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Video Course Study Guide

Finite Element Procedures for Solids and Structures–Linear Analysis

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PREFACE

The analysis of complex static and dynamic problems involves in essence three stages: selection of a mathematical model, analysis of the model, and interpretation of the results. During recent years the finite element method implemented on the digital computer has been used successfully in modeling very complex problems in various areas of engineering and has significantly increased the possibilities for safe and cost-effective design. However, the efficient use of the method is only possible if the basic assumptions of the procedures employed are known, and the method can be exercised confidently on the computer.

The objective in this course is to summarize modern and effective finite element procedures for the linear analyses of static and dynamic problems. The material discussed in the lectures includes the basic finite element formulations employed, the effective implementation of these formulations in computer programs, and recommendations on the actual use of the methods in engineering practice. The course is intended for practicing engineers and scientists who want to solve problems using modern and efficient finite element methods.

Finite element procedures for the nonlinear analysis of structures are presented in the follow-up course, Finite Element Procedures for Solids and Structures - Nonlinear Analysis.

In this study guide short descriptions of the lectures and the viewgraphs used in the lecture presentations are given. Below the short description of each lecture, reference is made to the accompanying textbook for the course: Finite Element Procedures in Engineering Analysis, by K.J. Bathe, Prentice-Hall, Inc., 1982.

The textbook sections and examples, listed below the short description of each lecture, provide important reading and study material to the course.

Contents

Lectures

SOME BASIC CONCEPTS OF ENGINEERING ANALYSIS

LECTURE 1

46 MINUTES

LECTURE 1

Introduction to the course, objective of lectures

Some basic concepts of engineering analysis, discrete and continuous systems, problem types: steady-state, propagation and eigenvalue problems

Analysis of discrete systems: example analysis of a spring system

Basic solution requirements

Use and explanation of the modern direct stiffness method

Variational formulation

TEXTBOOK: Sections: 3.1 and 3.2.1, 3.2.2, 3.2.3, 3.2.4

Examples: 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, 3.11, 3.12, 3.13, 3.14

INTRODUCTION TO LINEAR ANALYSIS OF SOLIDS AND STRUCTURES

- The finite element method is now widely used for analysis of structural engineering problems.
- In civil, aeronautical, mechanical, ocean, mining, nuclear, biomechanical,... engineering
- Since the first applications two decades ago,
    - we now see applications in linear, nonlinear, static and dynamic analysis.
    - various computer programs are available and in significant use

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My objective in this set of lectures is:

- to introduce to you finite element methods for the linear analysis of solids and structures.

  ["linear" meaning infinitesimally small displacements and linear elastic material properties (Hooke's law applies)]
- to consider
    - the formulation of the finite element equilibrium equations
    - the calculation of finite element matrices
    - methods for solution of the governing equations
    - computer implementations
- to discuss modern and effective techniques, and their practical usage.

REMARKS

- Emphasis is given to physical explanations rather than mathematical derivations
- Techniques discussed are those employed in the computer programs

  SAP and ADINA

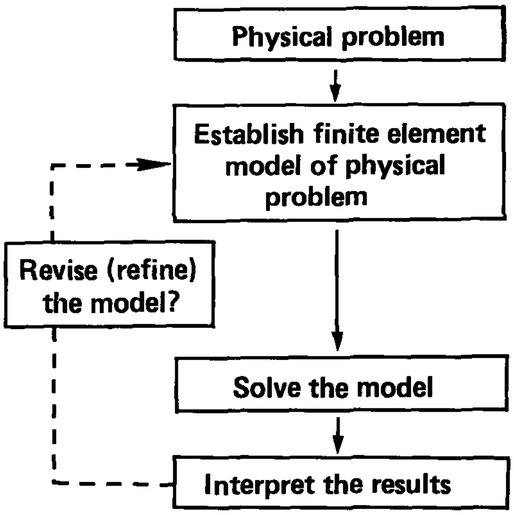
  SAP ≡ Structural Analysis Program

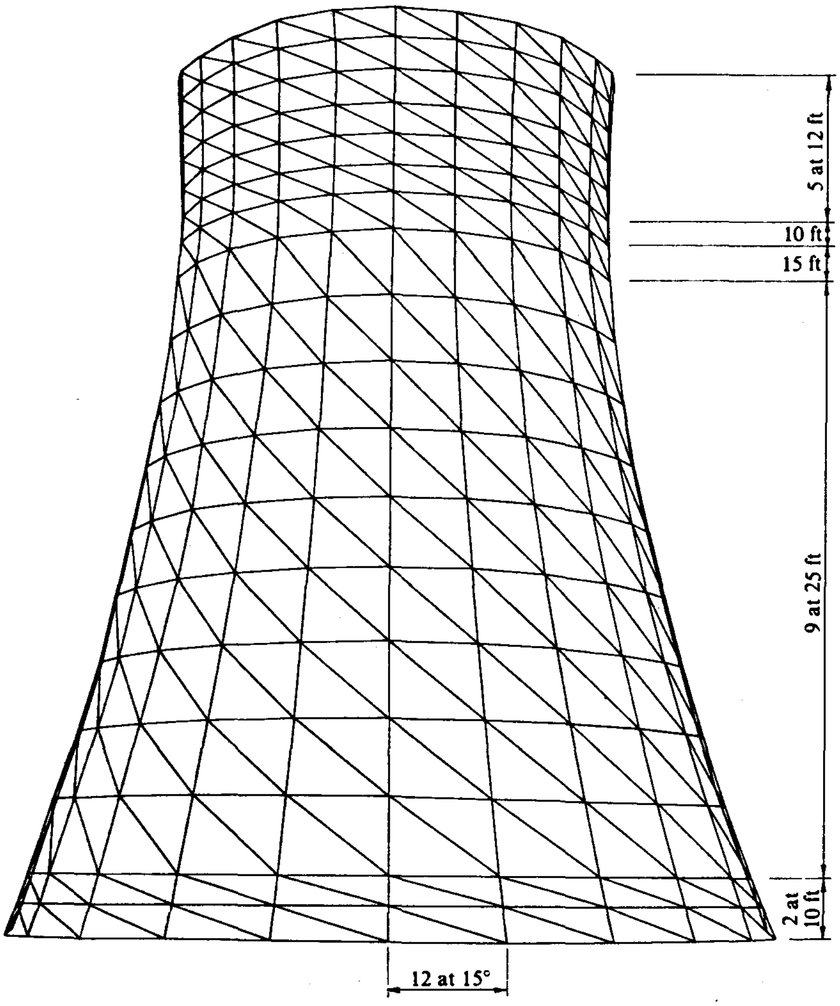
  ADINA ≡ Automatic Dynamic
  Incremental Nonlinear Analysis

- These few lectures represent a very brief and compact introduction to the field of finite element analysis
- We shall follow quite closely certain sections in the book

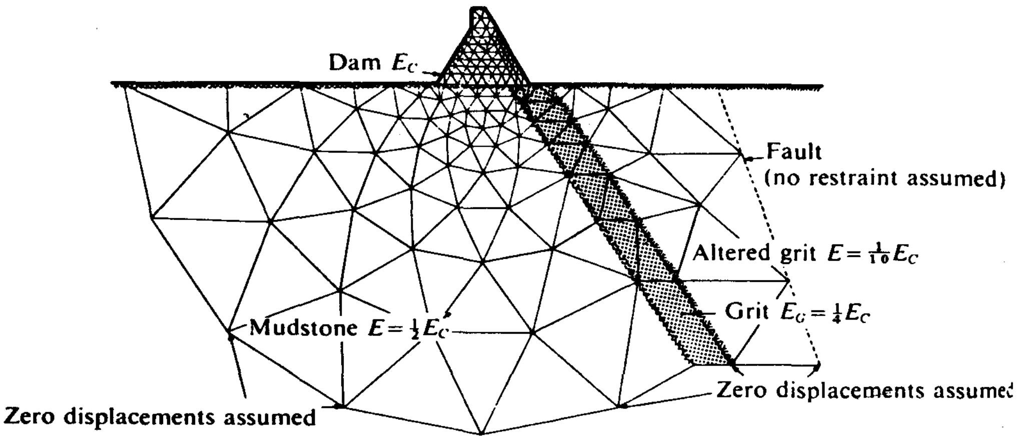
  Finite Element Procedures
  in Engineering Analysis,
  Prentice-Hall, Inc.
  (by K.J. Bathe).

Finite Element Solution Process

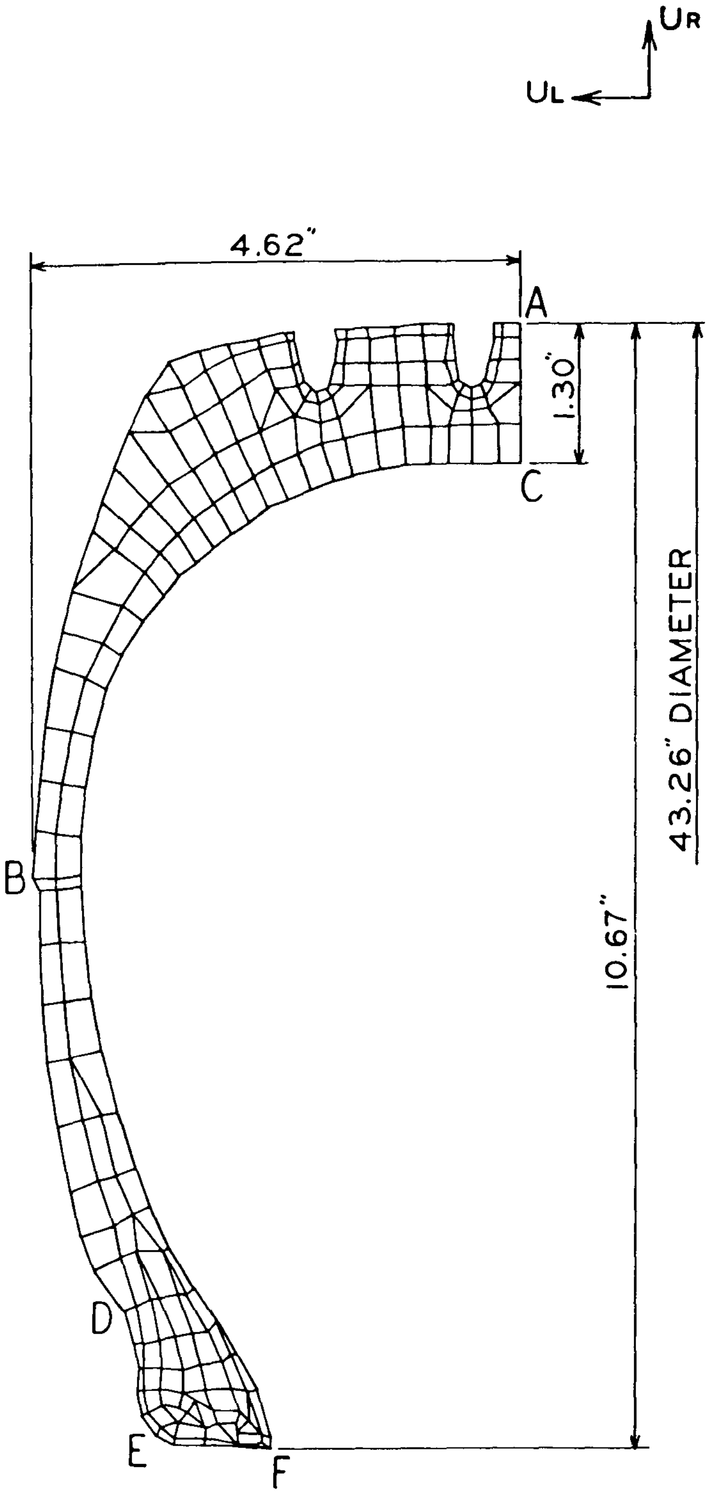




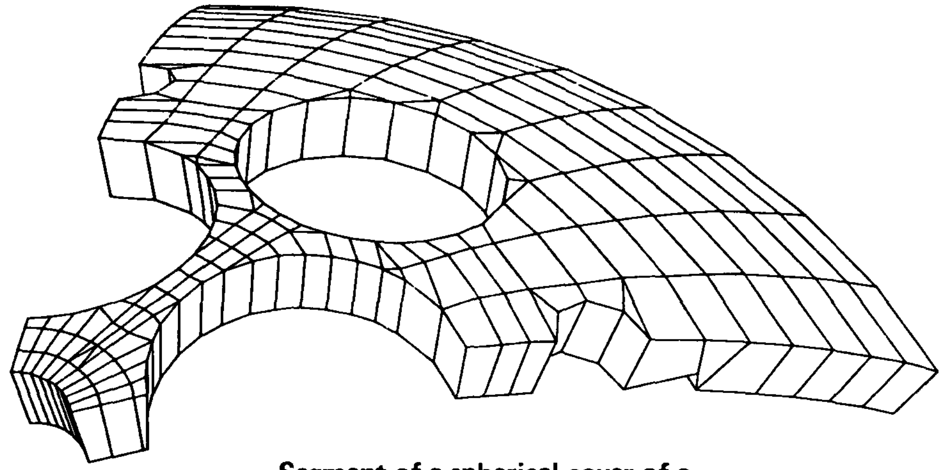
Analysis of cooling tower.



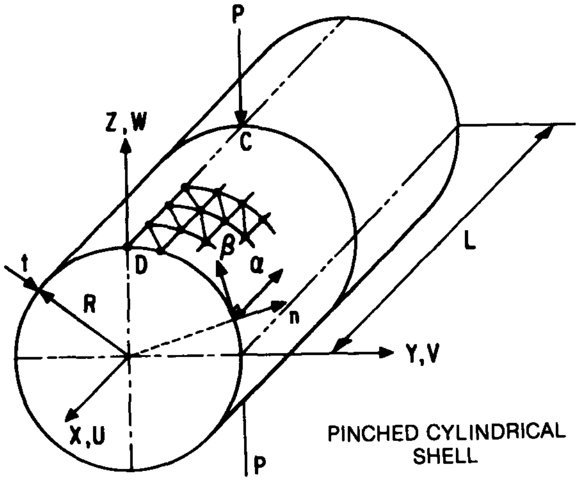
Analysis of dam.



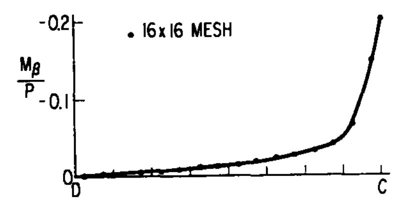
Finite element mesh for tire inflation analysis.



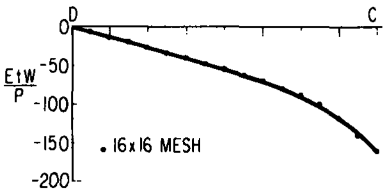
Segment of a spherical cover of a laser vacuum target chamber.



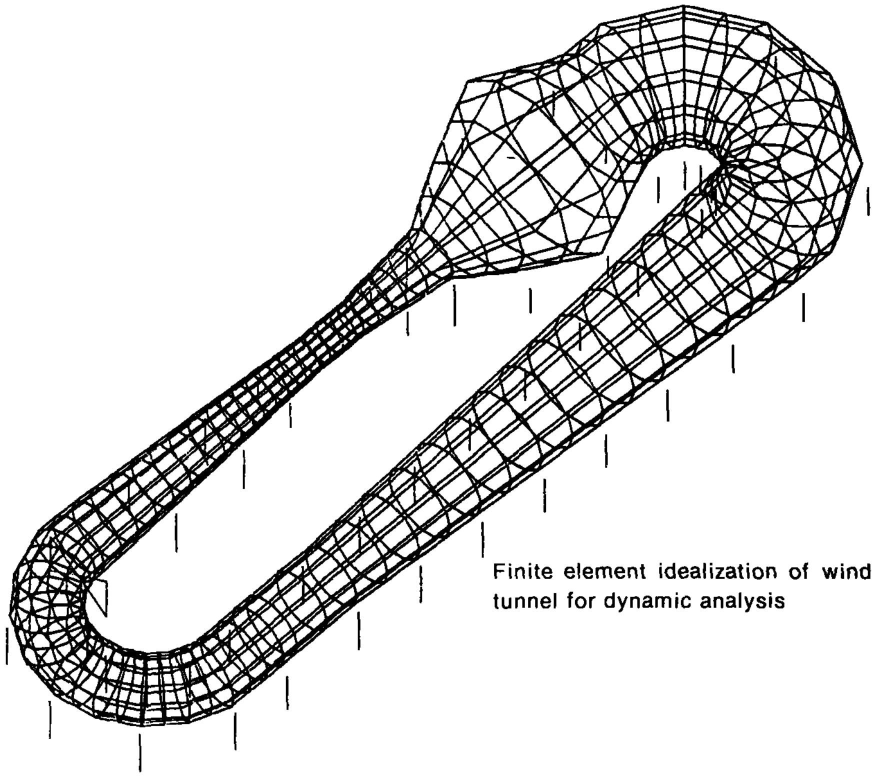
PINCHED CYLINDRICAL SHELL



BENDING MOMENT DISTRIBUTION ALONG DC OF PINCHED CYLINDRICAL SHELL



DISPLACEMENT DISTRIBUTION ALONG DC OF PINCHED CYLINDRICAL SHELL



Finite element idealization of wind tunnel for dynamic analysis

SOME BASIC CONCEPTS OF ENGINEERING ANALYSIS

The analysis of an engineering system requires:

- **idealization of system**
- **formulation of equilibrium equations**
- **solution of equations**
- **interpretation of results**

SYSTEMS

DISCRETE	CONTINUOUS
response is described by variables at a finite number of points	response is described by variables at an infinite number of points
set of algebraic equations	set of differential equations

PROBLEM TYPES ARE

- STEADY-STATE (statics)
- PROPAGATION (dynamics)
- EIGENVALUE

For discrete and continuous systems

Analysis of complex continuous system requires solution of differential equations using numerical procedures

reduction of continuous system to discrete form

powerful mechanism:

the finite element methods, implemented on digital computers

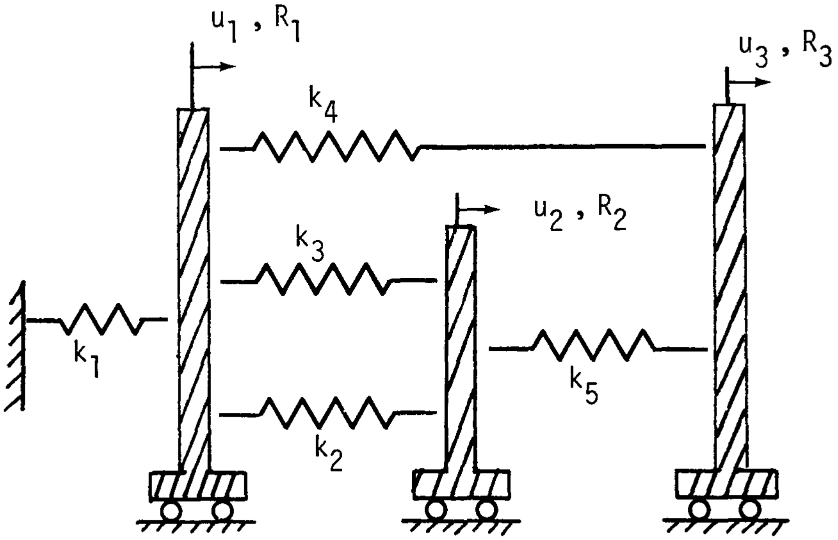
ANALYSIS OF DISCRETE SYSTEMS

Steps involved:

- system idealization into elements
- evaluation of element equilibrium requirements
- element assemblage
- solution of response

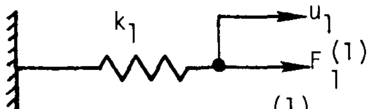
Example:

steady - state analysis of system of rigid carts interconnected by springs

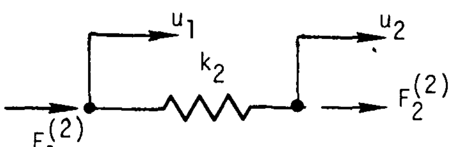


Physical layout

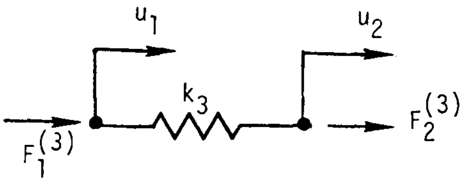
ELEMENTS



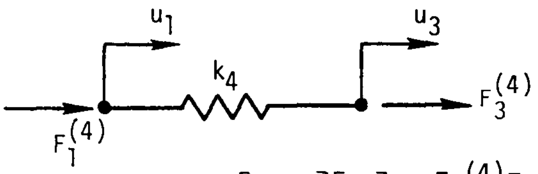
$$k_1 u_1 = F_1^{(1)}$$



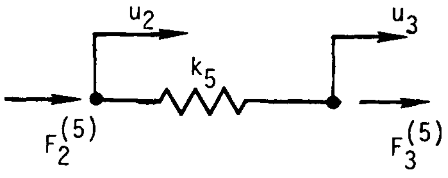
$$k_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1^{(2)} \\ F_2^{(2)} \end{bmatrix}$$



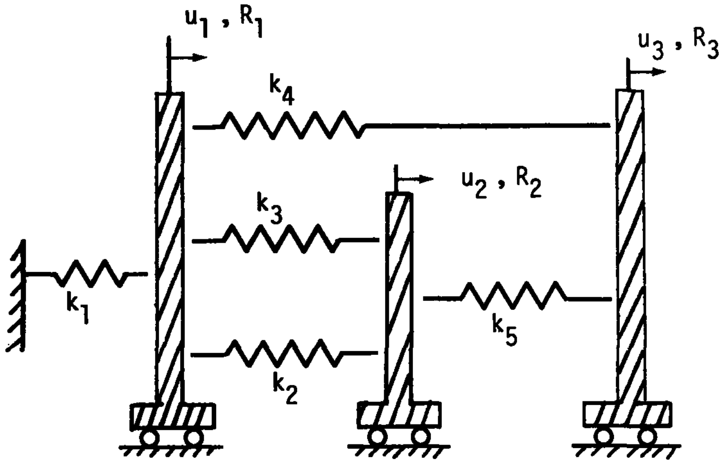
$$k_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1^{(3)} \\ F_2^{(3)} \end{bmatrix}$$



$$k_4 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_1^{(4)} \\ F_3^{(4)} \end{bmatrix}$$



$$k_5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_2^{(5)} \\ F_3^{(5)} \end{bmatrix}$$



Element interconnection requirements :

$$F_1^{(1)} + F_1^{(2)} + F_1^{(3)} + F_1^{(4)} = R_1$$

$$F_2^{(2)} + F_2^{(3)} + F_2^{(5)} = R_2$$

$$F_3^{(4)} + F_3^{(5)} = R_3$$

These equations can be written in the form

$$\underline{K}\ \underline{U} = \underline{R}$$

Equilibrium equations

$$\underline{K}\ \underline{U} = \underline{R} \qquad \text{(a)}$$

$$\underline{U}^T = [u_1 \quad u_2 \quad u_3] \ ;$$

$$\underline{R}^T = [R_1 \quad R_2 \quad R_3]$$

$$\underline{K} = \left[\begin{array}{c:c:c} \begin{matrix} +k_4 \\ k_1 + k_2 + k_3 \end{matrix} & -k_2 - k_3 & -k_4 \\ \hdashline -k_2 - k_3 & k_2 + k_3 + k_5 & -k_5 \\ \hdashline -k_4 & -k_5 & k_4 + k_5 \end{array}\right]$$

and we note that

$$\underline{K} = \sum_{i=1}^{5} \underline{K}^{(i)}$$

where

$$\underline{K}^{(1)} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

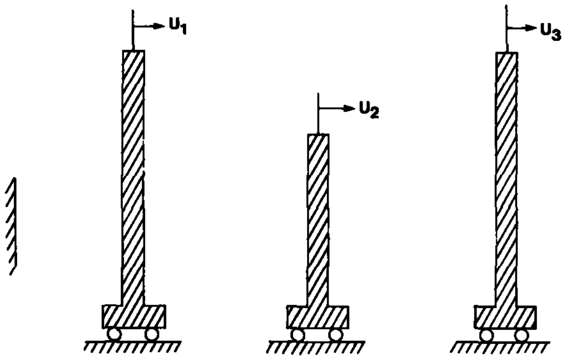
$$\underline{K}^{(2)} = \begin{bmatrix} k_2 & -k_2 & 0 \\ -k_2 & k_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

etc...

