Unit 8: Complex Integration, Part 1

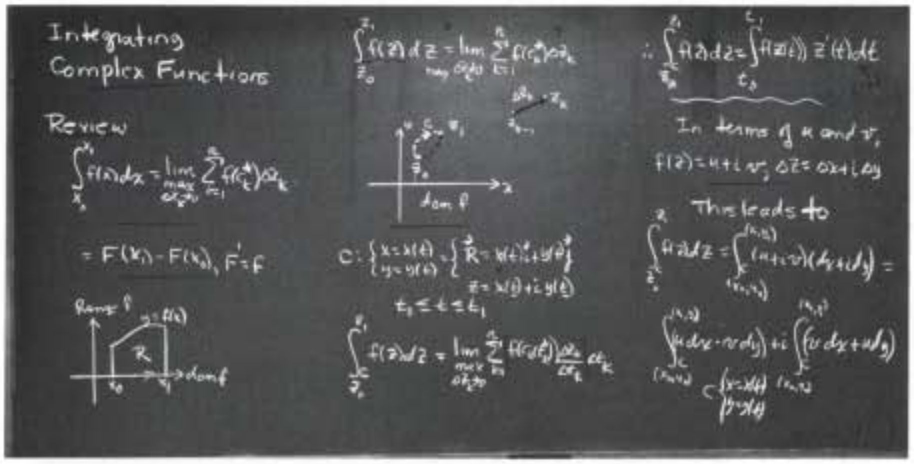
1. Overview

While our text does not discuss this topic at all, the fact is that integration plays a decidedly more crucial role in the calculus of complex functions than does differentiation. In fact, many important properties of analytic functions are proven through integration rather than differentiation.

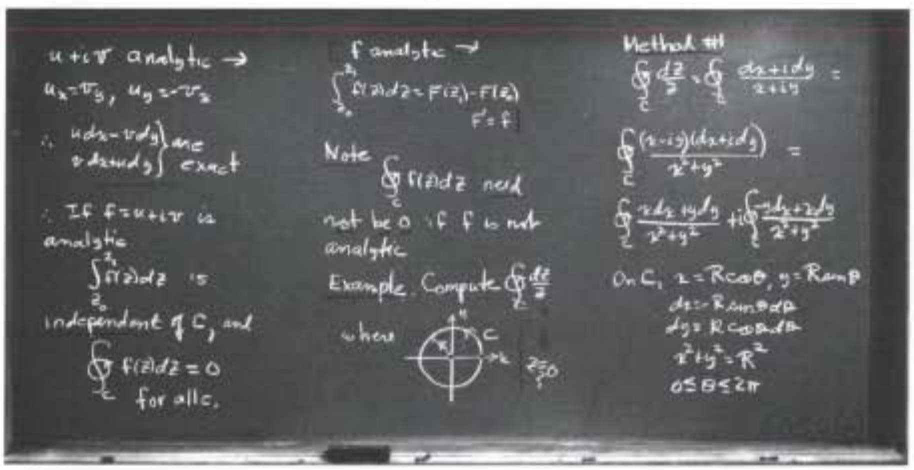
For this reason we wanted to present at least a minimum introduction to this topic as a closing point for our present discussion of complex variables. In all fairness, however, (and perhaps this is why the topic is omitted in the text) complex integration is a subtle and rather difficult concept and usually one needs much experience to feel at home with it. Our hope is to wet your appetite and try to allay any mental blocks which might arise in a regular course on complex variables.

Our approach is to have you watch the lecture and then try the exercises; with an additional batch of exercises comprising Unit 9 which is optional.

2. Lecture 1.050



a.



