0 & r_2 - r_1 & r_2^2 - r_1^2 & r_2^3 - r_1^3 \\ 0 & r_3 - r_1 & r_3^2 - r_1^2 & r_3^3 - r_1^3 \\ 0 & r_4 - r_1 & r_4^2 - r_1^2 & r_4^3 - r_1^3 \end{vmatrix}$$

$$= \begin{vmatrix} r_2 - r_1 & (r_2 - r_1)(r_2 + r_1) & (r_2 - r_1)(r_2^2 + r_2r_1 + r_1^2) \\ r_3 - r_1 & (r_3 - r_1)(r_3 + r_1) & (r_3 - r_1)(r_3^2 + r_3r_1 + r_1^2) \\ r_4 - r_1 & (r_4 - r_1)(r_4 + r_1) & (r_4 - r_1)(r_4^2 + r_4r_1 + r_1^2) \end{vmatrix}$$

$$= (r_2 - r_1)(r_3 - r_1)(r_4 - r_1) \begin{vmatrix} 1 & r_2 + r_1 & r_2^2 + r_2r_1 + r_1^2 \\ 1 & r_3 + r_1 & r_3^2 + r_3r_1 + r_1^2 \\ 1 & r_4 + r_1 & r_4^2 + r_4r_1 + r_1^2 \end{vmatrix}. \qquad (4)$$

Since the determinant in (4) has a second column, each of whose entries is the sum of two numbers, we may write the determinant as the sum of two simpler determinants, one of which is zero.

Namely,

$$\begin{vmatrix} 1 & r_2 + r_1 & r_2^2 + r_2r_1 + r_1^2 \\ 1 & r_3 + r_1 & r_3^2 + r_3r_1 + r_1^2 \\ 1 & r_4 + r_1 & r_4^2 + r_4r_1 + r_1^2 \end{vmatrix} = \begin{vmatrix} 1 & r_2 & r_2^2 + r_2r_1 + r_1^2 \\ 1 & r_3 & r_3^2 + r_3r_1 + r_1^2 \\ 1 & r_4 & r_4^2 + r_4r_1 + r_1^2 \end{vmatrix}$$

3.5.8 continued

$$+ \underbrace{\begin{vmatrix} 1 & r_1 & r_2 + r_2r_1 + r_1^2 \\ 1 & r_1 & r_3^2 + r_3r_1 + r_1^2 \\ 1 & r_1 & r_4^2 + r_4r_1 + r_1^2 \end{vmatrix}}$$

= 0 (since the second column is a multiple of the first)

$$= \begin{vmatrix} 1 & r_2 & r_2^2 + r_2r_1 + r_1^2 \\ 0 & r_3 - r_2 & r_3^2 + r_3r_1 - r_2^2 - r_2r_1 \\ 0 & r_4 - r_2 & r_4^2 + r_4r_1 - r_2^2 - r_2r_1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & r_2 & r_2^2 + r_2r_1 + r_1^2 \\ 0 & r_3 - r_2 & (r_3^2 - r_2^2) + (r_3 - r_2)r_1 \\ 0 & r_4 - r_2 & (r_4^2 - r_2^2) + (r_4 - r_2)r_1 \end{vmatrix}. \qquad (5)$$

Now,

$$(r_3^2 - r_2^2) + (r_3 - r_2)r_1 = (r_3 - r_2)(r_3 + r_2 + r_1)$$

$$(r_4^2 - r_2^2) + (r_4 - r_2)r_1 = (r_4 - r_2)(r_4 + r_2 + r_1).$$

Hence, (5) may be rewritten as

$$\begin{vmatrix} (r_3 - r_2) & (r_3 - r_2)(r_3 + r_2 + r_1) \\ (r_4 - r_2) & (r_4 - r_2)(r_4 + r_2 + r_1) \end{vmatrix}$$

$$= (r_3 - r_2)(r_4 - r_2) \begin{vmatrix} 1 & r_3 + r_2 + r_1 \\ 1 & r_4 + r_2 + r_1 \end{vmatrix}. \qquad (6)$$

3.5.8 continued

Again, the determinant in (6) has a column in which each entry is the sum of three numbers. Hence, we may rewrite it as the sum of three simpler determinants. Namely,

$$\begin{vmatrix} 1 & r_3 + r_2 + r_1 \\ 1 & r_4 + r_2 + r_1 \end{vmatrix} = \begin{vmatrix} 1 & r_3 \\ 1 & r_4 \end{vmatrix} + \underbrace{\begin{vmatrix} 1 & r_2 \\ 1 & r_2 \end{vmatrix} + \begin{vmatrix} 1 & r_1 \\ 1 & r_1 \end{vmatrix}} = r_4 - r_3. \quad (7)$$

These are both 0 since in each two rows are the same.

Combining the results of (5), (6), and (7) with (5) we obtain the desired result.

Perhaps the most significant part of this exercise (aside from the actual result) is to see how one can avoid much painful computation by having the patience to invoke the various theorems rather than to try to "bludgeon out" the answer from the basic definition of a determinant or from the method of cofactors. You might want to try part (b) by working directly from the method of cofactors without row-reducing or factoring. If you do, you will soon notice how involved the arithmetic gets. Moreover, in the case of an n by n determinant in which n exceeds 4, the arithmetic blows out of proportion very rapidly if we're not careful. The technique used by us in our solution of (a) and (b) generalizes very nicely to higher dimensions, and - at least relative to most other methods - it manages to keep the arithmetic fairly simple.

In summary, if you have to work much with numerical determinants, it is wise to cultivate techniques for row reduction, factoring, and rewriting determinants as a sum of simpler determinants, preferrably as a sum in which most of the terms are zero!