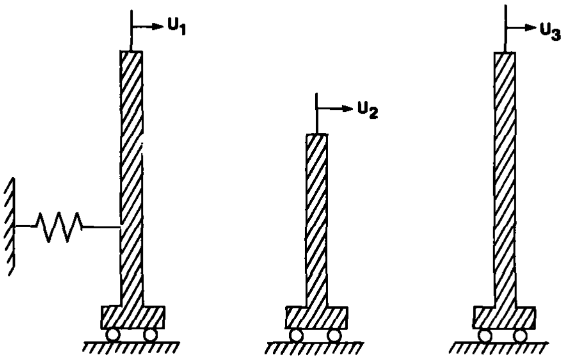
This assemblage process is called the <u>direct stiffness method</u>

The steady-state analysis is completed by solving the equations in (a)

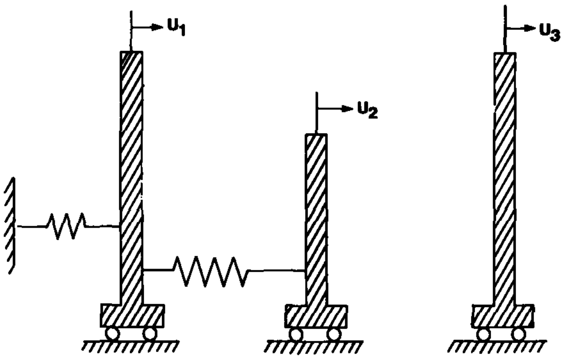
$$\left[\begin{array}{c:c:c} & & \\ \hdashline & & \\ \hdashline & & \end{array}\right]$$



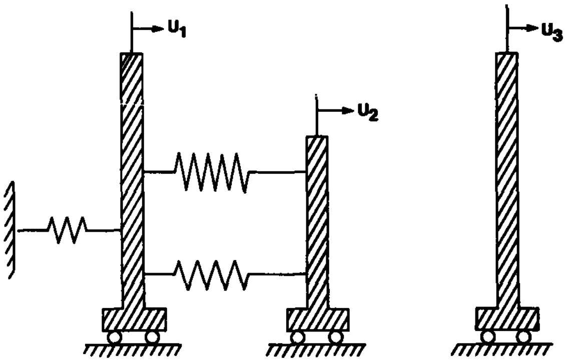
$$\underline{K} = \left[\begin{array}{c:c:c} K_1 & & \\ \hdashline & & \\ \hdashline & & \end{array}\right]$$



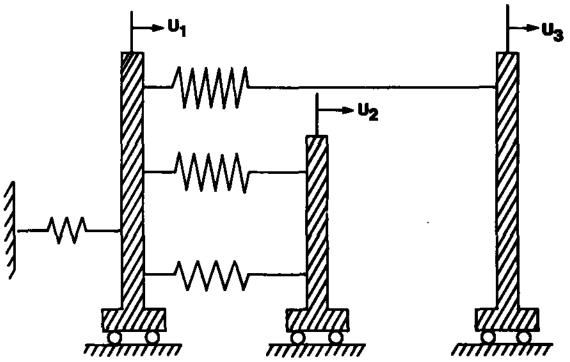
$$\underline{K} = \left[\begin{array}{c:c:c} K_1+K_2 & -K_2 & \\ \hdashline -K_2 & K_2 & \\ \hdashline & & \end{array}\right]$$



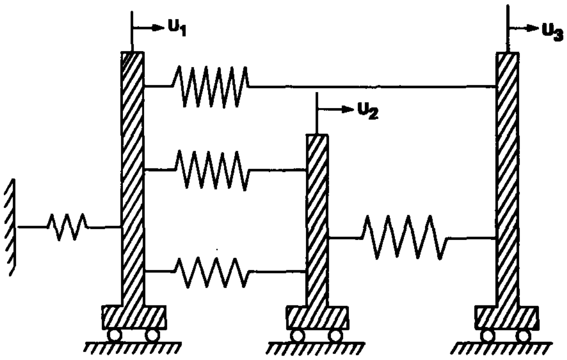
$$\underline{K} = \left[\begin{array}{c:c:c} K_1+K_2+K_3 & -K_2-K_3 & \\ \hdashline -K_2-K_3 & K_2+K_3 & \\ \hdashline & & \end{array}\right]$$



$$\underline{K} = \left[\begin{array}{c:c:c} \begin{array}{r} +K_4 \\ K_1+K_2+K_3 \end{array} & -K_2-K_3 & -K_4 \\ \hdashline -K_2-K_3 & K_2+K_3 & \\ \hdashline -K_4 & & K_4 \end{array}\right]$$



$$\underline{K} = \left[\begin{array}{c:c:c} \begin{array}{r} +K_4 \\ K_1+K_2+K_3 \end{array} & -K_2-K_3 & -K_4 \\ \hdashline -K_2-K_3 & K_2+K_3+K_5 & -K_5 \\ \hdashline -K_4 & -K_5 & K_4+K_5 \end{array}\right]$$



$$\boxed{\underline{K} = \sum_{i=1}^{5} \underline{K}^{(i)}}$$

In this example we used the direct approach; alternatively we could have used a variational approach.

In the variational approach we operate on an extremum formulation:

$$\Pi = \mathcal{U} - \mathcal{W}$$

$\mathcal{U}$ = strain energy of system

$\mathcal{W}$ = total potential of the loads

Equilibrium equations are obtained from

$$\frac{\partial \Pi}{\partial u_i} = 0 \qquad \text{(b)}$$

In the above analysis we have

$$\mathcal{U} = \tfrac{1}{2} \underline{U}^T \underline{K} \underline{U}$$

$$\mathcal{W} = \underline{U}^T \underline{R}$$

Invoking (b) we obtain

$$\underline{K} \underline{U} = \underline{R}$$

Note: to obtain $\mathcal{U}$ and $\mathcal{W}$ we again add the contributions from all elements

PROPAGATION PROBLEMS

main characteristic: the response changes with time ⇒ need to include the d'Alembert forces:

$$\underline{K}\ \underline{U}(t) = \underline{R}(t) - \underline{M}\ \ddot{\underline{U}}(t)$$

For the example:

$$\underline{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

EIGENVALUE PROBLEMS

we are concerned with the generalized eigenvalue problem (EVP)

$$\underline{A}\ \underline{v} = \lambda\ \underline{B}\ \underline{v}$$

$\underline{A}$, $\underline{B}$ are symmetric matrices of order n

$\underline{v}$ is a vector of order n

λ is a scalar

EVPs arise in dynamic and buckling analysis

Example: system of rigid carts

$$\underline{M}\,\underline{\ddot{U}} + \underline{K}\,\underline{U} = \underline{0}$$

Let

$$\underline{U} = \underline{\phi} \sin \omega(t-\tau)$$

Then we obtain

$$-\omega^2\,\underline{M}\,\underline{\phi} \sin \omega(t-\tau)$$

$$+ \underline{K}\,\underline{\phi} \sin \omega(t-\tau) = \underline{0}$$

Hence we obtain the equation

$$\underline{K}\,\underline{\phi} = \omega^2\,\underline{M}\,\underline{\phi}$$

There are 3 solutions

$$\left.\begin{array}{l}\omega_1 , \underline{\phi}_1\\ \omega_2 , \underline{\phi}_2\\ \omega_3 , \underline{\phi}_3\end{array}\right\}\text{ eigenpairs}$$

In general we have n solutions

ANALYSIS OF CONTINUOUS SYSTEMS; DIFFERENTIAL AND VARIATIONAL FORMULATIONS

LECTURE 2

59 MINUTES

LECTURE 2

Basic concepts in the analysis of continuous systems

Differential and variational formulations

Essential and natural boundary conditions

Definition of C^{m-1} variational problem

Principle of virtual displacements

Relation between stationarity of total potential, the principle of virtual displacements, and the differential formulation

Weighted residual methods, Galerkin, least squares methods

Ritz analysis method

Properties of the weighted residual and Ritz methods

Example analysis of a nonuniform bar, solution accuracy, introduction to the finite element method

TEXTBOOK: Sections: 3.3.1, 3.3.2, 3.3.3

Examples: 3.15, 3.16, 3.17, 3.18, 3.19, 3.20, 3.21, 3.22, 3.23, 3.24, 3.25

BASIC CONCEPTS OF FINITE ELEMENT ANALYSIS – CONTINUOUS SYSTEMS

- We discussed some basic concepts of analysis of discrete systems

- Some additional basic concepts are used in analysis of continuous systems

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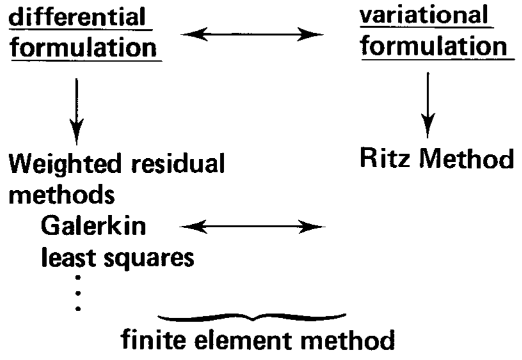
CONTINUOUS SYSTEMS



differential formulation ⟷ variational formulation

↓ Weighted residual methods

Galerkin

least squares

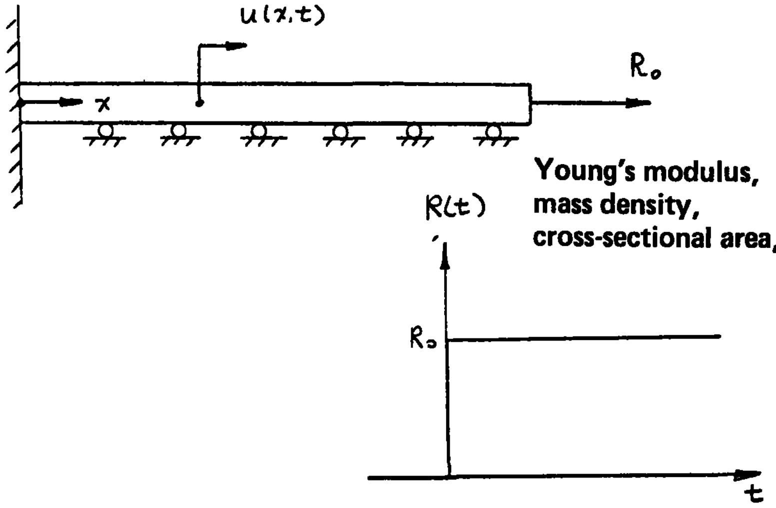
⋮

↓ Ritz Method

Galerkin ⟷ Ritz Method

finite element method

Example - Differential formulation



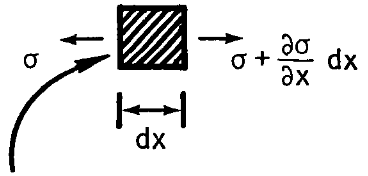
Young's modulus, E
mass density,
cross-sectional area, A

The problem governing differential equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\ , \quad c = \sqrt{\frac{E}{\rho}}$$

Derivation of differential equation

The element force equilibrium requirement of a typical differential element is using d'Alembert's principle



Area A, mass density ρ .

$$\sigma A\big|_x + A \left.\frac{\partial \sigma}{\partial x}\right|_x dx - \sigma A\big|_x = \rho A \frac{\partial^2 u}{\partial t^2}$$

The constitutive relation is

$$\sigma = E\,\frac{\partial u}{\partial x}$$

Combining the two equations above we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

The boundary conditions are

$$u(0,t) = 0 \quad \Rightarrow \text{essential (displ.) B.C.}$$

$$EA \frac{\partial u}{\partial x}(L,t) = R_0 \quad \Rightarrow \text{natural (force) B.C.}$$

with initial conditions

$$u(x,0) = 0$$

$$\frac{\partial u}{\partial t}(x,0) = 0$$

In general, we have

highest order of (spatial) derivatives in problem-governing differential equation is 2m.

highest order of (spatial) derivatives in essential b.c. is (m-1)

highest order of spatial derivatives in natural b.c. is (2m-1)

Definition:

We call this problem a C^{m-1} variational problem.

Example – Variational formulation

We have in general

$$\Pi = \mathcal{U} - \mathcal{W}$$

For the rod

$$\Pi = \int_0^L \frac{1}{2} EA \left(\frac{\partial u}{\partial x}\right)^2 dx - \int_0^L u \, f^B \, dx - u_L \, R$$

and

$$u_0 = 0$$

and we have $\delta \Pi = 0$

The stationary condition $\delta\Pi = 0$ gives

$$\int_0^L \left(EA \frac{\partial u}{\partial x}\right)\left(\delta \frac{\partial u}{\partial x}\right) dx - \int_0^L \delta u \, f^B \, dx - \delta u_L \, R = 0$$

This is the principle of virtual displacements governing the problem. In general, we write this principle as

$$\int_V \delta\underline{\epsilon}^T \underline{\tau} \, dV = \int_V \delta\underline{U}^T \underline{f}^B \, dV + \int_S \delta\underline{U}^{S^T} \underline{f}^S \, dS$$

or

$$\int_V \overline{\underline{\epsilon}}^T \underline{\tau} \, dV = \int_V \overline{\underline{U}}^T \underline{f}^B \, dV + \int_S \overline{\underline{U}}^{S^T} \underline{f}^S \, dS$$

(see also Lecture 3)

However, we can now derive the differential equation of equilibrium and the b.c. at $x = L$.

Writing $\frac{\partial \delta u}{\partial x}$ **for** $\frac{\delta \partial u}{\partial x}$ **, recalling that** EA **is constant and using integration by parts yields**

$$-\int_0^L (EA \frac{\partial^2 u}{\partial x^2} + f^B)\ \delta u\ dx + [EA \frac{\partial u}{\partial x}\Big|_{x=L} - R]\ \delta u_L$$

$$- EA \frac{\partial u}{\partial x}\Big|_{x=0}$$

Since δu_0 **is zero but** δu **is arbitrary at all other points, we must have**

$$EA \frac{\partial^2 u}{\partial x^2} + f^B = 0$$

and

$$EA \frac{\partial u}{\partial x}\Big|_{x=L} = R$$

Also, $f^B = -A\ \rho \frac{\partial^2 u}{\partial t^2}$ **and**

hence we have

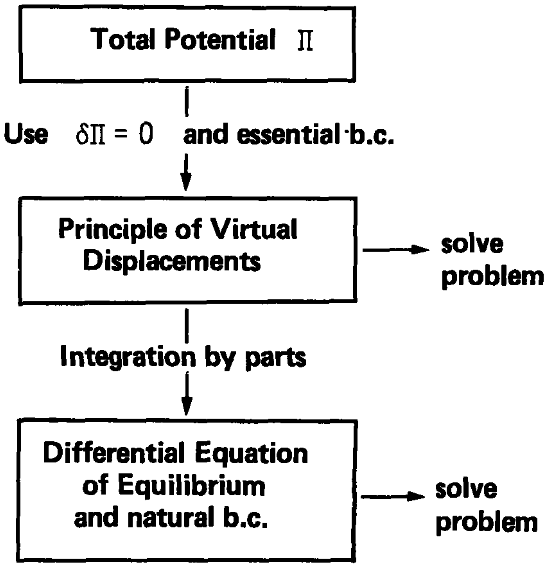
$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}\ ;\ c = \sqrt{\frac{E}{\rho}}$$

The important point is that invoking $\delta \Pi = 0$ and using the essential b.c. only we generate

- the principle of virtual displacements
- the problem-governing differential equation
- the natural b.c. (these are in essence "contained in" Π, i.e., in w).

In the derivation of the problem-governing differential equation we used integration by parts

- the highest spatial derivative in Π is of order m .
- We use integration by parts m-times.



Weighted Residual Methods

Consider the steady-state problem

$$L_{2m}[\phi] = r \qquad (3.6)$$

with the B.C.

$$B_i[\phi] = q_i \Big|_{\text{at boundary}} \qquad i = 1,2,... \qquad (3.7)$$

The basic step in the weighted residual (and the Ritz analysis) is to assume a solution of the form

$$\bar{\phi} = \sum_{i=1}^{n} a_i f_i \qquad (3.10)$$

where the f_i are linearly independent trial functions and the a_i are multipliers that are determined in the analysis.

Using the weighted residual methods, we choose the functions f_i in (3.10) so as to satisfy all boundary conditions in (3.7) and we then calculate the residual,

$$R = r - L_{2m}[\sum_{i=1}^{n} a_i \; f_i] \qquad (3.11)$$

The various weighted residual methods differ in the criterion that they employ to calculate the a_i such that R is small. In all techniques we determine the a_i so as to make a weighted average of R vanish.

Galerkin method

In this technique the parameters a_i are determined from the n equations

$$\int_D f_i \; R \; dD = 0 \qquad i = 1,2,\ldots,n \qquad (3.12)$$

Least squares method

In this technique the integral of the square of the residual is minimized with respect to the parameters a_i ,

$$\frac{\partial}{\partial a_i} \int_D R^2 dD = 0 \qquad i = 1,2,\ldots,n$$

[The methods can be extended to operate also on the natural boundary conditions, if these are not satisfied by the trial functions.]

RITZ ANALYSIS METHOD

Let Π be the functional of the C^{m-1} variational problem that is equivalent to the differential formulation given in (3.6) and (3.7). In the Ritz method we substitute the trial functions $\bar{\Phi}$ given in (3.10) into Π and generate n simultaneous equations for the parameters a_i using the stationary condition on Π ,

$$\frac{\partial \Pi}{\partial a_i} = 0 \qquad i = 1,2,\ldots,n \qquad (3.14)$$

Properties

- The trial functions used in the Ritz analysis need only satisfy the essential b.c.

- Since the application of $\delta\Pi = 0$ generates the principle of virtual displacements, we in effect use this principle in the Ritz analysis.

- By invoking $\delta\Pi = 0$ we minimize the violation of the internal equilibrium requirements and the violation of the natural b.c.

- A symmetric coefficient matrix is generated, of form

$$\underline{K}\,\underline{U} = \underline{R}$$

Example

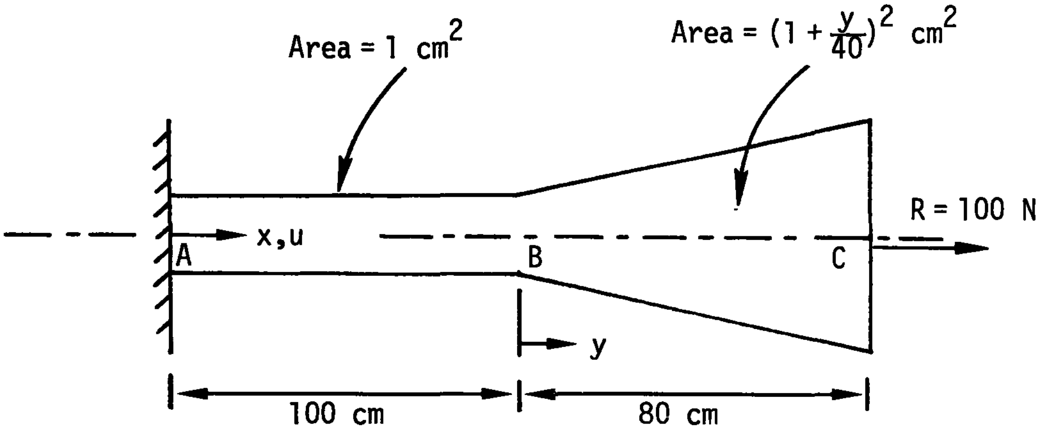


Fig. 3.19. Bar subjected to concentrated end force.

Here we have

$$\Pi = \int_0^{180} \frac{1}{2} EA\left(\frac{\partial u}{\partial x}\right)^2 dx \; - \; 100 \; u\big|_{x = 180}$$

and the essential boundary condition is $u\big|_{x=0} = 0$

Let us assume the displacements

Case 1

$$u = a_1 x + a_2 \, x^2$$

Case 2

$$u = \frac{x \; u_B}{100} \qquad 0 \leq x \leq 100$$

$$u = \left(1 - \frac{x-100}{80}\right) u_B + \left(\frac{x-100}{80}\right) u_C$$

$$100 \leq x \leq 180$$

We note that invoking $\delta\Pi = 0$
we obtain

$$\delta\Pi = \int_0^{180} \left(EA \frac{\partial u}{\partial x}\right) \delta\left(\frac{\partial u}{\partial x}\right) dx - 100 \; \delta u\big|_{x=180} = 0$$

or the principle of virtual displacements

$$\int_0^{180} \left(\frac{\partial \delta u}{\partial x}\right)\left(EA \; \frac{\partial u}{\partial x}\right) dx = 100 \;\; \delta u\big|_{x=180}$$

$$\int_V \underline{\bar{\varepsilon}}^T \; \underline{\tau} \; dV = \overline{U}_i \; F_i$$

Exact Solution

Using integration by parts we obtain

$$\frac{\partial}{\partial x}\left(EA\,\frac{\partial u}{\partial x}\right) = 0$$

$$EA\,\frac{\partial u}{\partial x}\Big|_{x=180} = 100$$

The solution is

$$u = \frac{100}{E}\,x \quad ; \quad 0 \leq x \leq 100$$

$$u = \frac{10000}{E} + \frac{4000}{E} - \frac{4000}{E\left(1 + \frac{x-100}{40}\right)} \quad ;$$

$$100 \leq x \leq 180$$

The stresses in the bar are

$$\sigma = 100 \quad ; \qquad 0 \leq x \leq 100$$

$$\sigma = \frac{100}{\left(1+\frac{x-100}{40}\right)^2} \quad ; \; 100 \leq x \leq 180$$

Performing now the Ritz analysis:

Case 1

$$\Pi = \frac{E}{2}\int_0^{100} (a_1+2a_2x)^2\ dx + \frac{E}{2}\int_{100}^{180} (1+\frac{x-100}{40})^2$$

$$(a_1+2a_2x)^2\ dx - 100\ u\Big|_{x=180}$$

Invoking that $\delta\Pi = 0$ **we obtain**

$$E\begin{bmatrix} 0.4467 & 116 \\ 116 & 34076 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 18 \\ 3240 \end{bmatrix}$$

and

$$a_1 = \frac{128.6}{E}\ ;\quad a_2 = -\frac{0.341}{E}$$

Hence, we have the approximate solution

$$u = \frac{128.6}{E}\ x - \frac{0.341}{E}\ x^2$$

$$\sigma = 128.6 - 0.682\ x$$

Case 2

Here we have

$$\Pi = \frac{E}{2}\int_{0}^{100}\left(\frac{1}{100}u_B\right)^2 dx + \frac{E}{2}\int_{100}^{180}\left(1+\frac{x-100}{40}\right)^2$$

$$\left(-\frac{1}{80}u_B + \frac{1}{80}u_C\right)^2 dx$$

Invoking again $\delta\Pi = 0$ we obtain

$$\frac{E}{240}\begin{bmatrix} 15.4 & -13 \\ -13 & 13 \end{bmatrix}\begin{bmatrix} u_B \\ u_C \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \end{bmatrix}$$

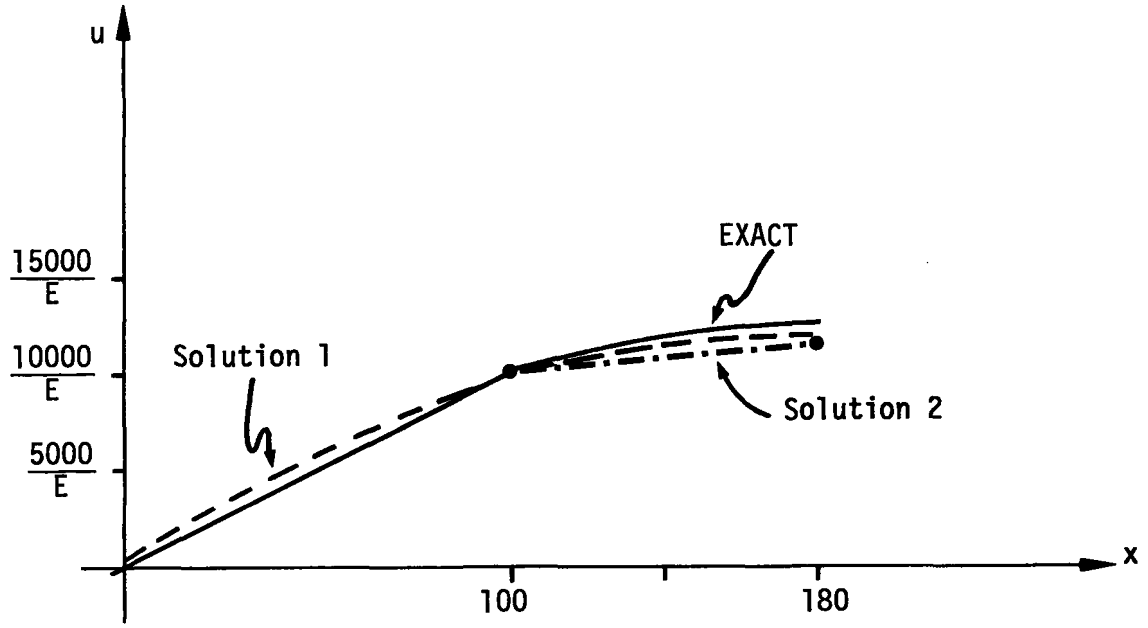
Hence, we now have

$$u_B = \frac{10000}{E} \quad ; \quad u_C = \frac{11846.2}{E}$$

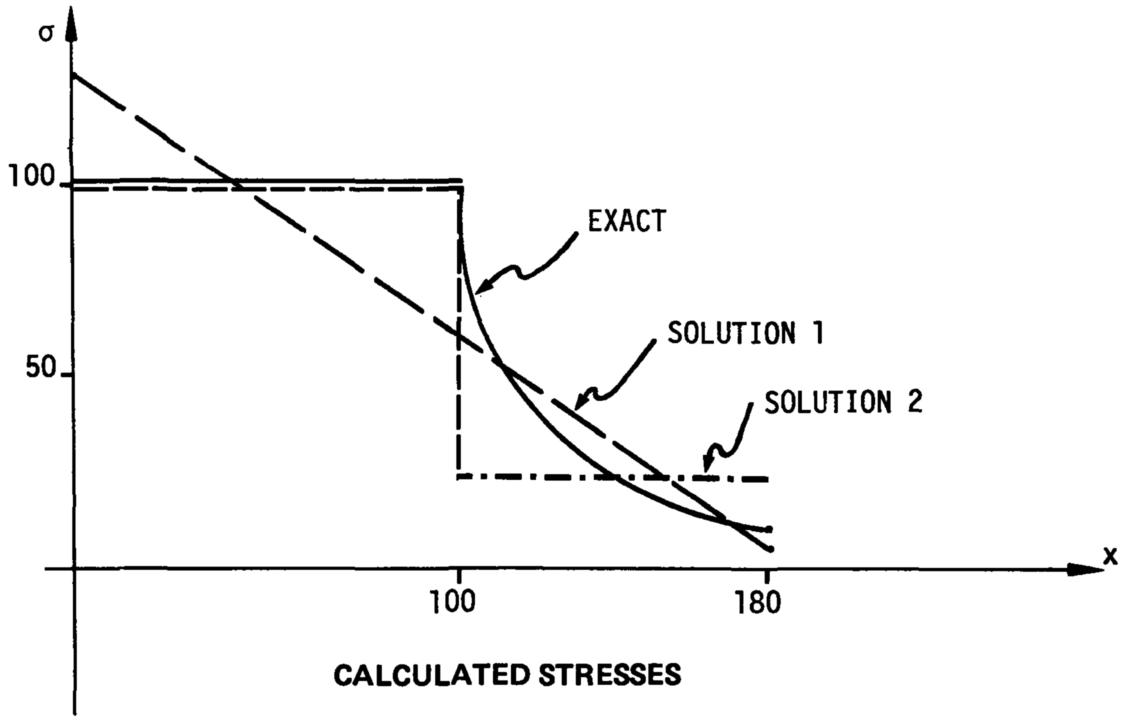
and

$$\sigma = 100 \quad ; \quad 0 \leq x \leq 100$$

$$\sigma = \frac{1846.2}{80} = 23.08 \qquad x \geq 100$$



CALCULATED DISPLACEMENTS



CALCULATED STRESSES

We note that in this last analysis

- we used trial functions that do not satisfy the natural b.c.
- the trial functions themselves are continuous, but the derivatives are discontinuous at point B.
  for a C^{m-1} variational problem we only need continuity in the (m-1)st derivatives of the functions; in this problem $m = 1$.
- domains A - B and B - C are finite elements and
  WE PERFORMED A FINITE ELEMENT ANALYSIS.