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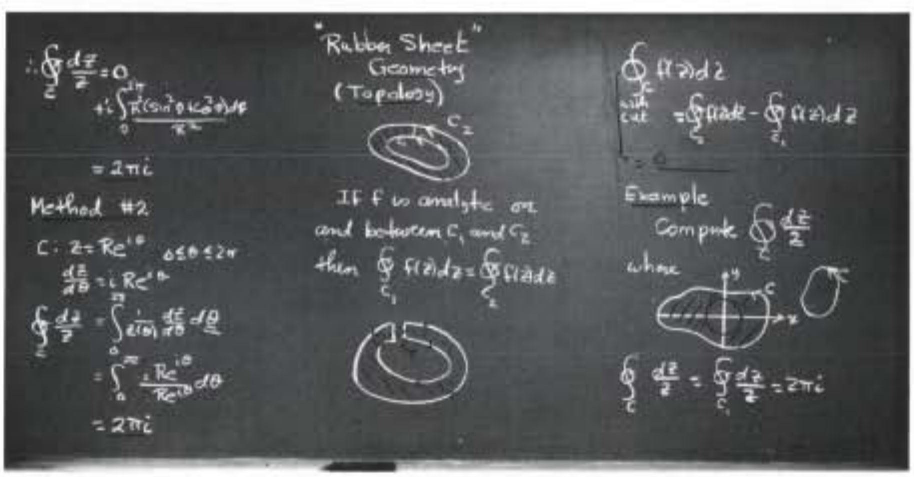

c.

3. Exercises:

1.8.1(L)

a. Compute

$$\int_0^{2i} z\,dz.$$

b. Compute

$$\int_0^{2i} \bar{z}\,dz$$

first along the line segment c_1 which joins 0 to 2i and then along the path c_2 where c_2 is the right half of the circle centered at i with radius 1.

1.8.2

Explain why

$\int_1^{i} 2e^{2z}dz$ is unambiguous, and then find the value of this integral.

1.8.3

Compute

$\int_1^{i} \bar{z}^2 dz$ along each of the following paths, c:

a. c is the line segment which joins 1 to i.

b. $c = \{z: z = e^{i\theta}, 0 \le \theta \le \frac{\pi}{2}\}$ [i.e., c is the first quadrant of the circle $|z| = 1$].

1.8.4(L)

a. Under what conditions is

$$\int_{-1}^{2} \frac{dz}{z^2}$$

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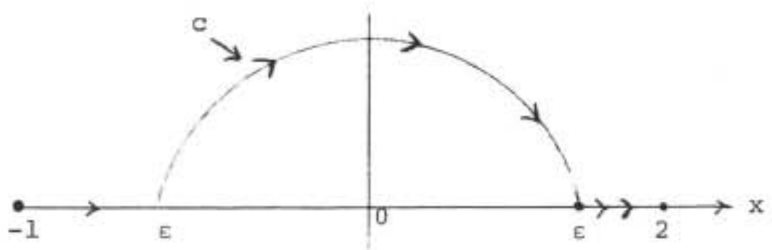
1.8.4(L) continued

unambiguous, and under these conditions what is the value of

$$\int_{-1}^{2} \frac{dz}{z^2}\ ?$$

b. Verify the answer to part (a) along the contour c where



1.8.5

a. Compute

$$P \int_{0}^{2} \frac{dx}{(x-1)^2}$$

b. Compute

$$P \int_{0}^{2} \frac{dx}{(x-1)^4}$$

1.8.6(L)

a. Suppose $|f(z)| \leq M$ for all z on the curve c and that the length of c is L. Prove that

$$\left| \int_{c} f(z)dz \right| \leq ML$$

b. Let $c_R = \{z: z = Re^{i\theta},\ 0 \leq \theta \leq \pi\}$. Use (a) to find an upper bound for

$$\left| \int_{c_R} \frac{e^{iz}}{z^2}\, dz \right|.$$

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1.8.6(L) continued

c. Use (b) to conclude that

$$\lim_{R\to\infty} \int_{c_R} \frac{e^{iz}dz}{z^2} = 0.$$

1.8.7

Show that

$$\left| \int_0^{3+4i} (x^2 + iy^2)dz \right| \leq 5\sqrt{337}$$

if the path is the straight line segment which joins 0 to 3 + 4i.

1.8.8(optional)

In this exercise we would like to show that the analog of

$$F(x) = \int_{x_0}^{x} f(t)dt \to F'(x) = f(x)$$

holds for complex functions as well, provided that f is analytic (in the real case we only had to assume that f was [piecewise] continuous). The exercise affords us a good review of several principles but may be omitted without loss of continuity.

Show that $f(z)$ is analytic in N and $z_0 \in N$, then

$$\frac{d}{dz} \int_{z_0}^{z} f(\zeta)d\zeta = f(z)$$

for any path lying in N which joins z_0 to z.