FORMULATION OF THE DISPLACEMENT-BASED FINITE ELEMENT METHOD

LECTURE 3

58 MINUTES

LECTURE 3

General effective formulation of the displacement-based finite element method

Principle of virtual displacements

Discussion of various interpolation and element matrices

Physical explanation of derivations and equations

Direct stiffness method

Static and dynamic conditions

Imposition of boundary conditions

Example analysis of a nonuniform bar, detailed discussion of element matrices

TEXTBOOK: Sections: 4.1, 4.2.1, 4.2.2

Examples: 4.1, 4.2, 4.3, 4.4

FORMULATION OF THE DISPLACEMENT-BASED FINITE ELEMENT METHOD

- A very general formulation
- Provides the basis of almost all finite element analyses performed in practice
- The formulation is really a modern application of the Ritz/Galerkin procedures discussed in lecture 2
- Consider static and dynamic conditions, but linear analysis

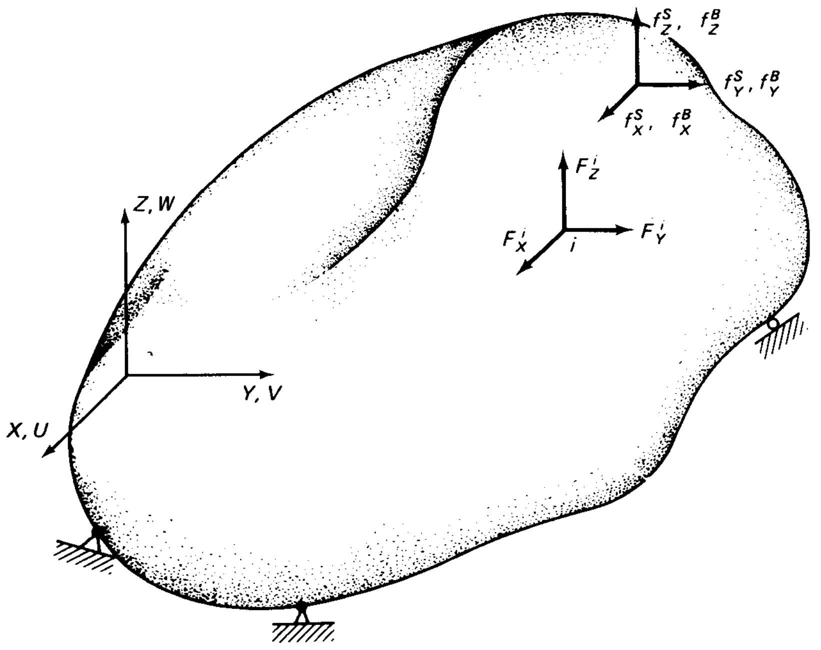


Fig. 4.2. General three-dimensional body.

The external forces are

$$\underline{f}^B = \begin{bmatrix} f_X^B \\ f_Y^B \\ f_Z^B \end{bmatrix} ; \quad \underline{f}^S = \begin{bmatrix} f_X^S \\ f_Y^S \\ f_Z^S \end{bmatrix} ; \quad \underline{F}^i = \begin{bmatrix} F_X^i \\ F_Y^i \\ F_Z^i \end{bmatrix} \qquad (4.1)$$

The displacements of the body from the unloaded configuration are denoted by U , where

$$\underline{U}^T = [U \quad V \quad W] \qquad (4.2)$$

The strains corresponding to U are,

$$\underline{\epsilon}^T = [\epsilon_{XX} \; \epsilon_{YY} \; \epsilon_{ZZ} \; \gamma_{XY} \; \gamma_{YZ} \; \gamma_{ZX}] \qquad (4.3)$$

and the stresses corresponding to ϵ are

$$\underline{\tau}^T = [\tau_{XX} \; \tau_{YY} \; \tau_{ZZ} \; \tau_{XY} \; \tau_{YZ} \; \tau_{ZX}] \qquad (4.4)$$

Principle of virtual displacements

$$\int_V \overline{\underline{\epsilon}}^T \underline{\tau} dV = \int_V \overline{\underline{U}}^T \underline{f}^B dV + \int_S \overline{\underline{U}}^{S^T} \underline{f}^S dS$$

$$+ \sum_i \overline{\underline{U}}^{i^T} \underline{F}^i \qquad (4.5)$$

where

$$\overline{\underline{U}}^T = [\overline{U} \quad \overline{V} \quad \overline{W}] \qquad (4.6)$$

$$\overline{\underline{\epsilon}}^T = [\overline{\epsilon}_{XX} \; \overline{\epsilon}_{YY} \; \overline{\epsilon}_{ZZ} \; \overline{\gamma}_{XY} \; \overline{\gamma}_{YZ} \; \overline{\gamma}_{ZX}] \qquad (4.7)$$

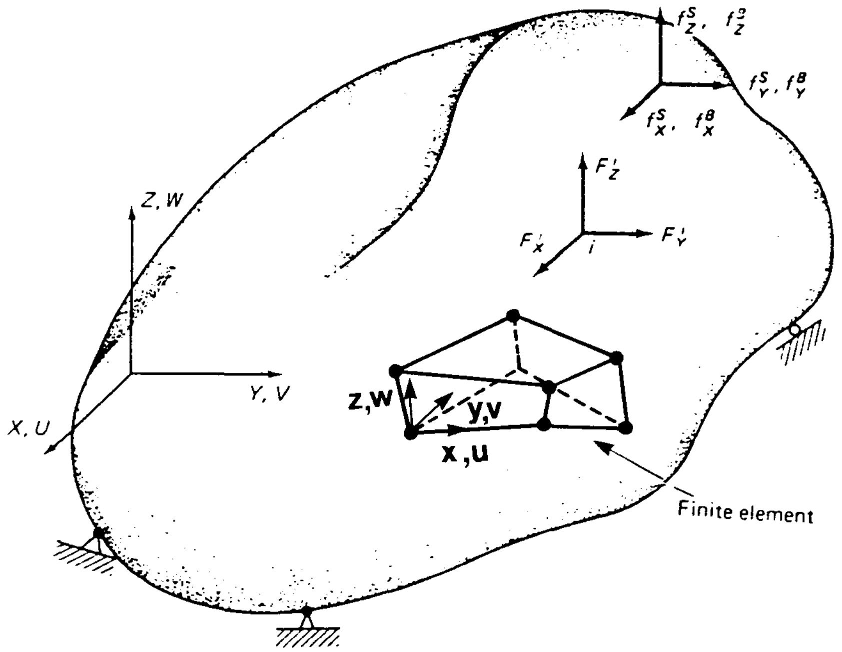
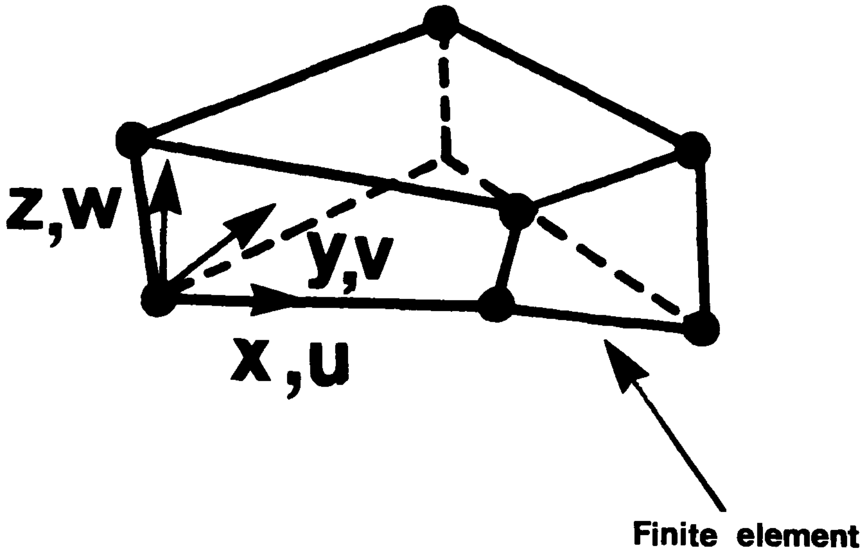


Fig. 4.2. General three-dimensional body.



For element (m) we use:

$$\underline{u}^{(m)}(x, y, z) = \underline{H}^{(m)}(x, y, z)\ \hat{\underline{U}} \tag{4.8}$$

$$\hat{\underline{U}}^T = [U_1 V_1 W_1 \quad U_2 V_2 W_2 \ \ldots \ U_N V_N W_N] \ ;$$

$$\hat{\underline{U}}^T = [U_1 U_2 U_3 \ \ldots \ U_n] \tag{4.9}$$

$$\underline{\epsilon}^{(m)}(x, y, z) = \underline{B}^{(m)}(x, y, z)\ \hat{\underline{U}} \tag{4.10}$$

$$\underline{\tau}^{(m)} = \underline{C}^{(m)} \underline{\epsilon}^{(m)} + \underline{\tau}^{I(m)} \tag{4.11}$$

Rewrite (4.5) as a sum of integrations over the elements

$$\sum_m \int_{V^{(m)}} \bar{\underline{\epsilon}}^{(m)^T} \underline{\tau}^{(m)} dV^{(m)} =$$

$$\sum_m \int_{V^{(m)}} \bar{\underline{U}}^{(m)^T} \underline{f}^{B^{(m)}} dV^{(m)}$$

$$+ \sum_m \int_{S^{(m)}} \bar{\underline{U}}^{S^{(m)T}} \underline{f}^{S^{(m)}} dS^{(m)}$$

$$+ \sum_i \bar{\underline{U}}^{i^T} \underline{F}^i \qquad (4.12)$$

Substitute into (4.12) for the element displacements, strains, and stresses, using (4.8), to (4.10),

$$\cancelto{\underline{I}}{\hat{\bar{\underline{U}}}}^T \left\{ \sum_m \int_{V^{(m)}} \overbrace{\underline{B}^{(m)^T}}^{\bar{\underline{\varepsilon}}^{(m)T}} \underbrace{\underline{C}^{(m)} \underline{B}^{(m)} dV^{(m)}\ \hat{\underline{U}}}_{\underline{\tau}^{(m)}} \right\} =$$

$$\underline{\tau}^{(m)} = \underline{C}^{(m)}\ \underline{\varepsilon}^{(m)}$$

$$(\bar{\underline{\varepsilon}})^{(m)} = \underline{B}^{(m)}\ (\hat{\bar{\underline{U}}})$$

$$\cancelto{\underline{I}}{\hat{\bar{\underline{U}}}}^T \Bigg[\underbrace{\left\{ \sum_m \int_{V^{(m)}} \underline{H}^{(m)^T}}_{\bar{\underline{U}}^{(m)T}} \underline{f}^{B^{(m)}} dV^{(m)} \right\}$$

$$\bar{\underline{u}}^{(m)} = \underline{H}^{(m)}\ \hat{\bar{\underline{U}}}$$

$$+ \underbrace{\sum_m \int_{V^{(m)}} \underline{H}^{S^{(m)T}} \underline{f}}_{\bar{\underline{U}}^{S^{(m)T}}}{}^{S^{(m)}} dS^{(m)} \Big\}$$

$$- \underbrace{\sum_m \int_{V^{(m)}} \underline{B}^{(m)^T}}_{\bar{\underline{\varepsilon}}^{(m)T}} \underline{\tau}^{I^{(m)}} dV^{(m)} \Big\}$$

$$+ \underline{F} \Bigg] \qquad (4.13)$$

We obtain

$$\underline{K}\,\underline{U} = \underline{R} \qquad (4.14)$$

where

$$\underline{R} = \underline{R}_B + \underline{R}_S - \underline{R}_I + \underline{R}_C \qquad (4.15)$$

$$\underline{K} = \sum_m \int_{V^{(m)}} \underline{B}^{(m)^T} \underline{C}^{(m)} \underline{B}^{(m)} dV^{(m)} \qquad (4.16)$$

$$\underline{R}_B = \sum_m \int_{V^{(m)}} \underline{H}^{(m)^T} \underline{f}^{B^{(m)}} dV^{(m)} \qquad (4.17)$$

$$\underline{R}_S = \sum_m \int_{V^{(m)}} \underline{H}^{S\,(m)^T} \underline{f}^{S^{(m)}} dS^{(m)} \qquad (4.18)$$

$$\underline{R}_I = \sum_m \int_{V^{(m)}} \underline{B}^{(m)^T} \underline{\tau}^{I^{(m)}} dV^{(m)} \qquad (4.19)$$

$$\underline{R}_C = \underline{F} \qquad (4.20)$$

In dynamic analysis we have

$$\underline{R}_B = \sum_m \int_{V^{(m)}} \underline{H}^{(m)^T} [\underline{\tilde{f}}^{B^{(m)}} - \rho^{(m)} \underline{H}^{(m)} \underline{\ddot{U}}] dV^{(m)} \qquad (4.21)$$

$$\underline{f}^{B^{(m)}} = \underline{\tilde{f}}^{B^{(m)}} - \rho\,\underline{\ddot{u}}^{(m)}$$

$$\underline{\ddot{u}}^{(m)} = \underline{H}^{(m)}\,\underline{\ddot{U}}$$

$$\underline{M}\,\underline{\ddot{U}} + \underline{K}\,\underline{U} = \underline{R} \qquad (4.22)$$

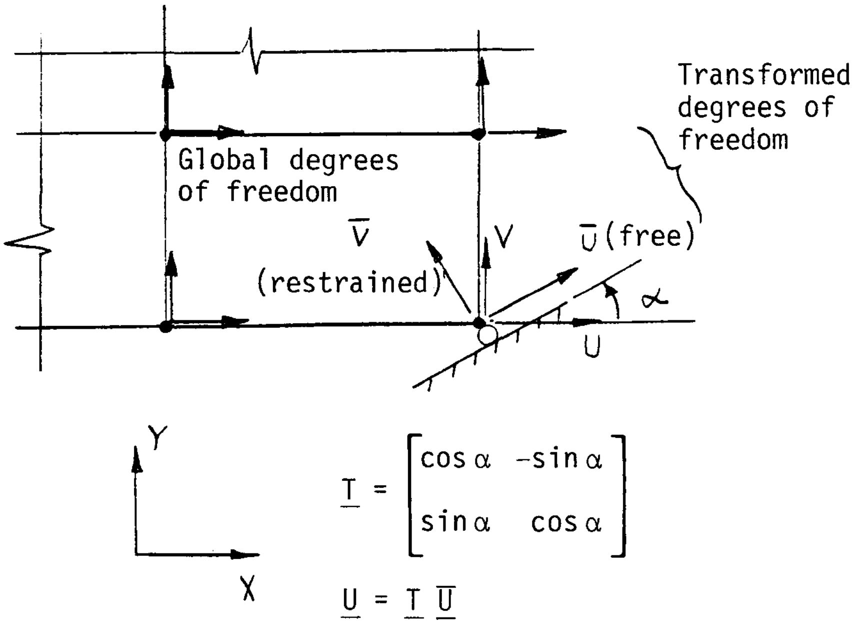
$$\underline{M} = \sum_m \int_{V^{(m)}} \rho^{(m)} \underline{H}^{(m)^T} \underline{H}^{(m)} dV^{(m)} \qquad (4.23)$$

To impose the boundary conditions, we use

$$\begin{bmatrix} \underline{M}_{aa} & \underline{M}_{ab} \\ \underline{M}_{ba} & \underline{M}_{bb} \end{bmatrix} \begin{bmatrix} \ddot{\underline{U}}_{a} \\ \ddot{\underline{U}}_{b} \end{bmatrix} + \begin{bmatrix} \underline{K}_{aa} & \underline{K}_{ab} \\ \underline{K}_{ba} & \underline{K}_{bb} \end{bmatrix} \begin{bmatrix} \underline{U}_{a} \\ \underline{U}_{b} \end{bmatrix} = \begin{bmatrix} \underline{R}_{a} \\ \underline{R}_{b} \end{bmatrix} \qquad (4.38)$$

$$\underline{M}_{aa}\, \ddot{\underline{U}}_{a} + \underline{K}_{aa}\, \underline{U}_{a} = \underline{R}_{a} - \underline{K}_{ab}\, \underline{U}_{b} - \underline{M}_{ab}\, \ddot{\underline{U}}_{b} \qquad (4.39)$$

$$\underline{R}_{b} = \underline{M}_{ba}\, \ddot{\underline{U}}_{a} + \underline{M}_{bb}\, \ddot{\underline{U}}_{b} + \underline{K}_{ba}\, \underline{U}_{a} + \underline{K}_{bb}\, \underline{U}_{b} \qquad (4.40)$$



$$\underline{T} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\underline{U} = \underline{T}\, \bar{\underline{U}}$$

Fig. 4.10. Transformation to skew boundary conditions

For the transformation on the total degrees of freedom we use

$$\underline{U} = \underline{T}\,\underline{\overline{U}} \tag{4.41}$$

so that

$$\underline{\overline{M}}\,\ddot{\underline{\overline{U}}} + \underline{\overline{K}}\,\underline{\overline{U}} = \underline{\overline{R}} \tag{4.42}$$

where

$$\underline{\overline{M}} = \underline{T}^T \underline{M}\,\underline{T}\,;\ \underline{\overline{K}} = \underline{T}^T \underline{K}\,\underline{T}\ ;\ \underline{\overline{R}} = \underline{T}^T \underline{R} \tag{4.43}$$

$$\underline{T} = \begin{array}{c|ccccccccccc|}
 & & & & i^{th}\text{ column} & & & & j^{th} & & & \\
 & 1 & & & & & & & & & & \\
 & & \ddots & & & & & & & & & \\
 & & & 1 & & & & & & & & \\
i^{th}\text{ row} & & & & \cos\alpha & & & & -\sin\alpha & & & \\
 & & & & & 1 & & & & & & \\
 & & & & & & \ddots & & & & & \\
 & & & & & & & 1 & & & & \\
j^{th} & & & & \sin\alpha & & & & \cos\alpha & & & \\
 & & & & & & & & & 1 & & \\
 & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & 1
\end{array}$$

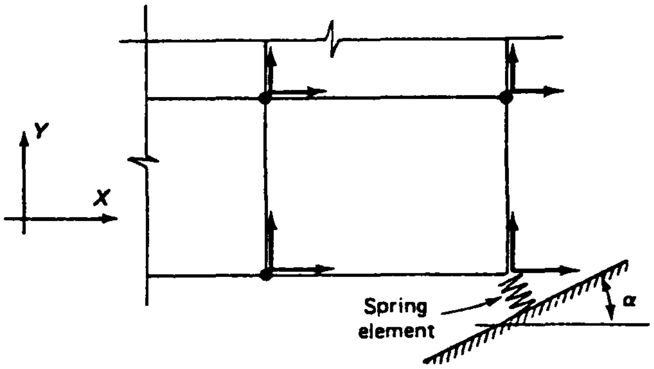


Fig. 4.11. Skew boundary condition imposed using spring element.

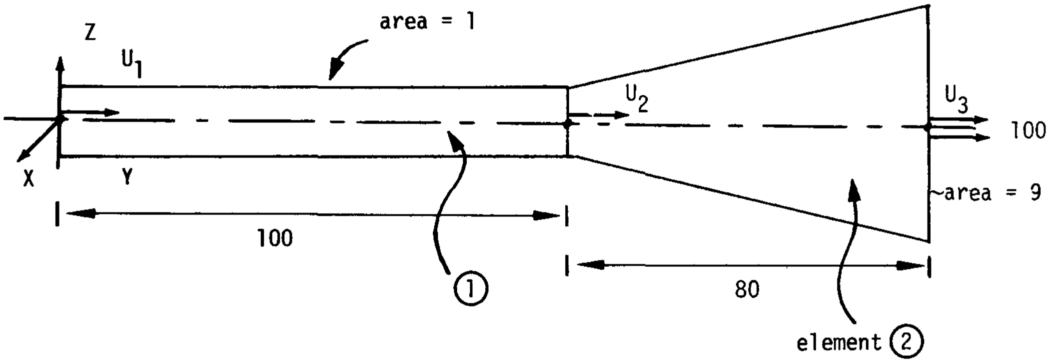
We can now also use this procedure (penalty method)
Say $U_i = b$, then the constraint equation is

$$k\,U_i = k\,b \tag{4.44}$$

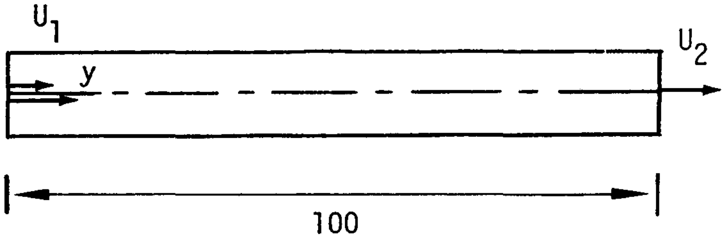
where

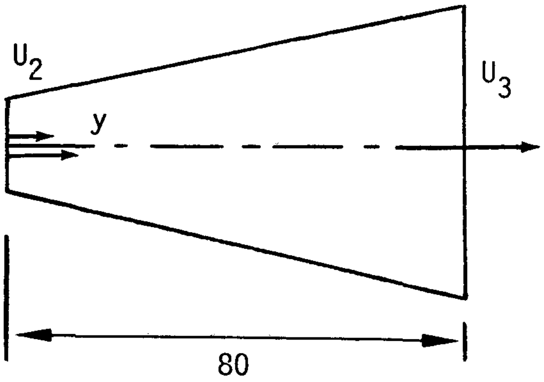
$$k >> \overline{k}_{ii}$$

Example analysis

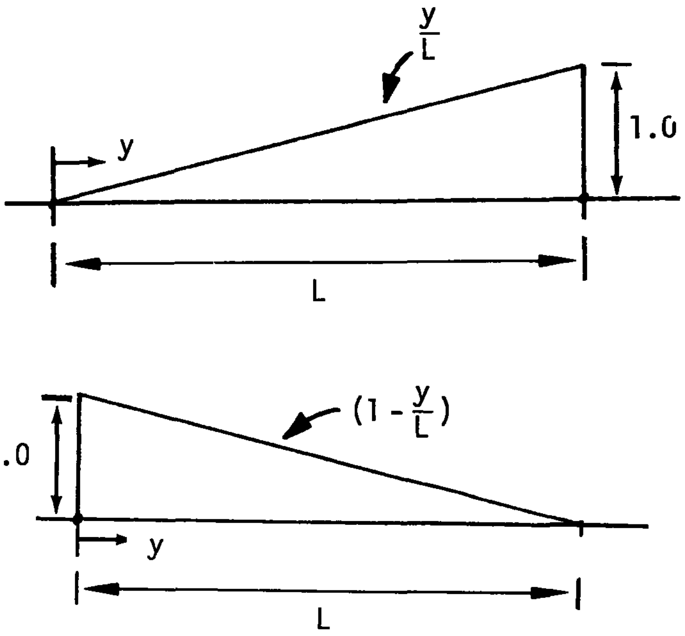


Finite elements





Element interpolation functions



Displacement and strain interpolation matrices:

$$\underline{H}^{(1)} = \left[\left(1 - \frac{y}{100}\right) \quad \frac{y}{100} \quad 0 \right]$$

$$\underline{H}^{(2)} = \left[\quad 0 \quad \left(1 - \frac{y}{80}\right) \quad \frac{y}{80} \right]$$

$$v^{(m)} = \underline{H}^{(m)} \underline{U}$$

$$\underline{B}^{(1)} = \left[\quad -\frac{1}{100} \quad \frac{1}{100} \quad 0 \right]$$

$$\underline{B}^{(2)} = \left[\quad 0 \quad -\frac{1}{80} \quad \frac{1}{80} \right]$$

$$\frac{\partial v}{\partial y} = \underline{B}^{(m)} \underline{U}$$

stiffness matrix

$$\underline{K} = (1)(E)\int_0^{100} \begin{bmatrix} -\frac{1}{100} \\ \frac{1}{100} \\ 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{100} & \frac{1}{100} & 0 \end{bmatrix} dy$$

$$+ E\int_0^{80} (1+\frac{y}{40})^2 \begin{bmatrix} 0 \\ -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{80} & \frac{1}{80} \end{bmatrix} dy$$

Hence

$$\underline{K} = \frac{E}{100} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{13E}{240} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \frac{E}{240} \begin{bmatrix} 2.4 & -2.4 & 0 \\ -2.4 & 15.4 & -13 \\ 0 & -13 & 13 \end{bmatrix}$$

Similarly for $\underline{M}$, $\underline{R}_B$, and so on. Boundary conditions must still be imposed.

GENERALIZED COORDINATE FINITE ELEMENT MODELS

LECTURE 4

57 MINUTES

LECTURE 4 Classification of problems; truss, plane stress, plane strain, axisymmetric, beam, plate and shell conditions; corresponding displacement, strain, and stress variables

Derivation of generalized coordinate models

One-, two-, three- dimensional elements, plate and shell elements

Example analysis of a cantilever plate, detailed derivation of element matrices

Lumped and consistent loading

Example results

Summary of the finite element solution process

Solution errors

Convergence requirements, physical explanations, the patch test

TEXTBOOK: Sections: 4.2.3, 4.2.4, 4.2.5, 4.2.6

Examples: 4.5, 4.6, 4.7, 4.8, 4.11, 4.12, 4.13, 4.14, 4.15, 4.16, 4.17, 4.18

DERIVATION OF SPECIFIC FINITE ELEMENTS

- **Generalized coordinate finite element models**

$$\underline{K}^{(m)} = \int_{V^{(m)}} \underline{B}^{(m)T} \underline{C}^{(m)} \underline{B}^{(m)} dV^{(m)}$$

$$\underline{R}_B^{(m)} = \int_{V^{(m)}} \underline{H}^{(m)T} \underline{f}^{B(m)} dV^{(m)}$$

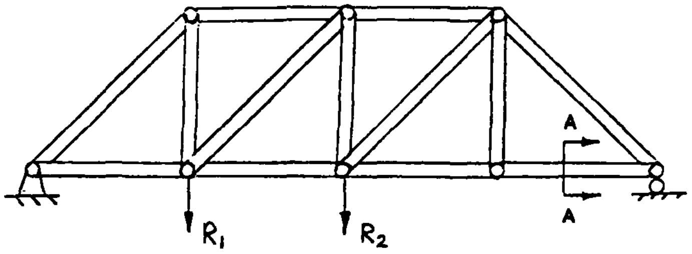
$$\underline{R}_S^{(m)} = \int_{S^{(m)}} \underline{H}^{S(m)T} \underline{f}^{S(m)} dS^{(m)}$$

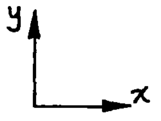
etc.

In essence, we need

$\underline{H}^{(m)}, \underline{B}^{(m)}, \underline{C}^{(m)}$

- **Convergence of analysis results**





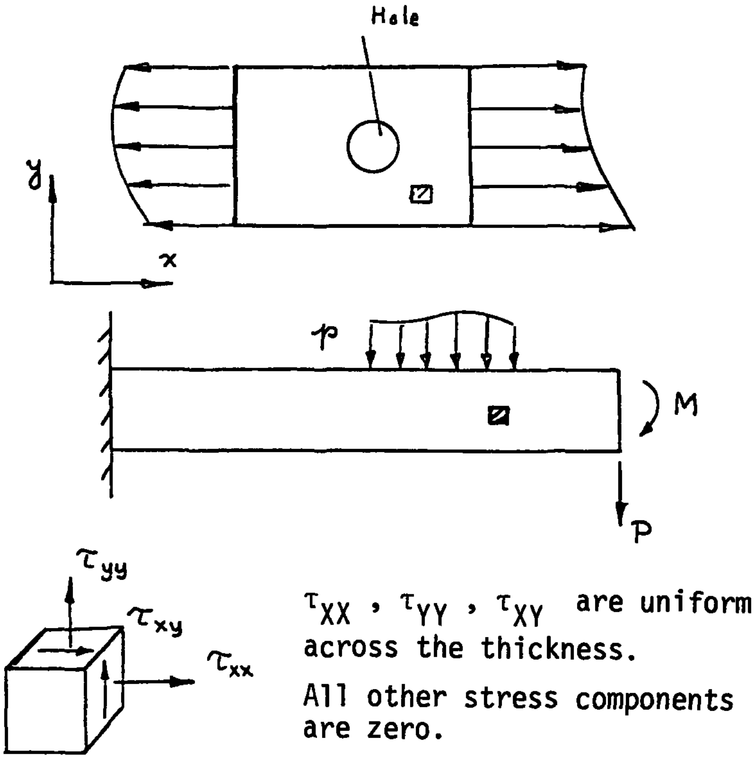
Across section A-A:

τ_{XX} is uniform.

All other stress components are zero.

Fig. 4.14. Various stress and strain conditions with illustrative examples.

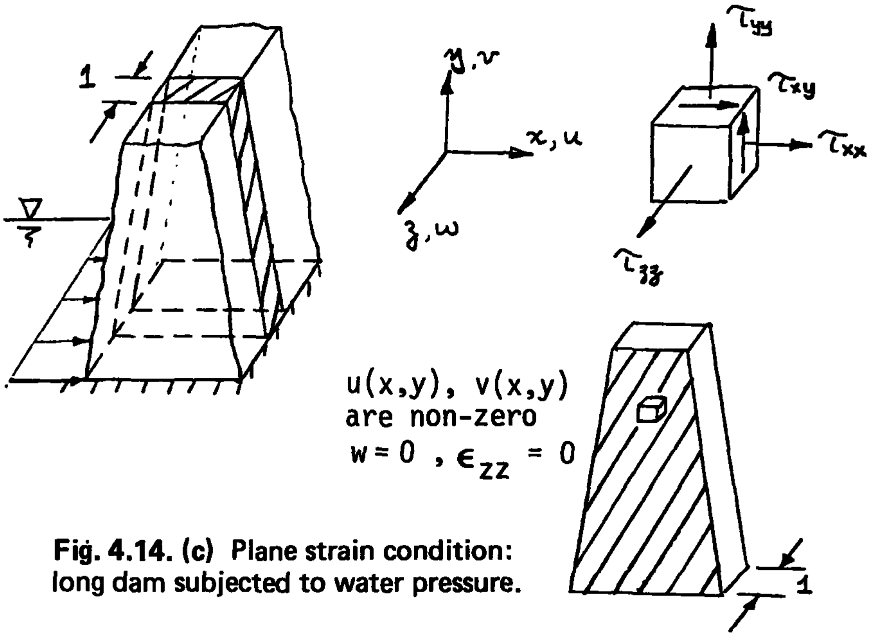
(a) Uniaxial stress condition: frame under concentrated loads.



τ_{XX} , τ_{YY} , τ_{XY} are uniform across the thickness.

All other stress components are zero.

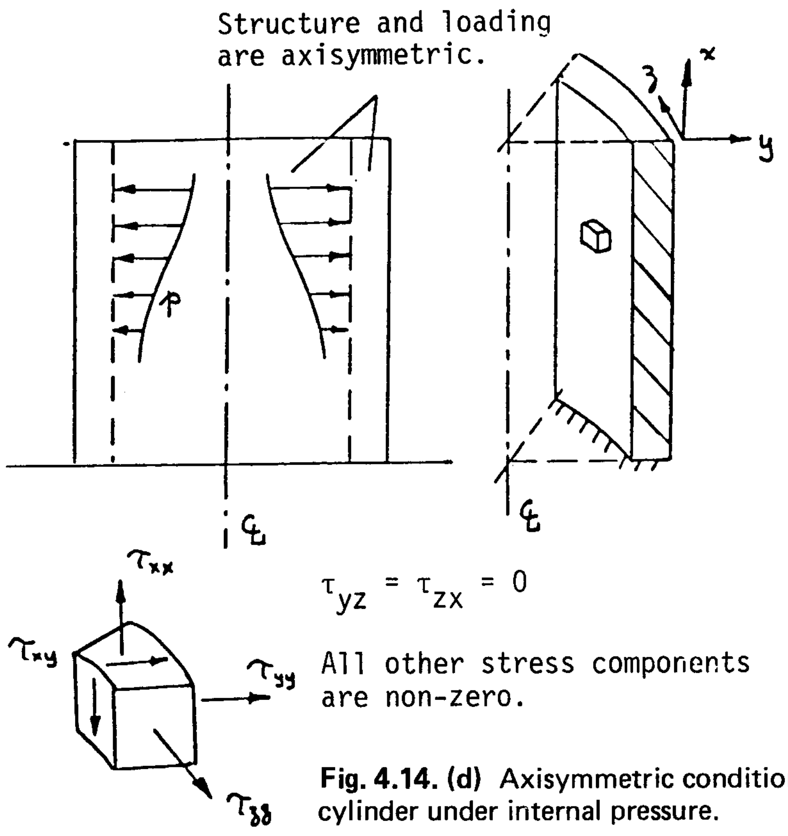
Fig. 4.14. (b) Plane stress conditions: membrane and beam under in-plane actions.



u(x,y), v(x,y) are non-zero

$w = 0$, $\epsilon_{zz} = 0$

Fig. 4.14. (c) Plane strain condition: long dam subjected to water pressure.



Structure and loading are axisymmetric.

$\tau_{yz} = \tau_{zx} = 0$

All other stress components are non-zero.

Fig. 4.14. (d) Axisymmetric condition: cylinder under internal pressure.

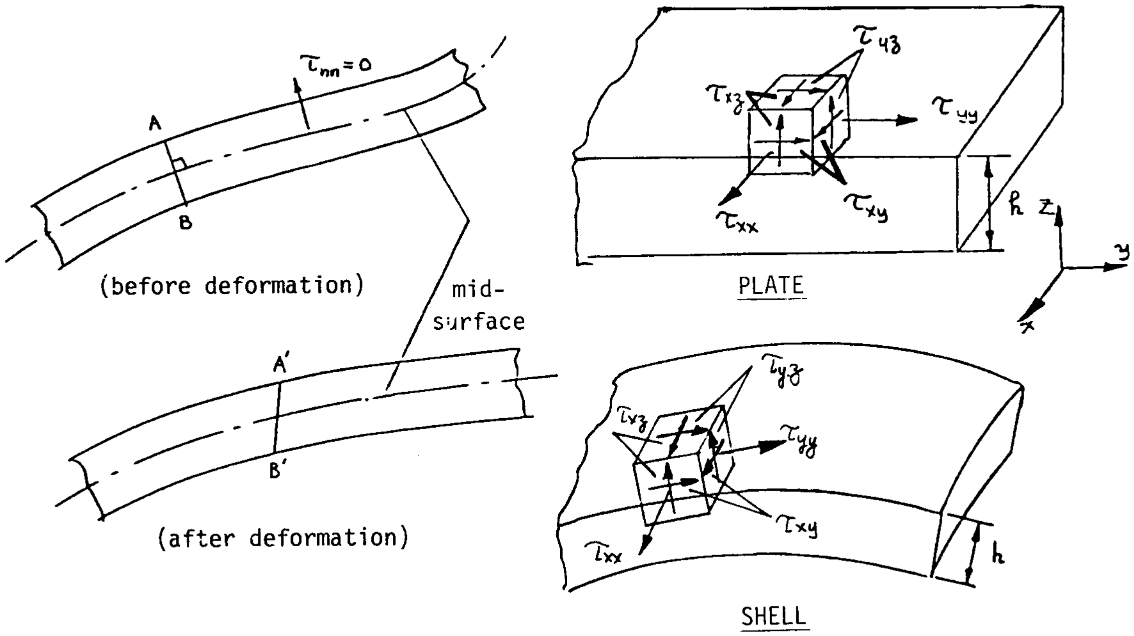


Fig. 4.14. (e) Plate and shell structures.

Problem	Displacement Components
Bar	u
Beam	w
Plane stress	u, v
Plane strain	u, v
Axisymmetric	u, v
Three-dimensional	u, v, w
Plate Bending	w

Table 4.2 (a) Corresponding Kinematic and Static Variables in Various Problems.

Problem	Strain Vector $\underset{\sim}{\epsilon}^T$
Bar	$[\epsilon_{xx}]$
Beam	$[\kappa_{xx}]$
Plane stress	$[\epsilon_{xx} \quad \epsilon_{yy} \quad \gamma_{xy}]$
Plane strain	$[\epsilon_{xx} \quad \epsilon_{yy} \quad \gamma_{xy}]$
Axisymmetric	$[\epsilon_{xx} \quad \epsilon_{yy} \quad \gamma_{xy} \quad \epsilon_{zz}]$
Three-dimensional	$[\epsilon_{xx} \quad \epsilon_{yy} \quad \epsilon_{zz} \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx}]$
Plate Bending	$[\kappa_{xx} \quad \kappa_{yy} \quad \kappa_{xy}]$

Notation: $\epsilon_x = \frac{\partial u}{\partial x}, \epsilon_y = \frac{\partial v}{\partial y}, \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},$

$$\ldots, \kappa_{xx} = -\frac{\partial^2 w}{\partial x^2}, \kappa_{yy} = -\frac{\partial^2 w}{\partial y^2}, \kappa_{xy} = 2\frac{\partial^2 w}{\partial x \, \partial y}$$

Table 4.2 (b) Corresponding Kinematic and Static Variables in Various Problems.

Problem	Stress Vector $\underline{\tau}^T$
Bar	$[\tau_{xx}]$
Beam	$[M_{xx}]$
Plane stress	$[\tau_{xx} \quad \tau_{yy} \quad \tau_{xy}]$
Plane strain	$[\tau_{xx} \quad \tau_{yy} \quad \tau_{xy}]$
Axisymmetric	$[\tau_{xx} \quad \tau_{yy} \quad \tau_{xy} \quad \tau_{zz}]$
Three-dimensional	$[\tau_{xx} \quad \tau_{yy} \quad \tau_{zz} \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{zx}]$
Plate Bending	$[M_{xx} \quad M_{yy} \quad M_{xy}]$

Table 4.2 (c) Corresponding Kinematic and Static Variables in Various Problems.

Problem	Material Matrix $\underline{C}$
Bar	E
Beam	EI
Plane Stress	$\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$

Table 4.3 Generalized Stress-Strain Matrices for Isotropic Materials and the Problems in Table 4.2.

ELEMENT DISPLACEMENT EXPANSIONS:

For one-dimensional bar elements

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \dots \quad (4.46)$$

For two-dimensional elements

$$u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 x^2 + \dots$$

$$v(x,y) = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy + \beta_5 x^2 + \dots$$

(4.47)

For plate bending elements

$$w(x,y) = \gamma_1 + \gamma_2 x + \gamma_3 y + \gamma_4 xy + \gamma_5 x^2 + \dots$$

(4.48)

For three-dimensional solid elements

$$u(x,y,z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 xy + \dots$$

$$v(x,y,z) = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 z + \beta_5 xy + \dots$$

$$w(x,y,z) = \gamma_1 + \gamma_2 x + \gamma_3 y + \gamma_4 z + \gamma_5 xy + \dots$$

(4.49)

Hence, in general

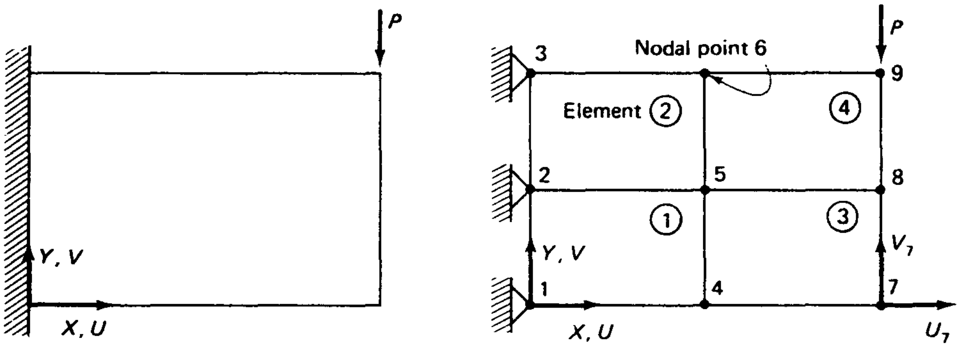
$$\underline{u} = \underline{\Phi}\,\underline{\alpha} \qquad (4.50)$$

$$\underline{\hat{u}} = \underline{A}\,\underline{\alpha}\ ;\ \underline{\alpha} = \underline{A}^{-1}\underline{\hat{u}} \qquad (4.51/52)$$

$$\underline{\epsilon} = \underline{E}\,\underline{\alpha}\ ;\ \underline{\tau} = \underline{C}\,\underline{\epsilon} \qquad (4.53/54)$$

$$\underline{H} = \underline{\Phi}\,\underline{A}^{-1}\ ;\ \underline{B} = \underline{E}\,\underline{A}^{-1} \qquad (4.55)$$

Example



(a) Cantilever plate

(b) Finite element idealization

Fig. 4.5. Finite element plane stress analysis; i.e. $\tau_{ZZ} = \tau_{ZY} = \tau_{ZX} = 0$

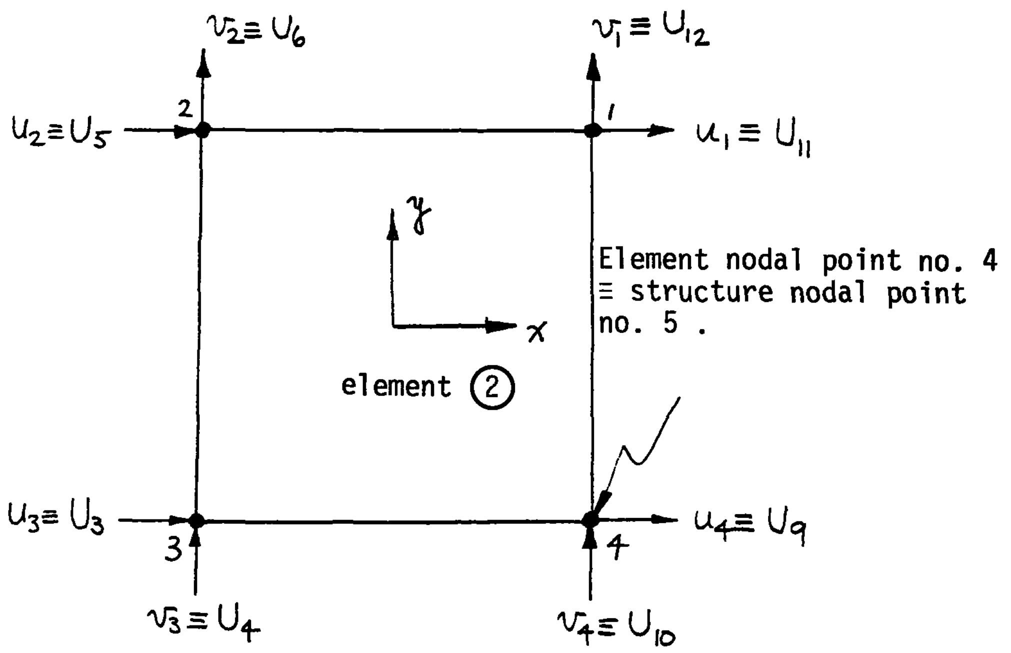


Fig. 4.6. Typical two-dimensional four-node element defined in local coordinate system.

For element 2 we have

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}^{(2)} = \underline{H}^{(2)} \underline{U}$$

where

$$\underline{U}^T = [U_1 \quad U_2 \quad U_3 \quad U_4 \cdots U_{17} \quad U_{18}]$$

To establish $\underline{H}^{(2)}$ we use:

$$u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

$$v(x,y) = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy$$

or

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \underline{\Phi}\ \underline{\alpha}$$

where

$$\underline{\Phi} = \begin{bmatrix} \underline{\phi} & \underline{0} \\ \underline{0} & \underline{\phi} \end{bmatrix}; \ \underline{\phi} = [1 \ \ x \ \ y \ \ xy]$$

and

$$\underline{\alpha}^T = [\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4 \ \beta_1 \ \beta_2 \ \beta_3 \ \beta_4]$$

Defining

$$\hat{\underline{u}}^T = [u_1 \ u_2 \ u_3 \ u_4 \ \ v_1 \ v_2 \ v_3 \ v_4]$$

we have

$$\hat{\underline{u}} = \underline{A}\,\underline{\alpha}$$

Hence

$$\underline{H} = \underline{\Phi}\,\underline{A}^{-1}$$

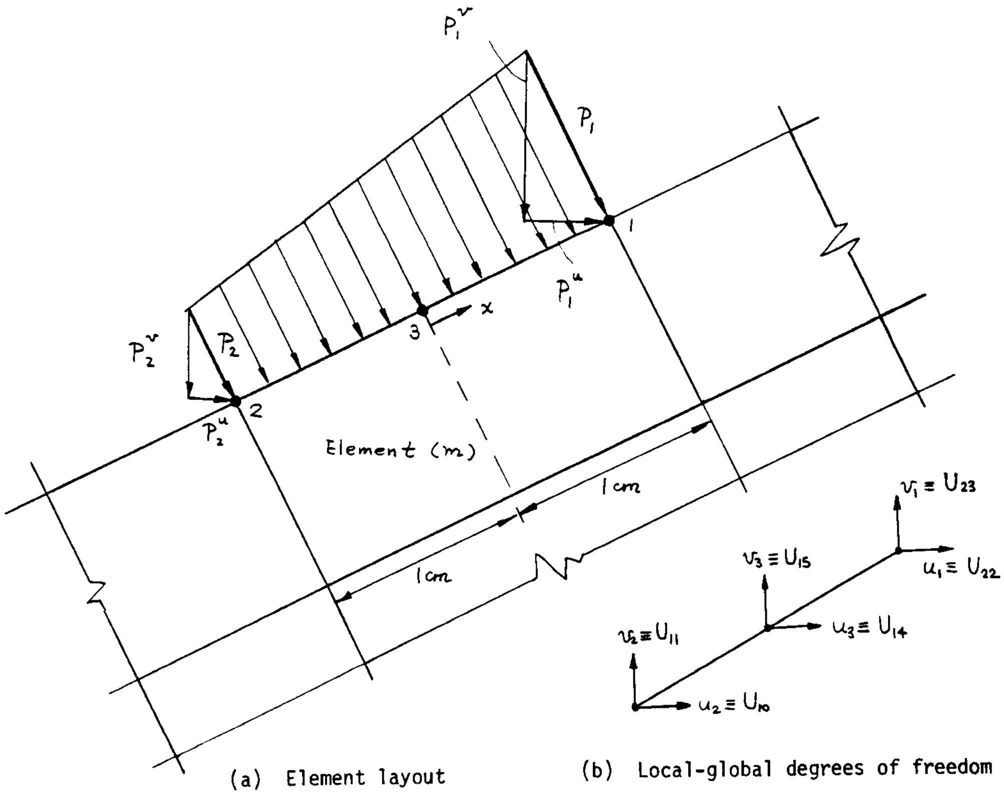
Hence

$$\underline{H} = \frac{1}{4}\left[\begin{array}{c:c:c:}
(1+x)(1+y) & \ldots & 0 \\
0 & & (1+x)(1+y)
\end{array}\right]_{2\times8}$$

and

$$\begin{array}{ccc:cc:cc:cc:cc:}
 & & & u_3 & v_3 & u_2 & v_2 & & & u_4 & v_4 \\
 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} \\
\underline{\mathbf{H}}^{(2)} = \Bigg[& 0 & 0 & H_{13} & H_{17} & H_{12} & H_{16} & 0 & 0 & H_{14} & H_{18} \\
 & 0 & 0 & H_{23} & H_{27} & H_{22} & H_{26} & 0 & 0 & H_{24} & H_{28}
\end{array}$$

$$\begin{array}{:cc:cc:cc}
u_1 & v_1 \leftarrow \text{element degrees of freedom} & & & & \\
U_{11} & U_{12} & U_{13} & U_{14} & & U_{18} \leftarrow \text{assemblage degrees of freedom} \\
H_{11} & H_{15} & 0 & 0 & \ldots\text{zeros}\ldots & 0 \\
H_{21} & H_{25} & 0 & 0 & \ldots\text{zeros}\ldots & 0
\end{array}\Bigg]_{2\times18}$$



(a) Element layout (b) Local-global degrees of freedom

Fig. 4.7. Pressure loading on element (m)

In plane-stress conditions the element strains are

$$\underline{\epsilon}^T = [\epsilon_{XX} \quad \epsilon_{YY} \quad \gamma_{XY}]$$

where

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \; ; \; \epsilon_{yy} = \frac{\partial v}{\partial y} \; ; \; \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Hence

$$\underline{B} = \underline{E}\,\underline{A}^{-1}$$

where

$$\underline{E} = \left[\begin{array}{cccc|cccc} 0 & 1 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 1 & x & 0 & 1 & 0 & y \end{array}\right]$$