1.8.9(L)

Suppose $f(z)$ is analytic and that c is any simple closed curve which contains z = a. Let $c_R = \{z: z = a + Re^{i\theta}\}$ where c_R lies within c.

a. Use the fact that $\oint_c \frac{f(z)dz}{z-a} = \oint_{c_R} \frac{f(z)dz}{z-a}$ to prove that

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1.8.9(L) continued

$$\int_c \frac{f(z)dz}{z - a} = i\int_o^{2\pi} f(a + Re^{i\theta})d\theta.$$

b. Use the result of part (a) and let $R \to 0$ to show that

$$f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)dz}{z - a} .$$

c. Use (b) to compute

$$\int_c \frac{e^z dz}{z - i}$$

where c is any simple closed curve which contains $z = i$ in its interior.

d. Let c be any simple closed curve which encloses the origin. Compute

$$\int_c \frac{\sin zdz}{z} .$$

Unit 9: Complex Integration, Part 2

1. Overview

It is not possible to present the complete flavor of complex integration in two units. All we wish to do here is present a little embellishment of the ideas developed in the previous unit so that the interested reader may gain a bit more insight to the importance of complex integration. While many of the results derived in the exercises may seem "artificial" to you, we try to add realism to our presentation by ending the unit with three exercises which show how complex integration may be used to evaluate real integrals.

This unit consists solely of exericses and their solutions, interspersed with some general comments and notes where we feel it important to do so. In no way, however, is this unit meant as a complete study of complex integration. Rather, this unit ends our treatment of complex calculus with the hope that you now have a general overview of complex variables, especially in regard to being able to view complex numbers, functions and calculus as being real.

2. Exercises:

1.9.1(L)

a. Let c be any simple closed curve which contains $z = a$ as an interior point. Prove that for any positive integer n,

$$\oint_c \frac{dz}{(z-a)^n} = \begin{cases} 0, \text{ if } n \neq 1 \\ 2\pi i, \text{ if } n = 1 \end{cases}$$

[Trivially, $\oint_c \frac{dz}{(z-a)^n} = 0$ if c excludes the point $z = a$ since then $\frac{1}{(z-a)^n}$ is analytic in and on c.]

b. Suppose c is any closed curve which contains $z = 0$ as an interior point. Use part (a) and the fact that

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1.9.1(L) continued

$$\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

converges uniformly to sin z, to compute $\oint_c \frac{(\sin z - z^2)dz}{z^6}$.

1.9.2

Expand e^{z^2} in a power series about the origin and use the procedure described in the previous exercise to compute

$$\oint_c \frac{e^{z^2} dz}{z^7}$$

where c is any simple closed curve containing z = 0 as an interior point.

1.9.3 (Optional)

[This exercise involves some "sticky" constructions but proves once and for all the very important result that if f is differentiable at z = a then $f^{(n)}(a)$ exists for any whole number n. If you wish to omit the exercise, you may. You should, however, find some way of remembering the rather interesting result that if f is analytic at z = a, then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_c \frac{f(z)dz}{(z - a)^{n+1}}$$

where c includes z = a in its interior.]

a. By writing $\frac{1}{\zeta - z}$ as

$$\frac{1}{(\zeta - a) - (z - a)} = \frac{1}{\zeta - a}\left[\frac{1}{1 - \frac{z - a}{\zeta - a}}\right]$$

and expanding the bracketed expression as a geometric series,

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1.9.3 continued

start with the fact that $f(z) = \frac{1}{2\pi i} \oint_c \frac{f(\zeta)d\zeta}{\zeta - z}$ where f is analytic in and on c to conclude that

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_c \frac{f(z)dz}{(z - a)^{n+1}}.$$

b. Use part (a) to compute $\frac{1}{2\pi i} \oint_c \frac{(e^z + z^4)dz}{(z - 2\pi i)^4}$ where c encloses $z = 2\pi i$.

1.9.4

Use the result established in part (a) of the previous exercise to evaluate $\oint_c \frac{e^z + \cos z - z^3}{(z - i)^4} dz$ where c is any simple closed curve which contains $z = i$ in its interior.

1.9.5(L)

Suppose c is any simple closed curve which contains $z = i$ and $z = -i$ as interior points. Evaluate $\oint_c \frac{zdz}{z^2 + 1}$.