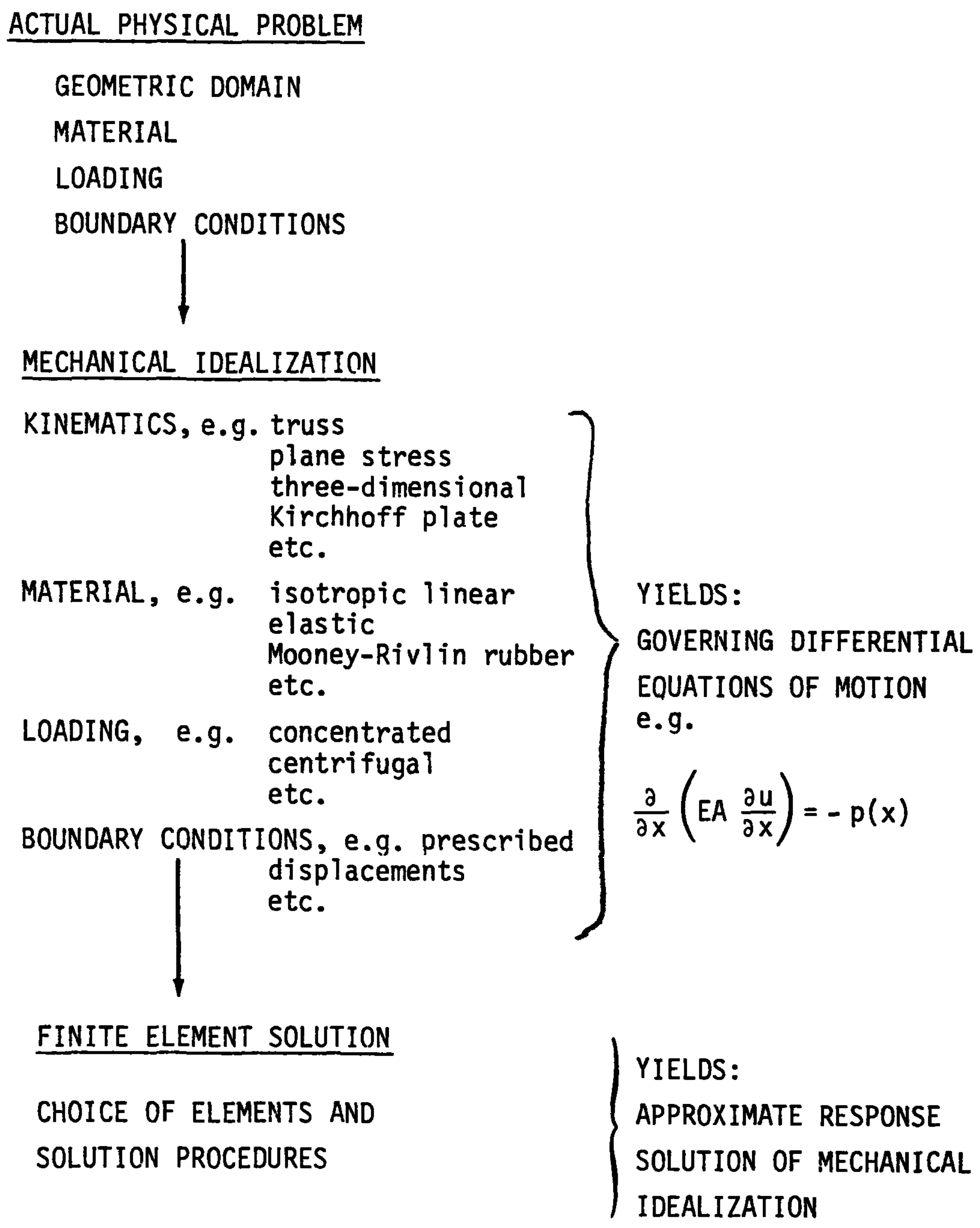
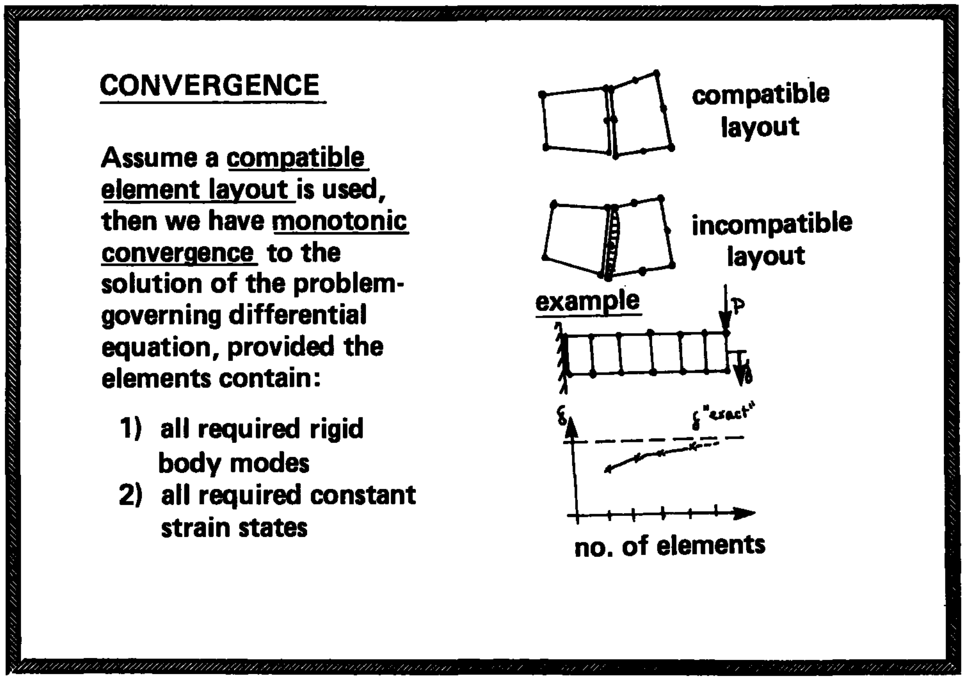


Fig. 4.23. Finite Element Solution Process

ERROR	ERROR OCCURRENCE IN	SECTION discussing error
DISCRETIZATION	use of finite element interpolations	4.2.5
NUMERICAL INTEGRATION IN SPACE	evaluation of finite element matrices using numerical integration	5.8.1 6.5.3
EVALUATION OF CONSTITUTIVE RELATIONS	use of nonlinear material models	6.4.2
SOLUTION OF DYNAMIC EQUILIBRIUM EQUATIONS	direct time integration, mode superposition	9.2 9.4
SOLUTION OF FINITE ELEMENT EQUATIONS BY ITERATION	Gauss-Seidel, Newton-Raphson, Quasi-Newton methods, eigensolutions	8.4 8.6 9.5 10.4
ROUND-OFF	setting-up equations and their solution	8.5

Table 4.4 Finite Element Solution Errors



CONVERGENCE

Assume a compatible element layout is used, then we have monotonic convergence to the solution of the problem-governing differential equation, provided the elements contain:

1) all required rigid body modes
2) all required constant strain states

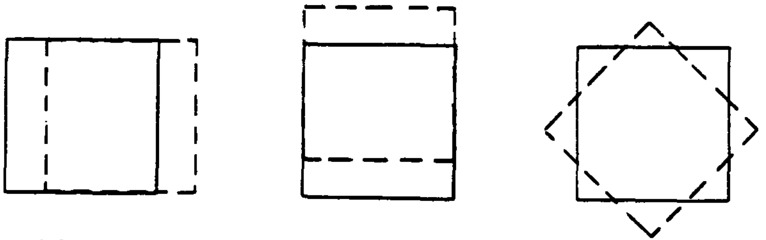
compatible layout

incompatible layout

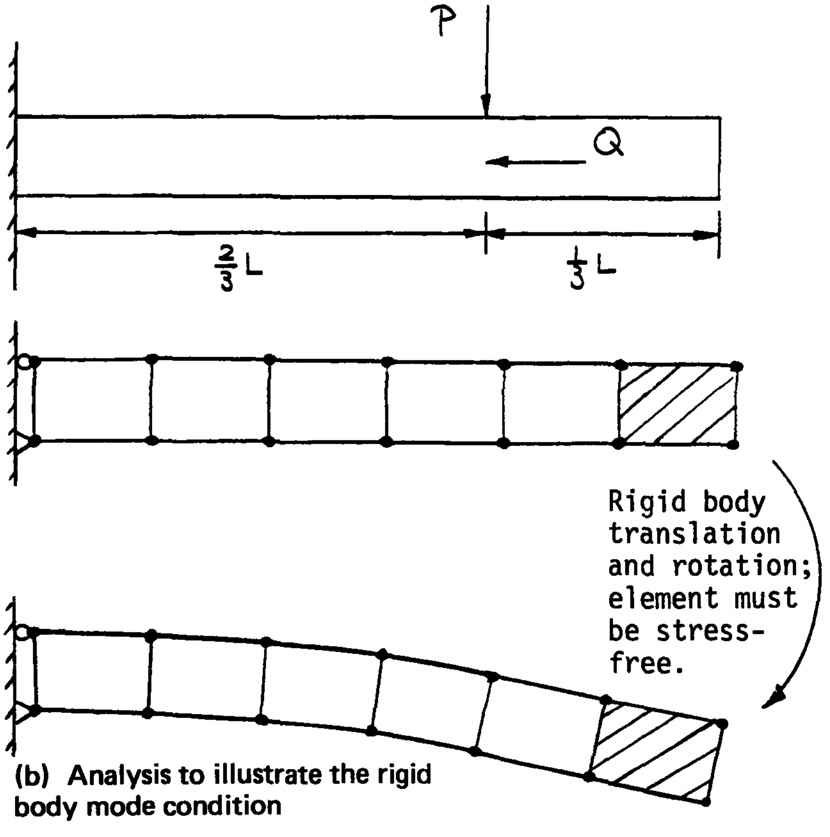
example

no. of elements

If an incompatible element layout is used, then in addition every patch of elements must be able to represent the constant strain states. Then we have convergence but non-monotonic convergence.

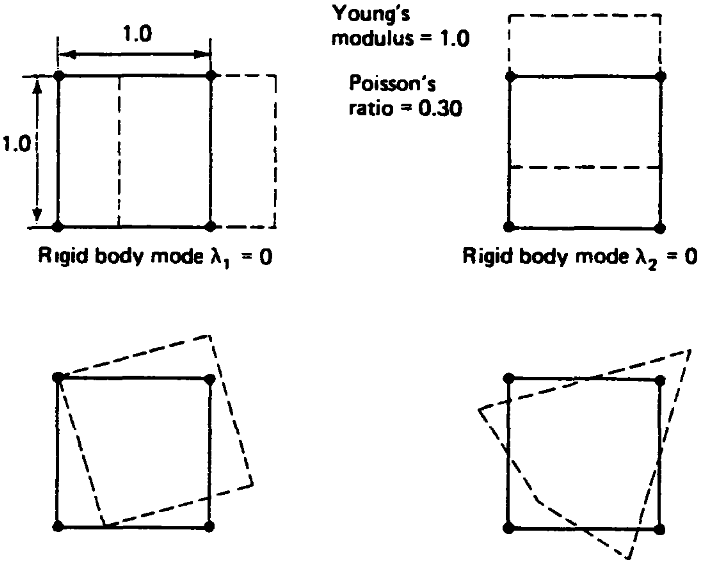


(a) Rigid body modes of a plane stress element



(b) Analysis to illustrate the rigid body mode condition

Fig. 4.24. Use of plane stress element in analysis of cantilever



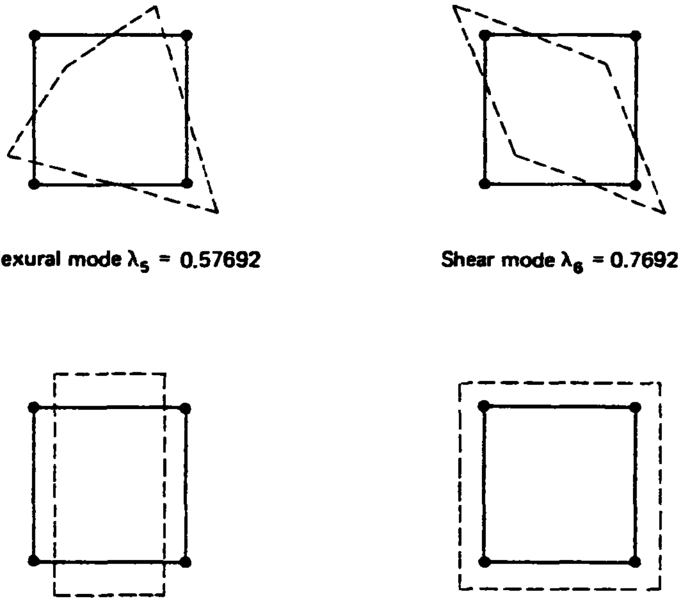
Rigid body mode $\lambda_1 = 0$

Rigid body mode $\lambda_2 = 0$

Rigid body mode $\lambda_3 = 0$

Flexural mode $\lambda_4 = 0.57692$

Fig. 4.25 (a) Eigenvectors and eigenvalues of four-node plane stress element



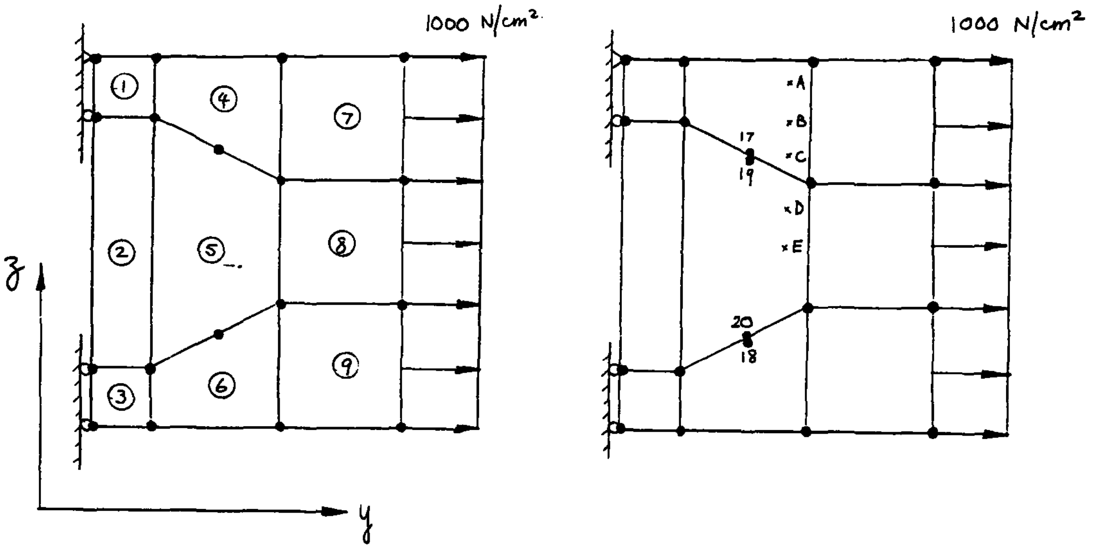
Flexural mode $\lambda_5 = 0.57692$

Shear mode $\lambda_6 = 0.76923$

Stretching mode $\lambda_7 = 0.76923$

Uniform extension mode $\lambda_8 = 1.92308$

Fig. 4.25 (b) Eigenvectors and eigenvalues of four-node plane stress element



a) compatible element mesh; constant stress $\sigma_{yy} = 1000$ N/cm^2 in each element.

b) incompatible element mesh; node 17 belongs to element 4, nodes 19 and 20 belong to element 5, and node 18 belongs to element 6.

Fig. 4.30 (a) Effect of displacement incompatibility in stress prediction

σ_{yy} stress predicted by the incompatible element mesh:

Point	σ_{yy}(N/m^2)
A	1066
B	716
C	359
D	1303
E	1303

Fig. 4.30 (b) Effect of displacement incompatibility in stress prediction

IMPLEMENTATION OF METHODS IN COMPUTER PROGRAMS; EXAMPLES SAP, ADINA

LECTURE 5

56 MINUTES

LECTURE 5

Implementation of the finite element method

The computer programs SAP and ADINA

Details of allocation of nodal point degrees of freedom, calculation of matrices, the assemblage process

Example analysis of a cantilever plate

Out-of-core solution

Effective nodal-point numbering

Flow chart of total solution process

Introduction to different effective finite elements used in one, two, three-dimensional, beam, plate and shell analyses

TEXTBOOK: Appendix A, Sections: 1.3, 8.2.3

Examples: A.1, A.2, A.3, A.4, Example Program STAP

IMPLEMENTATION OF THE FINITE ELEMENT METHOD

We derived the equilibrium equations

$$\underline{K}\,\underline{U} = \underline{R} \ ; \ \underline{R} = \underline{R}_B + \dots$$

where

$$\underline{K} = \sum_m \underline{K}^{(m)} \ ; \ \underline{R}_B = \sum_m \underline{R}_B^{(m)}$$

$$\underline{K}^{(m)} = \int_{V^{(m)}} \underline{B}^{(m)^T} \underline{C}^{(m)} \underline{B}^{(m)} \, dV^{(m)}$$

$$\underline{R}_B^{(m)} = \int_{V^{(m)}} \underline{H}^{(m)^T} \underline{f}^{B(m)} \, dV^{(m)}$$

$\underline{H}^{(m)}$ $k \times N$ $\underline{B}^{(m)}$ $\ell \times N$

N = no. of d.o.f. of total structure

<u>In practice</u>, we calculate <u>compacted</u> element matrices.

$\underline{K}$ $n \times n$, $\underline{R}_B$ $n \times 1$, . . .

n = no. of element d.o.f.

$\underline{H}$ $k \times n$ $\underline{B}$ $\ell \times n$

The stress analysis process can be understood to consist of essentially three phases:

1. Calculation of structure matrices K , M , C , and R , whichever are applicable.
2. Solution of equilibrium equations.
3. Evaluation of element stresses.

The calculation of the structure matrices is performed as follows:

1. The nodal point and element information are read and/or generated.

2. The element stiffness matrices, mass and damping matrices, and equivalent nodal loads are calculated.

3. The structure matrices K , M , C , and R , whichever are applicable, are assembled.

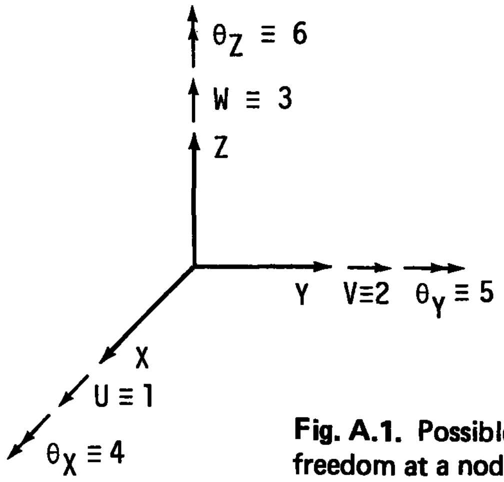
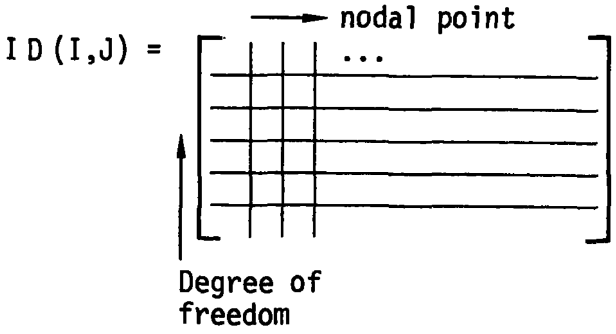


Fig. A.1. Possible degrees of freedom at a nodal point.



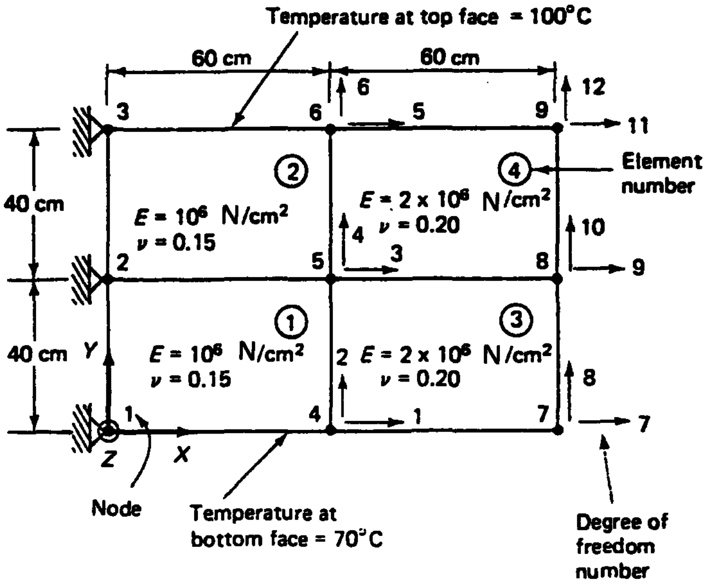


Fig. A.2. Finite element cantilever idealization.

In this case the ID array is given by

$$\mathrm{ID} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and then

$$ID = \begin{bmatrix} 0 & 0 & 0 & 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 0 & 0 & 2 & 4 & 6 & 8 & 10 & 12 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Also

$$X^T = [\ 0.0 \quad 0.0 \quad 0.0 \quad 60.0 \quad 60.0 \quad 60.0 \quad 120.0 \quad 120.0 \quad 120.0]$$

$$Y^T = [\ 0.0 \quad 40.0 \quad 80.0 \quad 0.0 \quad 40.0 \quad 80.0 \quad 0.0 \quad 40.0 \quad 80.0]$$

$$Z^T = [\ 0.0 \quad 0.0 \quad 0.0 \quad 0.0 \quad 0.0 \quad 0.0 \quad 0.0 \quad 0.0 \quad 0.0]$$

$$T^T = [70.0 \quad 85.0 \quad 100.0 \quad 70.0 \quad 85.0 \quad 100.0 \quad 70.0 \quad 85.0 \quad 100.0]$$

For the elements we have

Element 1: node numbers: 5,2,1,4;
material property set: 1

Element 2: node numbers: 6,3,2,5;
material property set: 1

Element 3: node numbers: 8,5,4,7;
material property set: 2

Element 4: node numbers: 9,6,5,8;
material property set: 2

CORRESPONDING COLUMN AND ROW NUMBERS

For compacted matrix	1	2	3	4	5	6	7	8
For $\underline{K}_1$	3	4	0	0	0	0	1	2

$LM^T = [3 \quad 4 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 2]$

Similarly, we can obtain the LM arrays that correspond to the elements 2,3, and 4. We have for element 2,

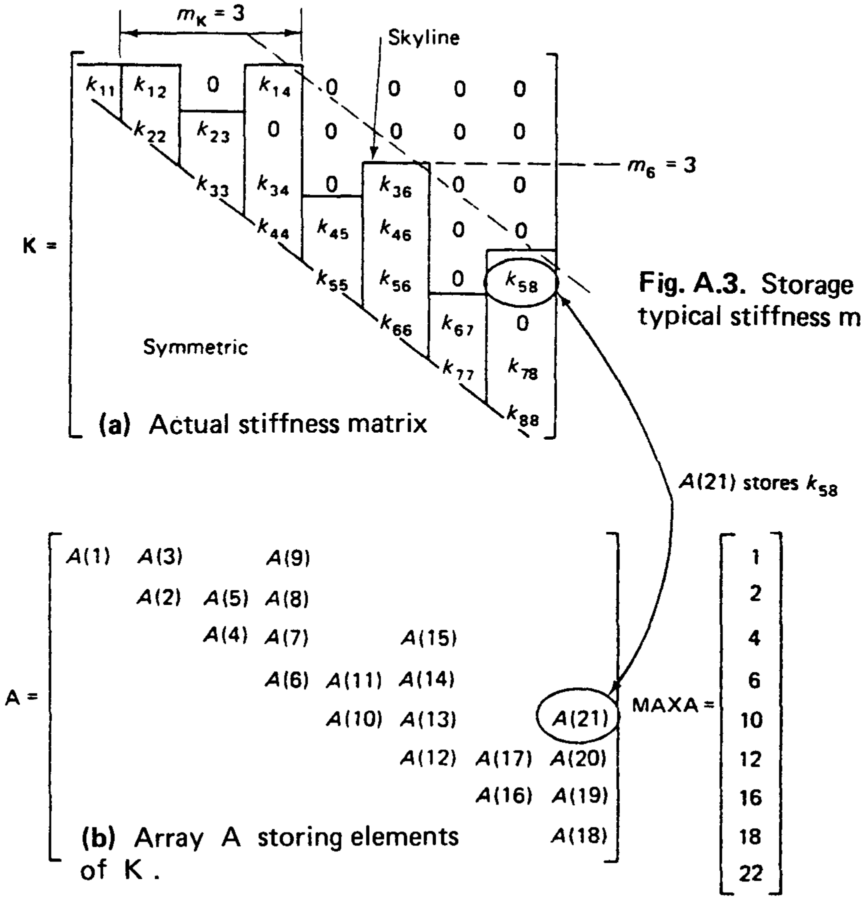
$$LM^T = [5 \quad 6 \quad 0 \quad 0 \quad 0 \quad 0 \quad 3 \quad 4]$$

for element 3,

$$LM^T = [9 \quad 10 \quad 3 \quad 4 \quad 1 \quad 2 \quad 7 \quad 8]$$

and for element 4,

$$LM^T = [11 \quad 12 \quad 5 \quad 6 \quad 3 \quad 4 \quad 9 \quad 10]$$



(a) Actual stiffness matrix

(b) Array A storing elements of K.

Fig. A.3. Storage scheme used for a typical stiffness matrix.

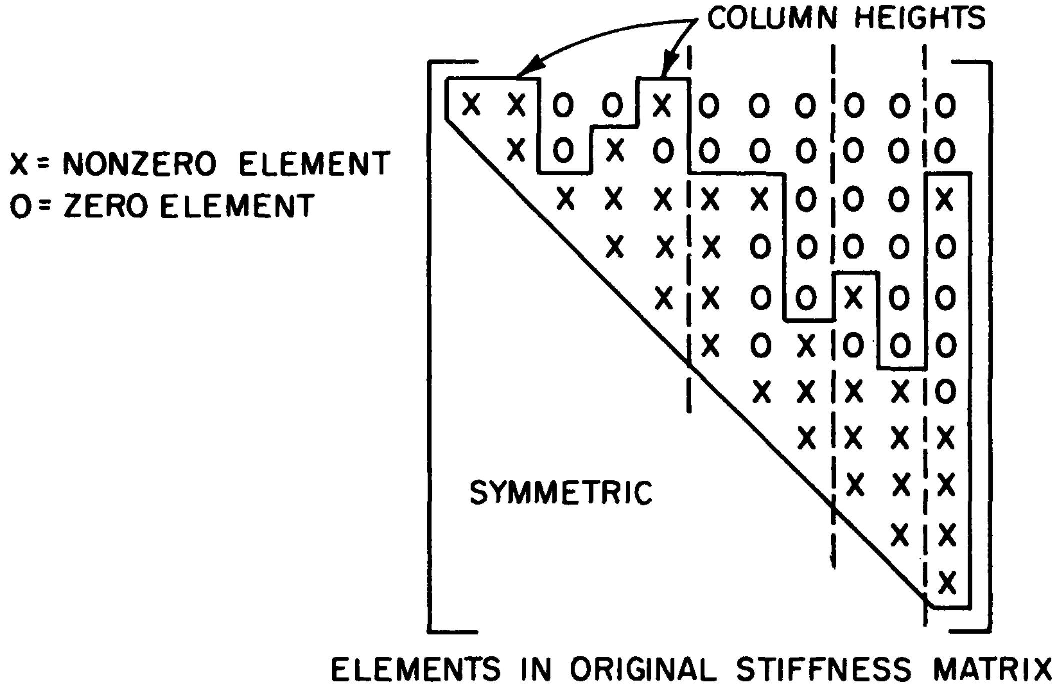


Fig. 10. Typical element pattern in a stiffness matrix using block storage.

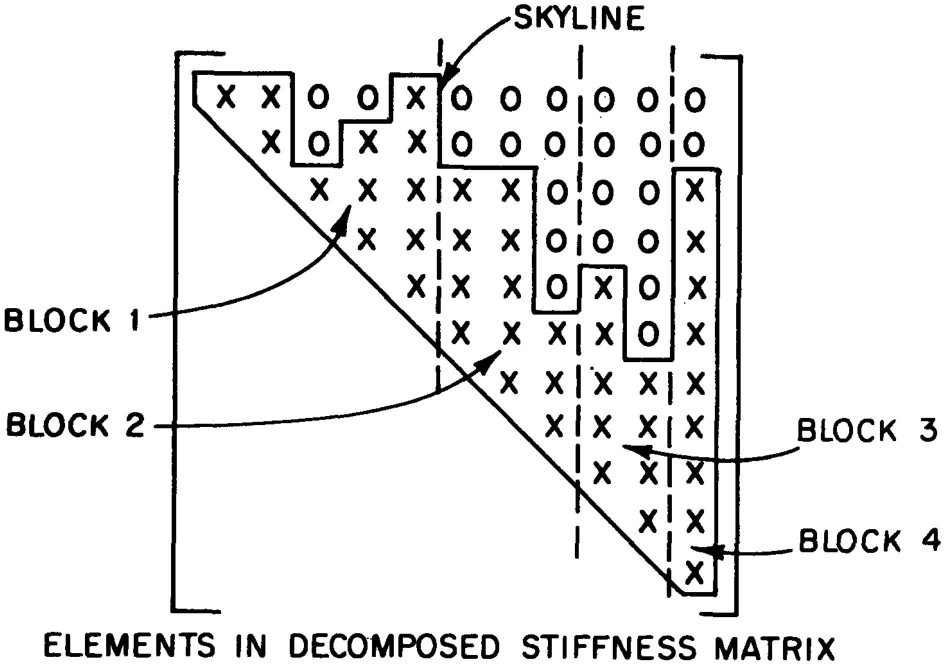
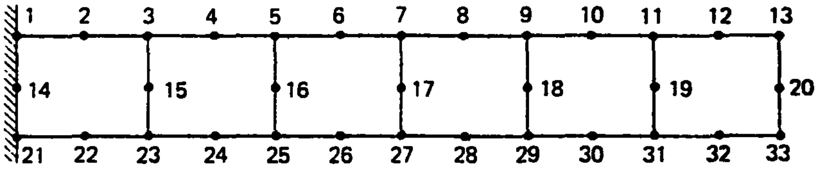
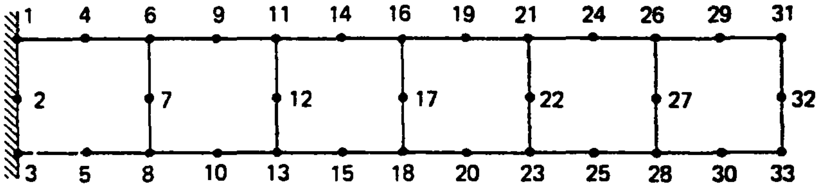


Fig. 10. Typical element pattern in a stiffness matrix using block storage.



(a) Bad nodal point numbering,
$m_k + 1 = 46$.



(b) Good nodal point numbering,
$m_k + 1 = 16$.

Fig. A.4. Bad and good nodal point numbering for finite element assemblage.

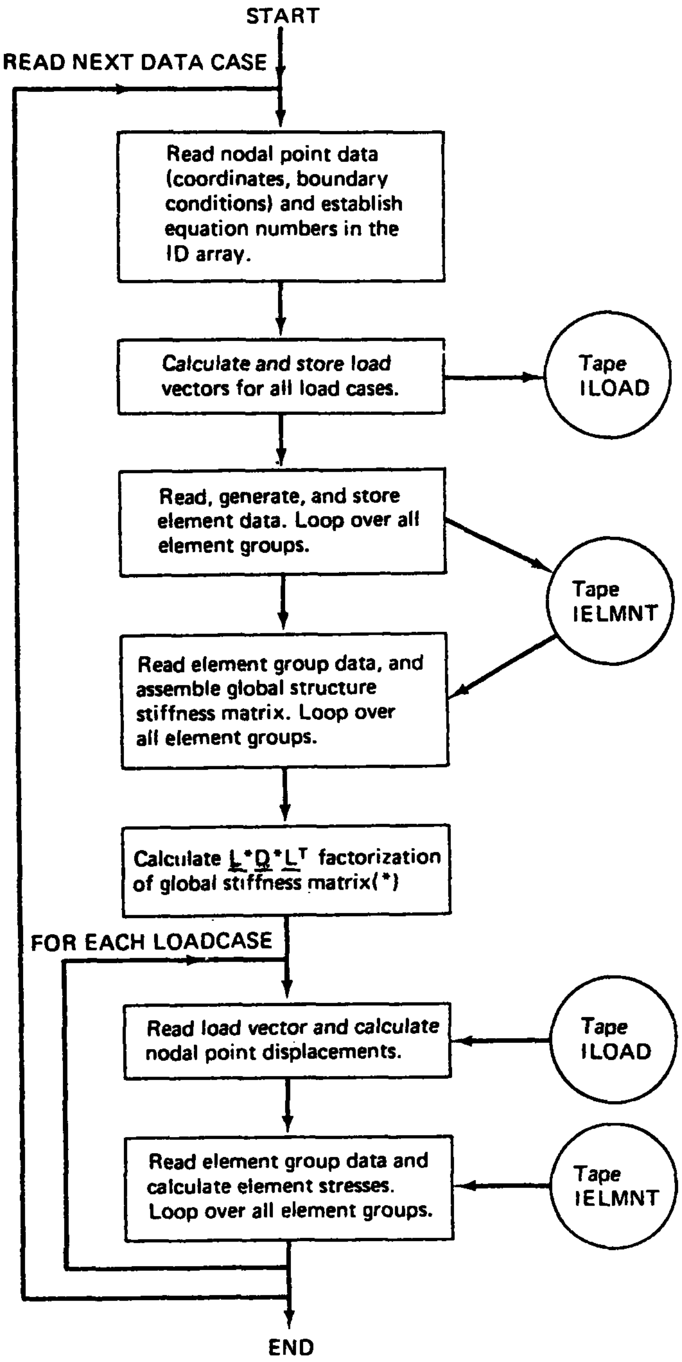


Fig. A.5. Flow chart of program STAP. *See Section 8.2.2.

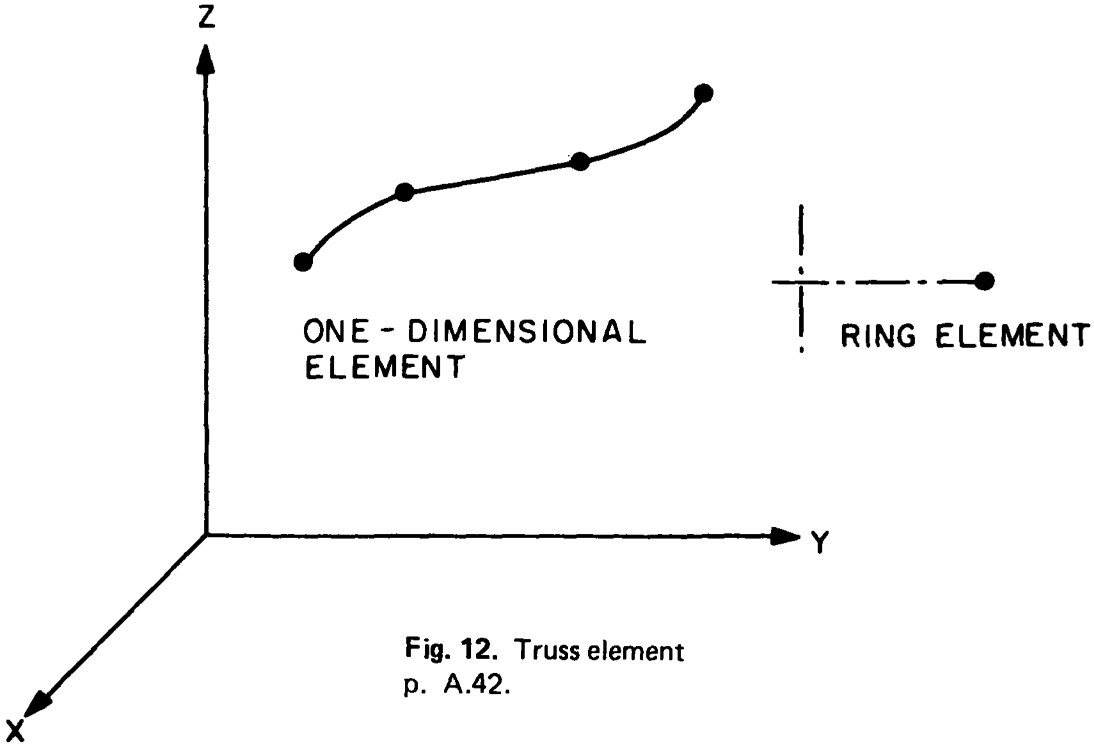


Fig. 12. Truss element
p. A.42.

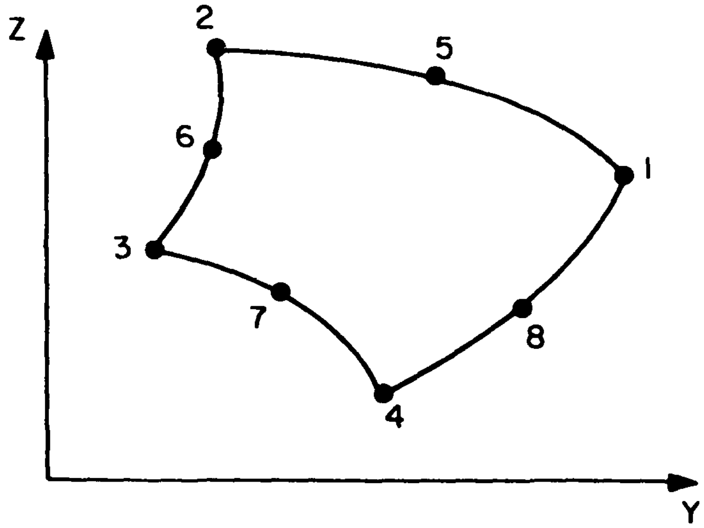


Fig. 13. Two-dimensional plane stress, plane strain and axisymmetric elements.
p..A.43.

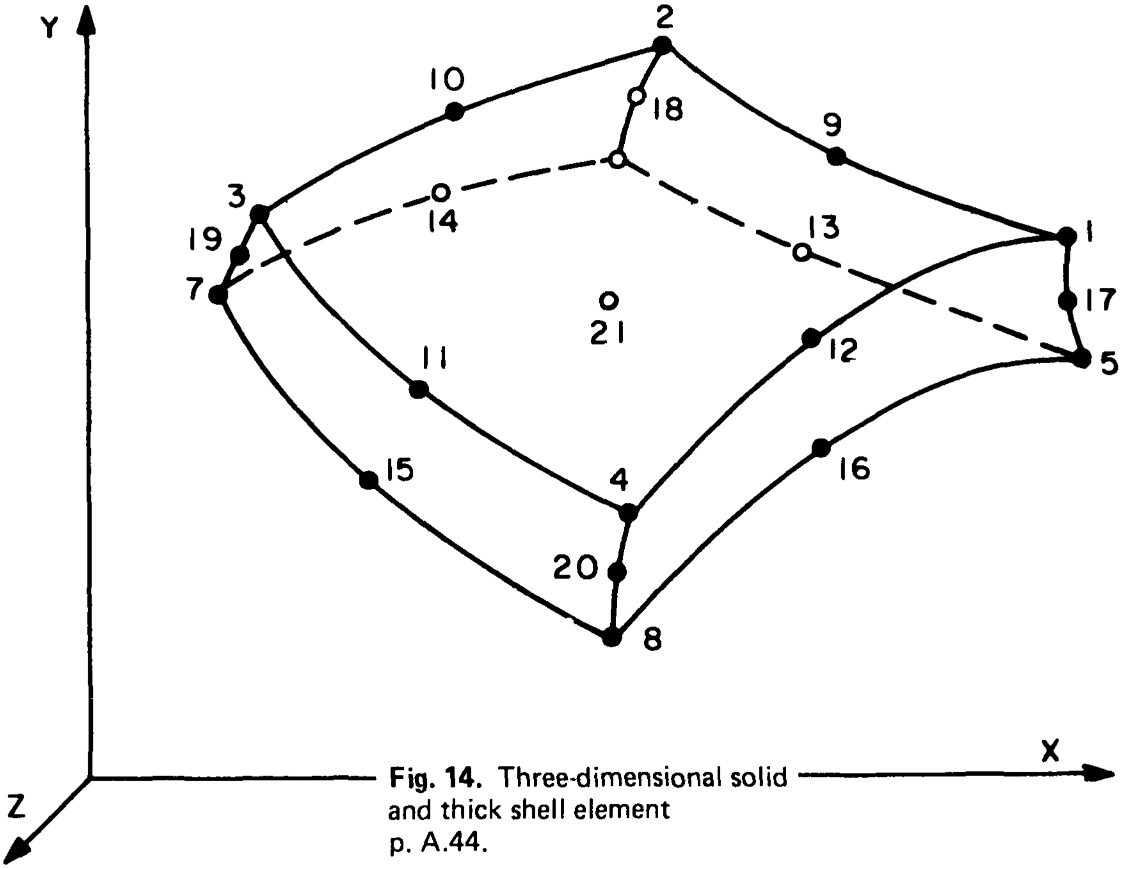


Fig. 14. Three-dimensional solid and thick shell element
p. A.44.

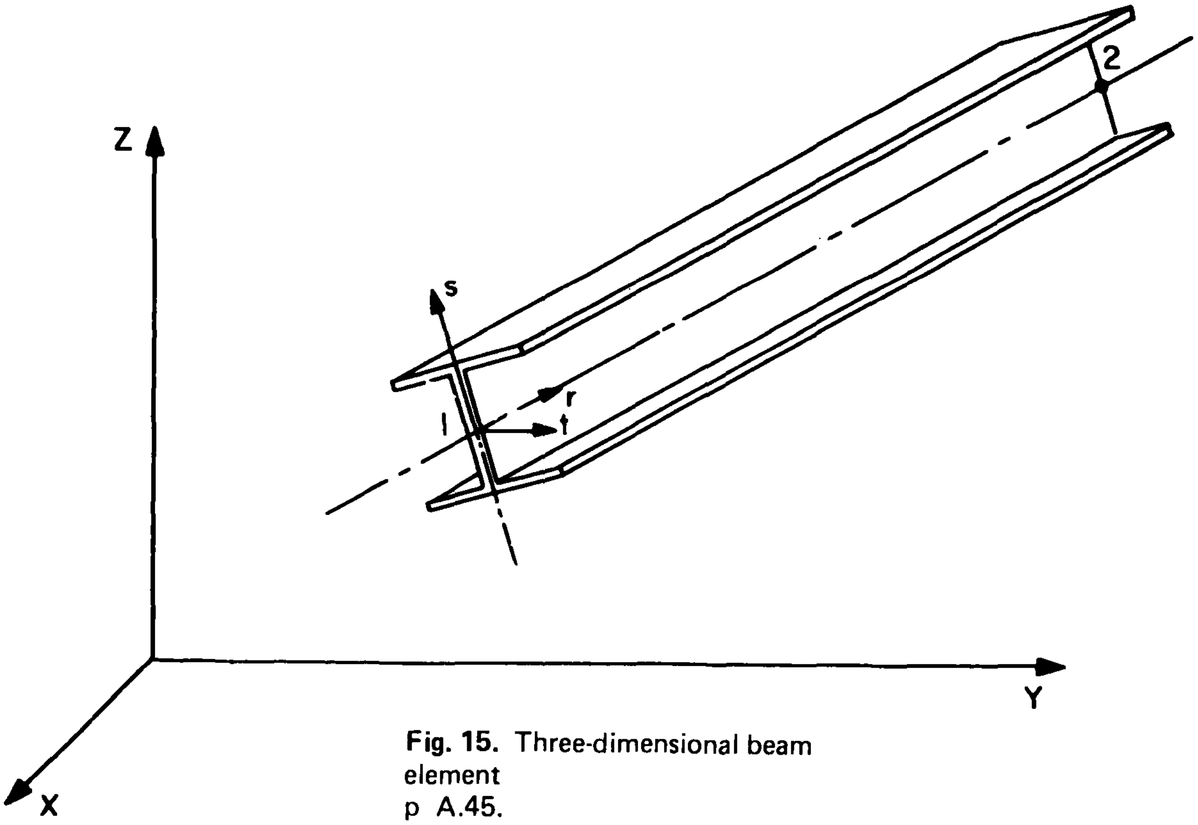


Fig. 15. Three-dimensional beam element
p A.45.

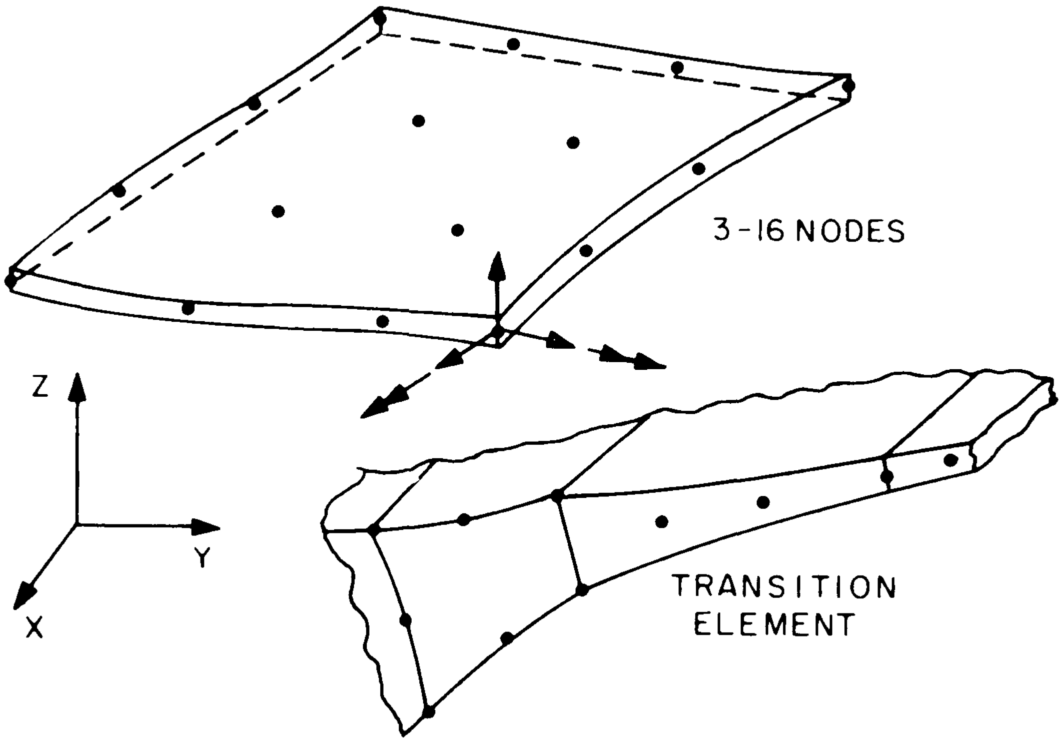


Fig. 16. Thin shell element
(variable-number-nodes)
p. A.46.