1.9.6(L)

Let $f(z) = \frac{z^3}{(z - 1)(z^2 + 1)^2}$. Compute $\oint f(z)dz$ around each of the following contours.

a. $c_1 = \{z: z = \frac{1}{2}e^{i\theta}, 0 \leqslant \theta \leqslant 2\pi\}$.

b. c_2 where c_2 is the square with vertices at $(\frac{1}{2},-\frac{1}{2})$, $(\frac{3}{2},-\frac{1}{2})$, $(\frac{3}{2},\frac{1}{2})$, and $(\frac{1}{2},\frac{1}{2})$.

c. $c_3 = \{z: z = i + \frac{3}{4}e^{i\theta}, 0 \leqslant \theta \leqslant 2\pi\}$.

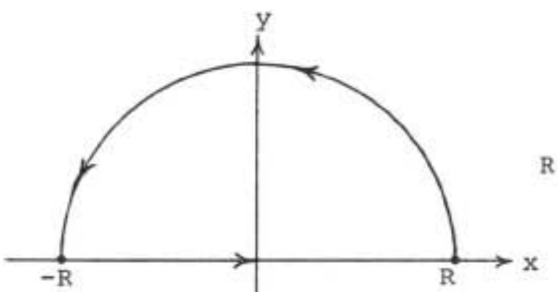
d. c_4 where c_4 is the circle of radius 1 centered at the midpoint of the line segment which joins $z = 1$ to $z = i$.

1.9.7

Compute $\displaystyle\oint_{|z| = 1} \frac{(1 + z)dz}{z(2 - z)}$

1.9.8(L)

Let c be the contour



Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$ by computing $\displaystyle\lim_{R\to\infty} \oint_c \frac{dz}{1 + z^2}$.

[The remaining two exercises are optional since they require a type of computational dexterity with complex variables which in a course as brief as ours may not have given you. Nevertheless, even if you can't do these exercises, it might be of benefit to read the solutions if only to get a feeling for how complex integration is used to evaluate real definite integrals.]

1.9.9 (Optional)

a. Describe all the singularities of $\dfrac{1}{z^6 + 1}$.

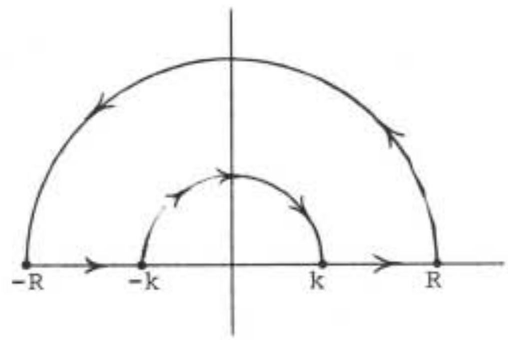
b. Use the fact that the roots of a polynomial equation with real coefficients must occur in pairs of complex conjugates to factor $z^6 + 1$ into the product of three <u>quadratic</u> polynomials, each with real coefficients.

c. Use the technique of the previous exercise to compute $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^6 + 1}$.

[This problem differs from Exercise 1.9.8 only in that it is a bit more difficult computationally.]

1.9.10 (Optional)

Evaluate $\int_0^\infty \frac{\sin x}{x}dx$ (which is an improper, convergent real integral) by computing $\oint_c \frac{e^{iz}dz}{z}$, where



and letting $k \to 0$ and $R \to \infty$.

Quiz

1. Use DeMoivre's Theorem to express $\sin 7\theta$ in terms of $\sin\theta$ and $\cos\theta$.

2. (a) Find the four 4th roots of $\frac{1+i}{\sqrt{2}}$.

   (b) [Optional] Find an integral polynomial equation which contains the four 4th roots of $\frac{1+i}{\sqrt{2}}$ as roots of the equation.

3. (a) Express $\frac{1}{z^2}$ in the form $u(x,y) + iv(x,y)$ where u and v are real-valued functions of x and y.

   (b) Use Cauchy-Riemann conditions to show that $\frac{1}{z^2}$ is analytic except at $z = 0$.

   (c) Let C be any simple curve which connects i and 2i but which does not pass through $z = 0$. Compute

$$\int_C \frac{dz}{z^2}.$$

4. Find an analytic function which has $x^4 - 6x^2y^2 + y^4$ as its real part.

5. Can $x^5y + y^5$ be the real part of an analytic function? Explain.

6. Use the power series expansion for $\sin z$ to compute

$$\int_C \frac{\sin z\, dz}{z^2}$$

where C is any simple closed curve which contains $z = 0$ in its interior.