FORMULATION AND CALCULATION OF ISOPARAMETRIC MODELS

LECTURE 6

57 MINUTES

LECTURE 6 Formulation and calculation of isoparametric continuum elements

Truss, plane-stress, plane-strain, axisymmetric and three-dimensional elements

Variable-number-nodes elements, curved elements

Derivation of interpolations, displacement and strain interpolation matrices, the Jacobian transformation

Various examples; shifting of internal nodes to achieve stress singularities for fracture mechanics analysis

TEXTBOOK: Sections: 5.1, 5.2, 5.3.1, 5.3.3, 5.5.1

Examples: 5.1, 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8, 5.9, 5.10, 5.11, 5.12, 5.13, 5.14, 5.15, 5.16, 5.17

FORMULATION AND CALCULATION OF ISO-PARAMETRIC FINITE ELEMENTS

- We considered earlier (lecture 4) generalized coordinate finite element models
- We now want to discuss a more general approach to deriving the required interpolation matrices and element matrices $\Rightarrow$ isoparametric elements

---

Isoparametric Elements
Basic Concept: (Continuum Elements)

Interpolate Geometry

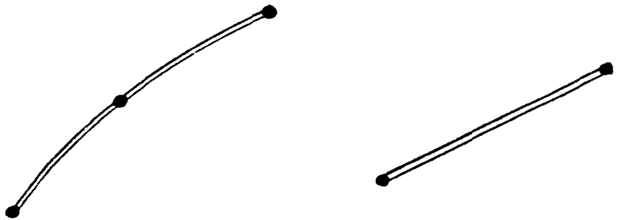
$$x=\sum_{i=1}^{N} h_i x_i \; ; \quad y=\sum_{i=1}^{N} h_i y_i \; ; \quad z=\sum_{i=1}^{N} h_i z_i$$

Interpolate Displacements

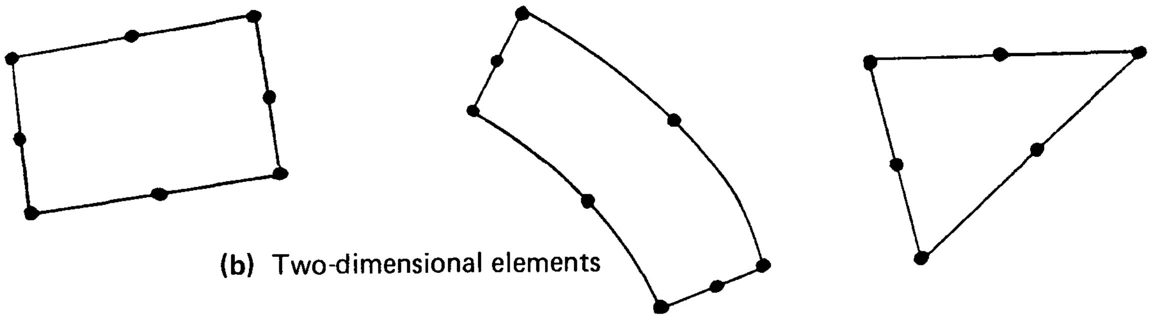
$$u=\sum_{i=1}^{N} h_i u_i \quad v=\sum_{i=1}^{N} h_i v_i \quad w=\sum_{i=1}^{N} h_i w_i$$

N = number of nodes

1/D Element	Truss	Continuum Elements
2/D Elements	Plane stress Plane strain Axisymmetric Analysis	
3/D Elements	Three-dimensional Thick Shell	

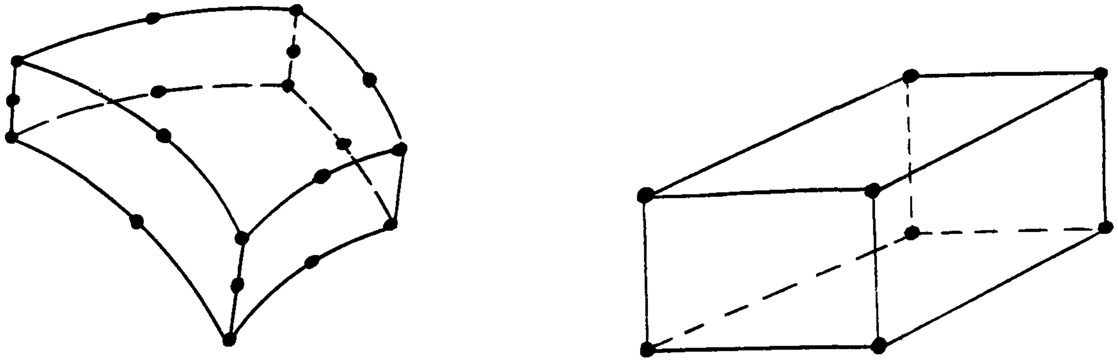


(a) Truss and cable elements



(b) Two-dimensional elements

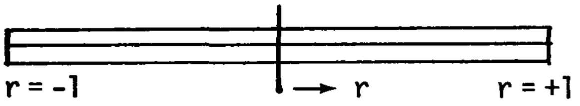
Fig. 5.2. Some typical continuum elements



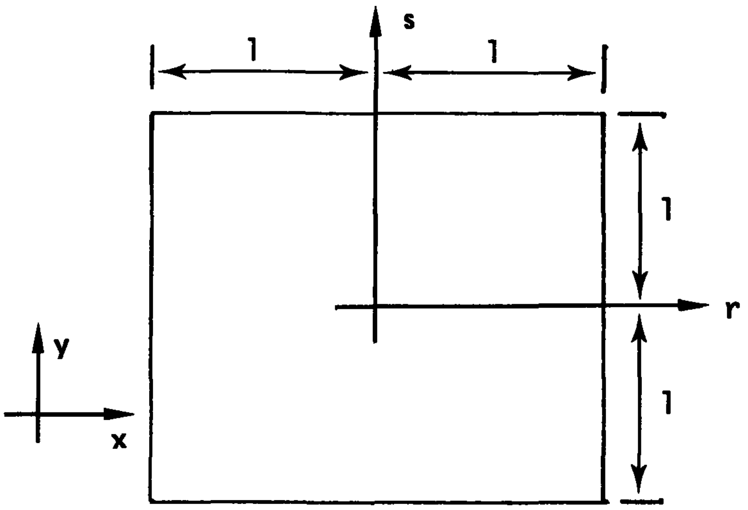
(c) Three-dimensional elements

Fig. 5.2. Some typical continuum elements

Consider special geometries first:



Truss, 2 units long



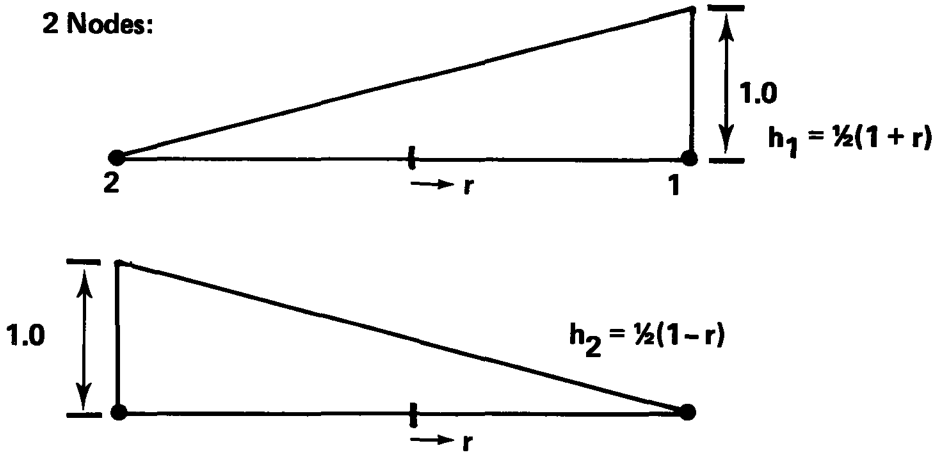
2/D element, 2x2 units

Similarly 3/D element 2x2x2 units
(r - s - t axes)

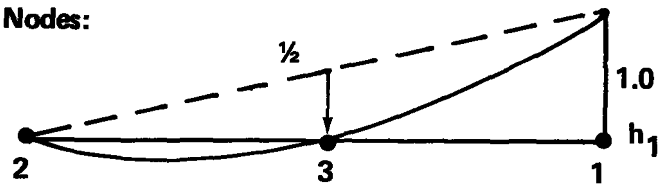
---

1 - D Element

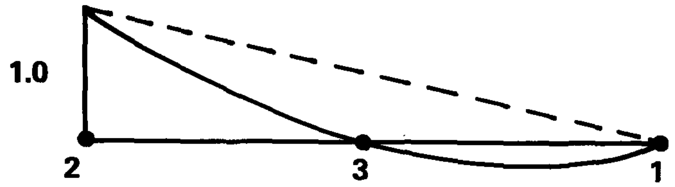
2 Nodes:



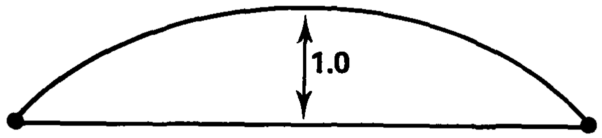
3 Nodes:



$h_1 = ½(1 + r) - ½(1 - r^2)$



$h_2 = ½(1 - r) - ½(1 - r^2)$

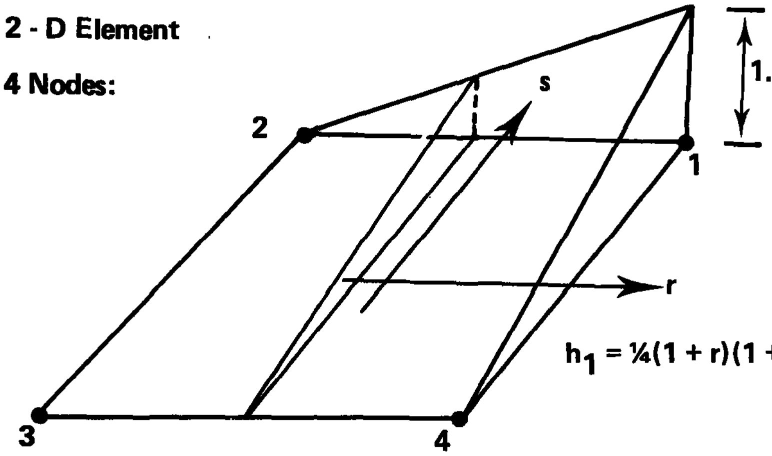


$h_3 = 1 - r^2$

---

2 - D Element

4 Nodes:



$h_1 = ¼(1 + r)(1 + s)$

Similarly

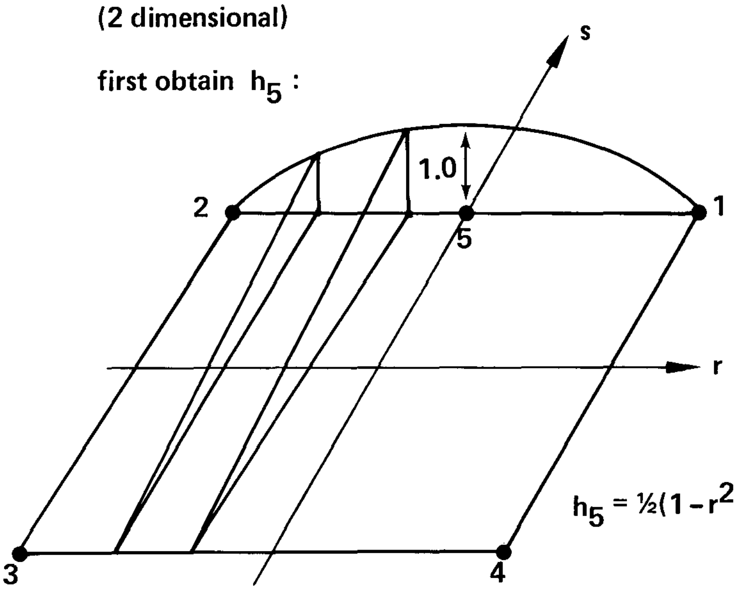
$h_2 = ¼(1 - r)(1 + s)$

$h_3 = ¼(1 - r)(1 - s)$

$h_4 = ¼(1 + r)(1 - s)$

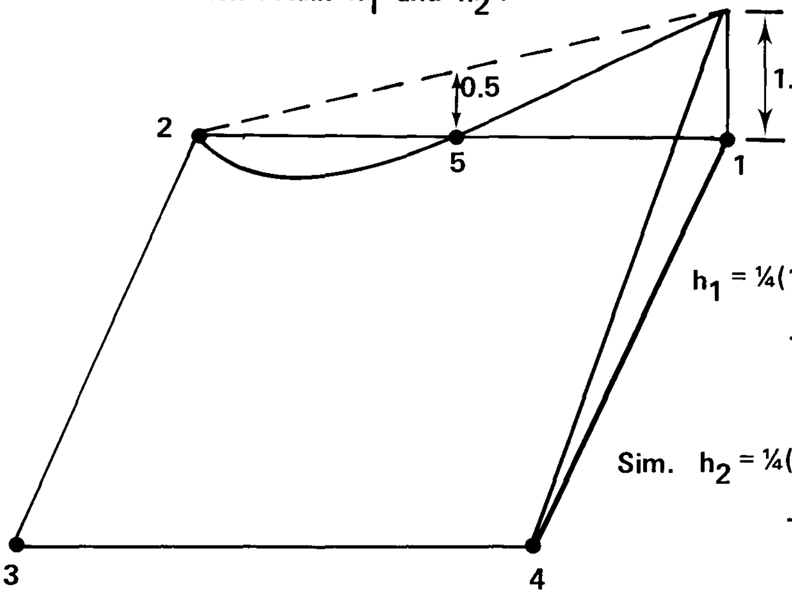
Construction of 5 node element
(2 dimensional)

first obtain h_5 :



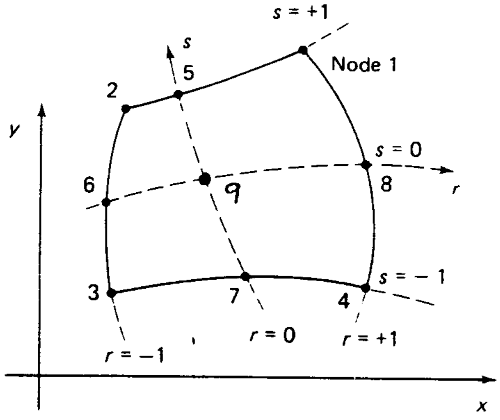
$h_5 = ½(1 - r^2)(1 + s)$

Then obtain h_1 and h_2 :



$h_1 = ¼(1 + r)(1 + s)$
$- ½h_5$

Sim. $h_2 = ¼(1 - r)(1 + s)$
$-½h_5$



(a) Four to 9 variable-number-nodes two-dimensional element

Fig. 5.5. Interpolation functions of four to nine variable-number-nodes two-dimensional element.

		Include only if node i is defined				
		$i = 5$	$i = 6$	$i = 7$	$i = 8$	$i = q$
$h_1 =$	$\frac{1}{4}(1+r)(1+s)$	$-\frac{1}{2}h_5$	....	....	$..-\frac{1}{2}h_8$	$-\frac{1}{4}h_q$
$h_2 =$	$\frac{1}{4}(1-r)(1+s)$	$-\frac{1}{2}h_5$	$-\frac{1}{2}h_6$			$-\frac{1}{4}h_q$
$h_3 =$	$\frac{1}{4}(1-r)(1-s)$	....	$...-\frac{1}{2}h_6$	$-\frac{1}{2}h_7$		$-\frac{1}{4}h_q$
$h_4 =$	$\frac{1}{4}(1+r)(1-s)$	....	....	$...-\frac{1}{2}h_7$	$-\frac{1}{2}h_8$	$-\frac{1}{4}h_q$
$h_5 =$	$\frac{1}{2}(1-r^2)(1+s)$	- - -	- - - - -	- - - - -	- - - -	$-\frac{1}{2}h_q$
$h_6 =$	$\frac{1}{2}(1-s^2)(1-r)$	- - -	- - - - -	- - - - -	- - - -	$-\frac{1}{2}h_q$
$h_7 =$	$\frac{1}{2}(1-r^2)(1-s)$	- - -	- - - - -	- - - - -	- - -	$-\frac{1}{2}h_q$
$h_8 =$	$\frac{1}{2}(1-s^2)(1+r)$	- - - -	- - - - -	- - - - -	- - - -	$-\frac{1}{2}h_q$
$h_q =$	$(1-r^2)(1-s^2)$					

(b) Interpolation functions

Fig. 5.5. Interpolation functions of four to nine variable-number-nodes two-dimensional element.

Having obtained the h_i we can construct the matrices $\underline{H}$ and $\underline{B}$:

- The elements of $\underline{H}$ are the h_i (or zero)
- The elements of $\underline{B}$ are the derivatives of the h_i (or zero),

Because for the $\underbrace{\underbrace{2x2}x2}$ elements we can use

$$x \equiv r$$
$$y \equiv s$$
$$z \equiv t$$

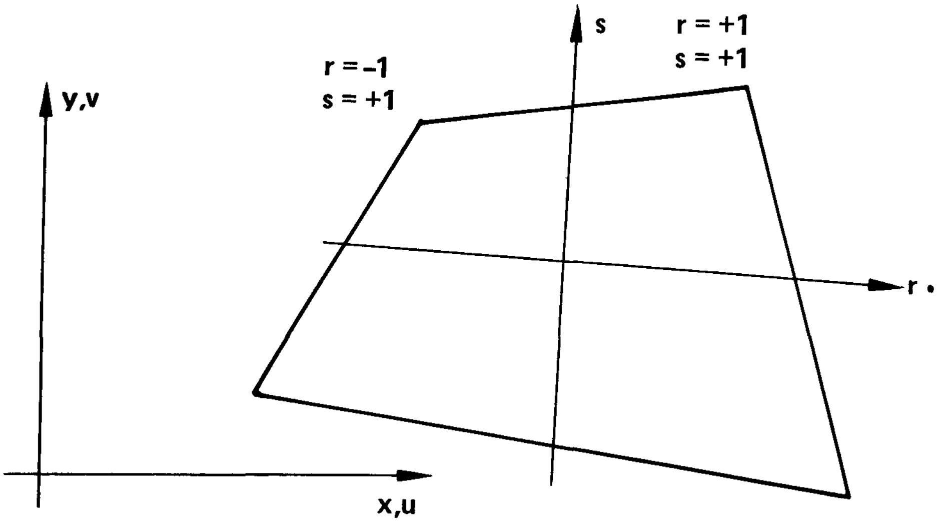
EXAMPLE 4 node 2 dim. element

$$\begin{bmatrix} u(r,s) \\ v(r,s) \end{bmatrix} = \underbrace{\left[\begin{array}{cc|cc|cc|cc} h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 & 0 \\ 0 & h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 \end{array}\right]}_{\underline{H}} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ \vdots \\ v_4 \end{bmatrix}$$

$$\begin{bmatrix}\varepsilon_{rr}\\ \varepsilon_{ss}\\ \gamma_{rs}\end{bmatrix} = \underbrace{\left[\begin{array}{cc|c|cc} \frac{\partial h_1}{\partial r} & 0 & & \frac{\partial h_4}{\partial r} & 0 \\ 0 & \frac{\partial h_1}{\partial s} & \cdots & 0 & \frac{\partial h_4}{\partial s}\\ \frac{\partial h_1}{\partial s} & \frac{\partial h_1}{\partial r} & & \frac{\partial h_4}{\partial s} & \frac{\partial h_4}{\partial r}\end{array}\right]}_{\underline{B}} \begin{bmatrix}u_1\\ v_1\\ u_2\\ \vdots\\ v_4\end{bmatrix}$$

We note again $r \equiv x$
$s \equiv y$

GENERAL ELEMENTS



Displacement and geometry interpolation as before, but

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$$

or

$$\underline{\frac{\partial}{\partial r}} = \underline{J}\ \underline{\frac{\partial}{\partial x}} \quad \text{(in general)}$$

$$\underline{\frac{\partial}{\partial x}} = \underline{J}^{-1}\ \underline{\frac{\partial}{\partial r}} \qquad (5.25)$$

Aside:
cannot use

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\ \frac{\partial r}{\partial x} + \ldots$$

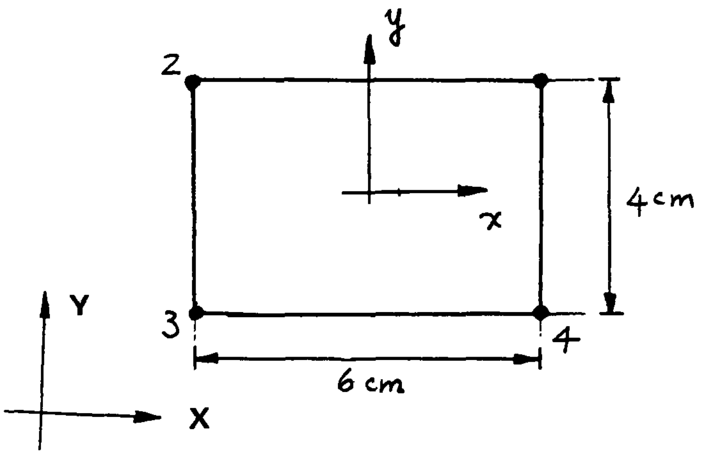
Using (5.25) we can find the matrix $\underline{B}$ of general elements

The $\underline{H}$ and $\underline{B}$ matrices are a function of r , s , t ; for the integration thus use

$$dv = \det \underline{J}\ dr\ ds\ dt$$

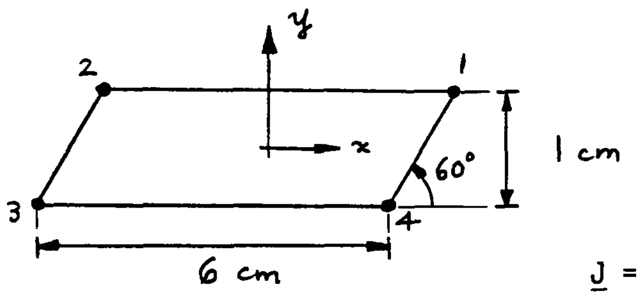
Fig. 5.9. Some two-dimensional elements

Element 1



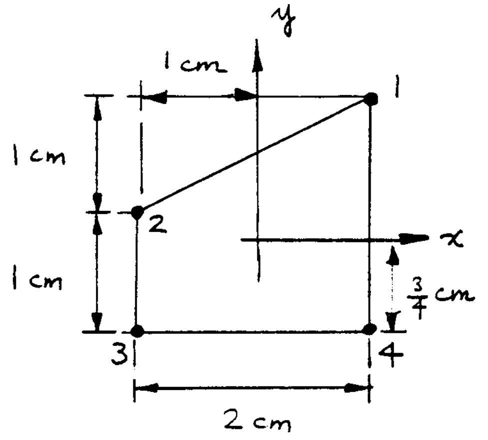
$$\underline{J} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Element 2

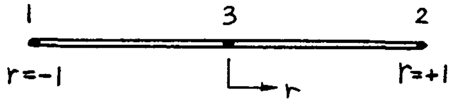


$$\underline{J} = \begin{bmatrix} 3 & 0 \\ \frac{1}{2\sqrt{3}} & \frac{1}{2} \end{bmatrix}$$

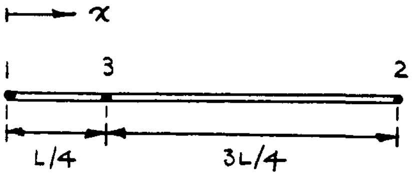
Element 3



$$\underline{J} = \begin{bmatrix} 4 & (1+s) \\ 0 & (3+r) \end{bmatrix}$$



Natural space



Actual physical space

Fig. 5.23. Quarter-point one-dimensional element.

Here we have

$$x = \sum_{i=1}^{3} h_i x_i \Rightarrow x = \frac{L}{4}(1+r)^2$$

hence

$$\underline{J} = \left[\frac{L}{2} + \frac{r}{2}L\right]$$

and

$$\underline{B} = \frac{1}{\frac{L}{2}+\frac{r}{2}L}\left[h_{1,r} \quad h_{2,r} \quad h_{3,r}\right]$$

or

$$\underline{B} = \frac{1}{\frac{L}{2}+\frac{r}{2}L}\left[(-\tfrac{1}{2}+r) \quad (\tfrac{1}{2}+r) \quad -2r\right]$$

Since

$$r = 2\sqrt{\frac{x}{L}} - 1$$

$$\underline{B} = [(\frac{2}{L} - \frac{3}{2\sqrt{L}}\ \frac{1}{\sqrt{x}}) \quad (\frac{2}{L} - \frac{1}{2\sqrt{L}}\ \frac{1}{\sqrt{x}}$$

$$(\frac{2}{\sqrt{L}\ \sqrt{x}} - \frac{4}{L})]$$

We note

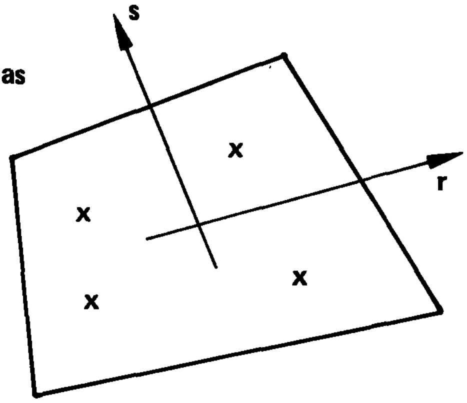
$\frac{1}{\sqrt{x}}$ singularity at X = 0 !

Numerical Integration

Gauss Integration
Newton-Cotes Formulas

$$\underline{K} = \sum_{i,j,k} \alpha_{ijk} \underline{F}_{ijk}$$

$$\underline{F} = \underline{B}^T \underline{C} \, \underline{B} \det \underline{J}$$



FORMULATION OF STRUCTURAL ELEMENTS

LECTURE 7

52 MINUTES

LECTURE 7

Formulation and calculation of isoparametric structural elements

Beam, plate and shell elements

Formulation using Mindlin plate theory and unified general continuum formulation

Assumptions used including shear deformations

Demonstrative examples: two-dimensional beam, plate elements

Discussion of general variable-number-nodes elements

Transition elements between structural and continuum elements

Low- versus high-order elements

TEXTBOOK: Sections: 5.4.1, 5.4.2, 5.5.2, 5.6.1

Examples: 5.20, 5.21, 5.22, 5.23, 5.24, 5.25, 5.26, 5.27

FORMULATION OF STRUCTURAL ELEMENTS

- beam, plate and shell elements
- isoparametric approach for interpolations

Strength of Materials Approach

- straight beam elements

  use beam theory including shear effects

- plate elements

  use plate theory including shear effects
  (Reissner/Mindlin)

Continuum Approach

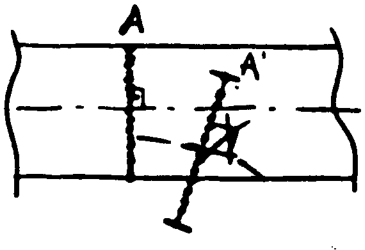
Use the general principle of virtual displacements, but

-- exclude the stress components not applicable

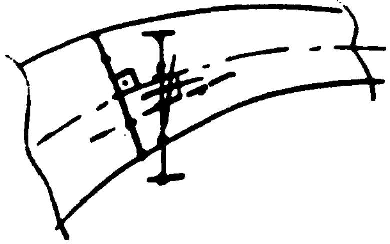
-- use kinematic constraints for particles on sections originally normal to the mid-surface

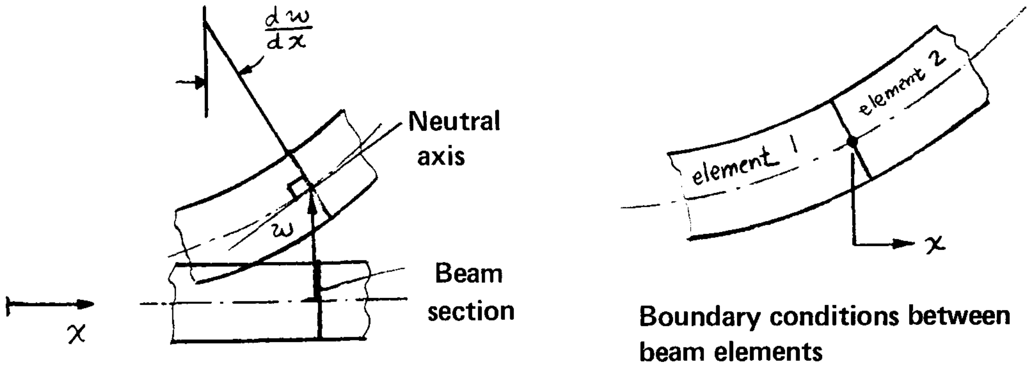
" particles remain on a straight line during deformation"

e.g. beam



e.g. shell





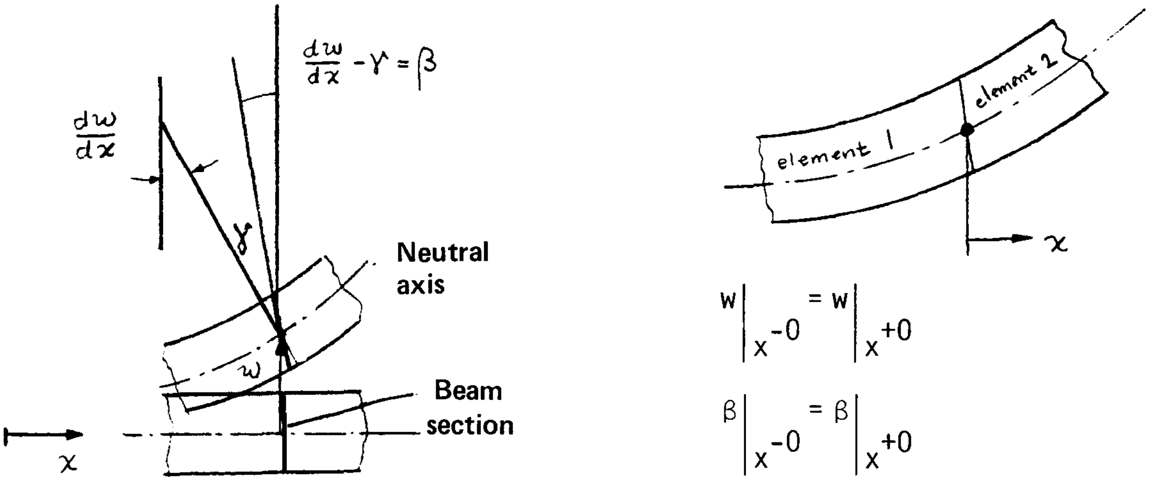
Deformation of cross-section

Boundary conditions between beam elements

$$w\Big|_{x^{-0}} = w\Big|_{x^{+0}} \;;\; \frac{dw}{dx}\Big|_{x^{-0}} = \frac{dw}{dx}\Big|_{x^{+0}}$$

a) Beam deformations excluding shear effect

Fig. 5.29. Beam deformation mechanisms



$$w\Big|_{x^{-0}} = w\Big|_{x^{+0}}$$

$$\beta\Big|_{x^{-0}} = \beta\Big|_{x^{+0}}$$

Deformation of cross-section

Boundary conditions between beam elements

b) Beam deformations including shear effect

Fig. 5.29. Beam deformation mechanisms

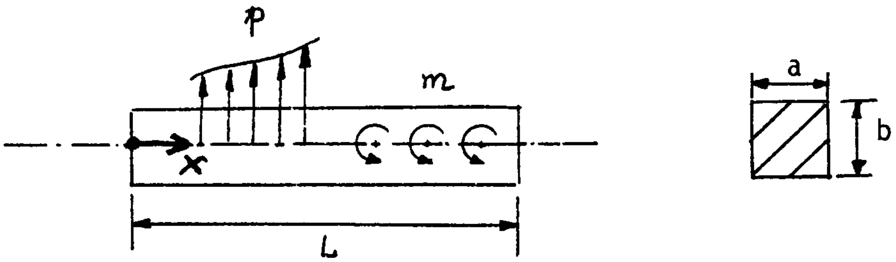
We use

$$\beta = \frac{dw}{dx} - \gamma \tag{5.48}$$

$$\tau = \frac{V}{A_s} \; ; \; \gamma = \frac{\tau}{G} \; ; \; k = \frac{A_s}{A} \tag{5.49}$$

$$\Pi = \frac{EI}{2} \int_0^L \left(\frac{d\beta}{dx}\right)^2 dx + \frac{GAk}{2} \int_0^L \left(\frac{dw}{dx} - \beta\right)^2 dx - \int_0^L pw \; dx - \int_0^L m \; \beta \; dx \tag{5.50}$$

$$EI \int_0^L \left(\frac{d\beta}{dx}\right) \delta\left(\frac{d\beta}{dx}\right) dx + GAk \int_0^L \left(\frac{dw}{dx} - \beta\right) \delta\left(\frac{dw}{dx} - \beta\right) dx - \int_0^L p \; \delta w \; dx - \int_0^L m \; \delta\beta \; dx = 0 \tag{5.51}$$

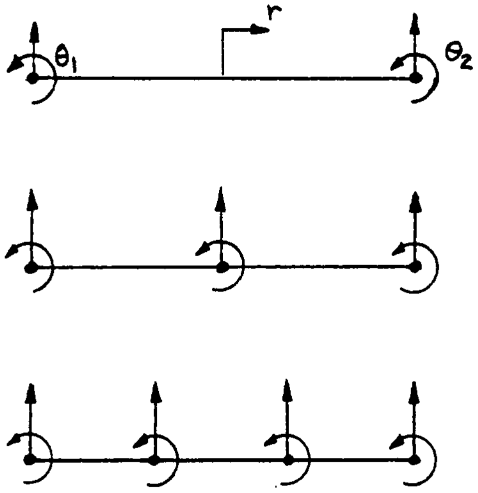


(a) Beam with applied loading

E = Young's modulus, G = shear modulus

$k = \frac{5}{6}$, $A = ab$, $I = \frac{ab^3}{12}$

Fig. 5.30. Formulation of two-dimensional beam element



(b) Two, three- and four-node models; $\theta_i = \beta_i$, i=1,...,q (Interpolation functions are given in Fig. 5.4)

Fig. 5.30. Formulation of two-dimensional beam element

The interpolations are now

$$w = \sum_{i=1}^{q} h_i w_i \; ; \; \beta = \sum_{i=1}^{q} h_i \theta_i \qquad (5.52)$$

$$w = \underset{\sim}{H}_w \underset{\sim}{U} \; ; \qquad \beta = \underset{\sim}{H}_\beta \underset{\sim}{U}$$

$$(5.53)$$

$$\frac{\partial w}{\partial x} = \underset{\sim}{B}_w \underset{\sim}{U} \; ; \quad \frac{\partial \beta}{\partial x} = \underset{\sim}{B}_\beta \underset{\sim}{U}$$

Where

$$\underset{\sim}{U}^T = [w_1 \ldots w_q \; \theta_1 \ldots \theta_q]$$

$$\underset{\sim}{H}_w = [h_1 \ldots h_q \; 0 \ldots 0]$$

$$\underset{\sim}{H}_\beta = [0 \ldots 0 \; h_1 \ldots h_q] \qquad (5.54)$$

and

$$\underset{\sim}{B}_w = J^{-1} \left[\frac{\partial h_1}{\partial r} \ldots \frac{\partial h_q}{\partial r} \; 0 \ldots 0 \right]$$

$$\underset{\sim}{B}_\beta = J^{-1} \left[0 \ldots 0 \; \frac{\partial h_1}{\partial r} \ldots \frac{\partial h_q}{\partial r} \right] (5.55)$$

So that

$$\underline{K} = EI \int_{-1}^{1} \underline{B}_{\beta}^{T}\, \underline{B}_{\beta} \det J\, dr + GAk \int_{-1}^{1} (\underline{B}_{w}-\underline{H}_{\beta})^{T}(\underline{B}_{w}-\underline{H}_{\beta}) \det J\, dr \qquad (5.56)$$

and

$$\underline{R} = \int_{-1}^{1} \underline{H}_{w}^{T}\, p \det J\, dr + \int_{-1}^{1} \underline{H}_{\beta}^{T}\, m \det J\, dr \qquad (5.57)$$

Considering the order of interpolations required, we study

$$\Pi = \int_{0}^{L} \left(\frac{d\beta}{dx}\right)^{2} dx + \alpha \int_{0}^{L} \left(\frac{dw}{dx} - \beta\right)^{2} dx \; ; \qquad \alpha = \frac{GAk}{EI} \qquad (5.60)$$

Hence

- use parabolic (or higher-order) elements
- discrete Kirchhoff theory
- reduced numerical integration

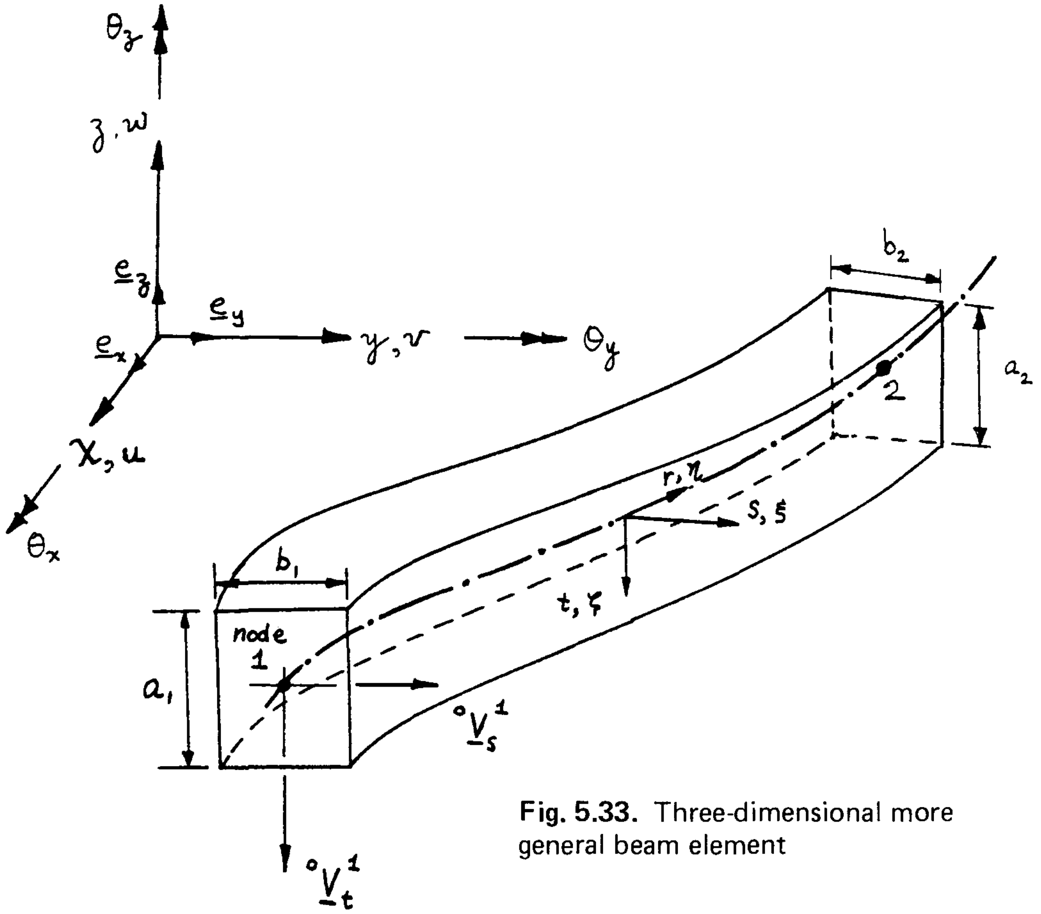


Fig. 5.33. Three-dimensional more general beam element

Here we use

$$
\begin{aligned}
{}^{\ell}x(r,s,t) &= \sum_{k=1}^{q} h_k \, {}^{\ell}x_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^{\ell}V_{tx}^{k} \\
&\quad + \frac{s}{2} \sum_{k=1}^{q} b_k h_k \, {}^{\ell}V_{sx}^{k} \\
{}^{\ell}y(r,s,t) &= \sum_{k=1}^{q} h_k \, {}^{\ell}y_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^{\ell}V_{ty}^{k} \\
&\quad + \frac{s}{2} \sum_{k=1}^{q} b_k h_k \, {}^{\ell}V_{sy}^{k} \qquad (5.61) \\
{}^{\ell}z(r,s,t) &= \sum_{k=1}^{q} h_k \, {}^{\ell}z_k + \frac{t}{2} \sum_{k=1}^{q} a_k \, h_k \, {}^{\ell}V_{tz}^{k} \\
&\quad + \frac{s}{2} \sum_{k=1}^{q} b_k h_k \, {}^{\ell}V_{sz}^{k}
\end{aligned}
$$

So that

$$u\ (r,s,t) = {}^1x - {}^0x$$

$$v\ (r,s,t) = {}^1y - {}^0y \qquad (5.62)$$

$$w\ (r,s,t) = {}^1z - {}^0z$$

---

and

$$u(r,s,t) = \sum_{k=1}^{q} h_k u_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k\ V_{tx}^k + \frac{s}{2} \sum_{k=1}^{q} b_k h_k\ V_{sx}^k$$

$$v(r,s,t) = \sum_{k=1}^{q} h_k v_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k\ V_{ty}^k + \frac{s}{2} \sum_{k=1}^{q} b_k h_k\ V_{sy}^k$$

$$w(r,s,t) = \sum_{k=1}^{q} h_k w_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k\ V_{tz}^k + \frac{s}{2} \sum_{k=1}^{q} b_k h_k\ V_{sz}^k$$

(5.63)

Finally, we express the vectors $\underline{V}_t^k$ and $\underline{V}_s^k$ in terms of rotations about the Cartesian axes x , y , z ,

$$\underline{V}_t^k = \underline{\theta}_k \times {}^0\underline{V}_t^k$$

$$\underline{V}_s^k = \underline{\theta}_k \times {}^0\underline{V}_s^k \qquad (5.65)$$

where

$$\underline{\theta}_k = \begin{bmatrix} \theta_x^k \\ \theta_y^k \\ \theta_z^k \end{bmatrix} \qquad (5.66)$$

We can now find

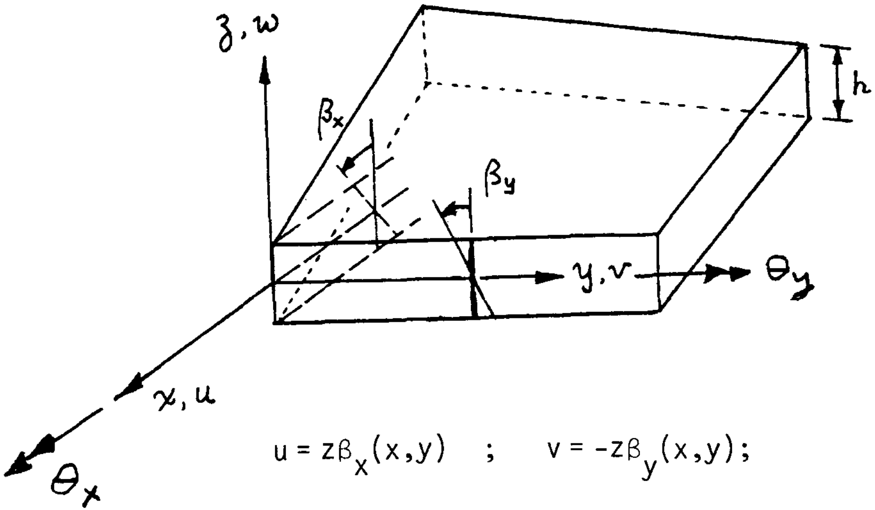
$$\begin{bmatrix} \varepsilon_{\eta\eta} \\ \gamma_{\eta\xi} \\ \gamma_{\eta\zeta} \end{bmatrix} = \sum_{k=1}^{q} \underline{B}_k \, \underline{u}_k \qquad (5.67)$$

where

$$\underline{u}_k^T = [u_k \; v_k \; w_k \quad \theta_x^k \, \theta_y^k \, \theta_z^k] \qquad (5.68)$$

and then also have

$$\begin{bmatrix} \tau_{\eta\eta} \\ \tau_{\eta\xi} \\ \tau_{\eta\zeta} \end{bmatrix} = \begin{bmatrix} E & 0 & 0 \\ 0 & Gk & 0 \\ 0 & 0 & Gk \end{bmatrix} \begin{bmatrix} \varepsilon_{\eta\eta} \\ \gamma_{\eta\xi} \\ \gamma_{\eta\zeta} \end{bmatrix} \qquad (5.77)$$



$$u = z\beta_x(x,y) \quad ; \quad v = -z\beta_y(x,y) ;$$

$$\text{and} \quad w = w(x,y) \qquad (5.78)$$

Fig. 5.36. Deformation mechanisms in analysis of plate including shear deformations

Hence

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = z \begin{bmatrix} \dfrac{\partial \beta_x}{\partial x} \\ -\dfrac{\partial \beta_y}{\partial y} \\ \dfrac{\partial \beta_x}{\partial y} - \dfrac{\partial \beta_y}{\partial x} \end{bmatrix} \qquad (5.79)$$

$$\begin{bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial w}{\partial y} - \beta_y \\ \dfrac{\partial w}{\partial x} + \beta_x \end{bmatrix} \qquad (5.80)$$

and

$$\begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix} = z \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \frac{\partial \beta_x}{\partial x} \\ - \frac{\partial \beta_y}{\partial y} \\ \frac{\partial \beta_x}{\partial y} - \frac{\partial \beta_y}{\partial x} \end{bmatrix}$$

(5.81)

$$\begin{bmatrix} \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \frac{E}{2(1+\nu)} \begin{bmatrix} \frac{\partial w}{\partial y} - \beta_y \\ \frac{\partial w}{\partial x} + \beta_x \end{bmatrix}$$

(5.82)

The total potential for the element is:

$$\Pi = \frac{1}{2} \int_A \int_{-h/2}^{h/2} [\varepsilon_{xx} \; \varepsilon_{yy} \; \gamma_{xy}] \begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix} dz \; dA$$

$$+ \frac{k}{2} \int_A \int_{-h/2}^{h/2} [\gamma_{yz} \; \gamma_{zx}] \begin{bmatrix} \tau_{yz} \\ \tau_{zx} \end{bmatrix} dx \; dA$$

$$- \int_A w \; p \; dA$$

(5.83)

or performing the integration
through the thickness

$$\Pi = \frac{1}{2} \int_A \underline{\kappa}^T \underline{C}_b \, \underline{\kappa} \, dA + \frac{1}{2} \int_A \underline{\gamma}^T \underline{C}_s \, \underline{\gamma} \, dA - \int_A w \, p \, dA \qquad (5.84)$$

---

where

$$\underline{\kappa} = \begin{bmatrix} \frac{\partial \beta_x}{\partial x} \\ - \frac{\partial \beta_y}{\partial y} \\ \frac{\partial \beta_x}{\partial y} - \frac{\partial \beta_y}{\partial x} \end{bmatrix} ; \; \underline{\gamma} = \begin{bmatrix} \frac{\partial w}{\partial y} - \beta_y \\ \frac{\partial w}{\partial x} + \beta_x \end{bmatrix} \qquad (5.86)$$

$$\underline{C}_b = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} ;$$

$$\underline{C}_s = \frac{Ehk}{2(1+\nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (5.87)$$

Using the condition $\delta\Pi = 0$ we obtain the principle of virtual displacements for the plate element.

$$\int_A \delta \underline{\kappa}^T \underline{C}_b \underline{\kappa} \, dA + \int_A \delta \underline{\gamma}^T \underline{C}_s \underline{\gamma} \, dA - \int_A \delta w \, p \, dA = 0 \qquad (5.88)$$

We use the interpolations

$$w = \sum_{i=1}^{q} h_i w_i \quad ; \quad \beta_x = \sum_{i=1}^{q} h_i \, \theta_y^i$$

$$\beta_y = \sum_{i=1}^{q} h_i \, \theta_x^i \qquad (5.89)$$

and

$$x = \sum_{i=1}^{q} h_i x_i \quad ; \quad y = \sum_{i=1}^{q} h_i y_i$$

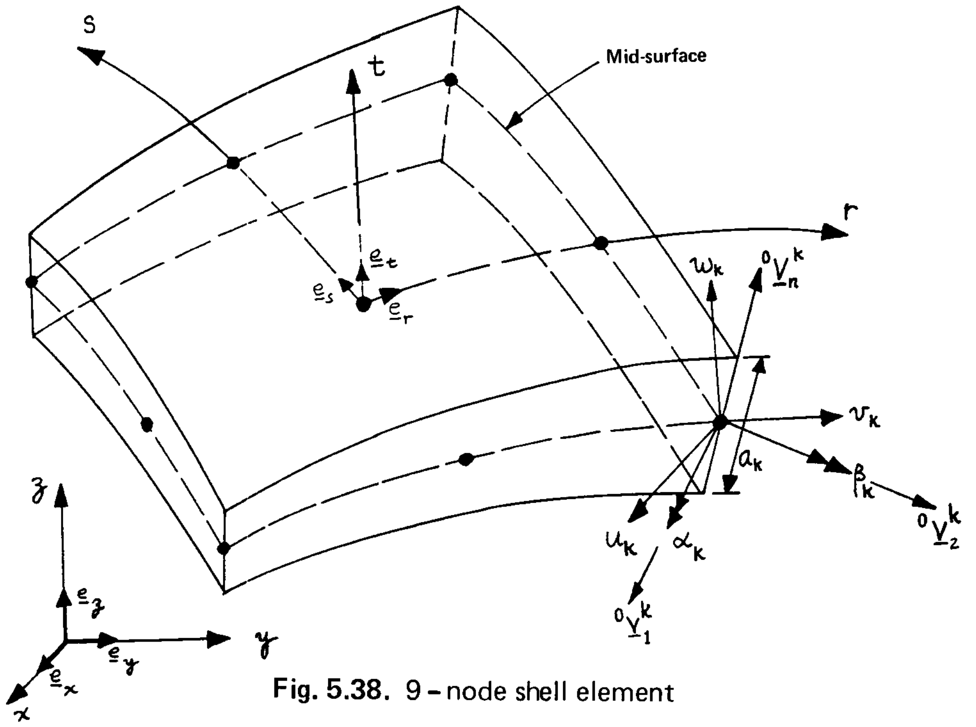


Fig. 5.38. 9 - node shell element

For shell elements we proceed as in the formulation of the general beam elements,

$$
{}^{\ell}x(r,s,t) = \sum_{k=1}^{q} h_k \, {}^{\ell}x_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^{\ell}V^{k}_{nx}
$$

$$
{}^{\ell}y(r,s,t) = \sum_{k=1}^{q} h_k \, {}^{\ell}y_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^{\ell}V^{k}_{ny}
$$

$$
{}^{\ell}z(r,s,t) = \sum_{k=1}^{q} h_k \, {}^{\ell}z_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^{\ell}V^{k}_{nz}
$$

(5.90)

Therefore,

$$u(r,s,t) = \sum_{k=1}^{q} h_k u_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, V_{nx}^k$$

$$v(r,s,t) = \sum_{k=1}^{q} h_k v_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, V_{ny}^k$$

$$w(r,s,t) = \sum_{k=1}^{q} h_k w_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, V_{nz}^k$$

where (5.91)

$$\underline{V}_n^k = {}^1\underline{V}_n^k - {}^0\underline{V}_n^k \qquad (5.92)$$

To express $\underline{V}_n^k$ in terms of rotations at the nodal-point k we define

$${}^0\underline{V}_1^k = \left(\underline{e}_y \times {}^0\underline{V}_n^k\right) \Big/ \left|\underline{e}_y \times {}^0\underline{V}_n^k\right| \qquad (5.93a)$$

$${}^0\underline{V}_2^k = {}^0\underline{V}_n^k \times {}^0\underline{V}_1^k \qquad (5.93b)$$

then

$$\underline{V}_n^k = -{}^0\underline{V}_2^k \, \alpha_k + {}^0\underline{V}_1^k \, \beta_k \qquad (5.94)$$

Finally, we need to recognize the use of the following stress-strain law

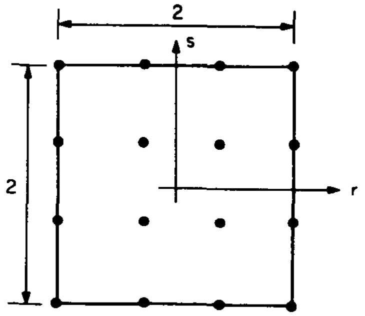
$$\underline{\tau} = \underline{C}_{sh}\, \underline{\varepsilon} \qquad (5.100)$$

$$\underline{\varepsilon}^T = [\varepsilon_{xx}\ \varepsilon_{yy}\ \varepsilon_{zz}\ \gamma_{xy}\ \gamma_{yz}\ \gamma_{zx}]$$

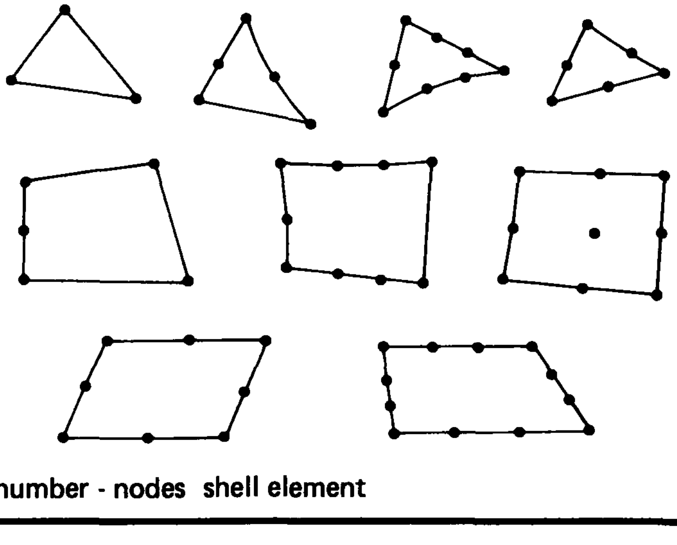
$$\underline{C}_{sh} = \underline{Q}_{sh}^T \left(\frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & \frac{1-\nu}{2} & 0 & 0 \\ & & & & \frac{1-\nu}{2} & 0 \\ \text{symmetric} & & & & & \frac{1-\nu}{2} \end{bmatrix}\right) \underline{Q}_{sh}$$

$$(5.101)$$

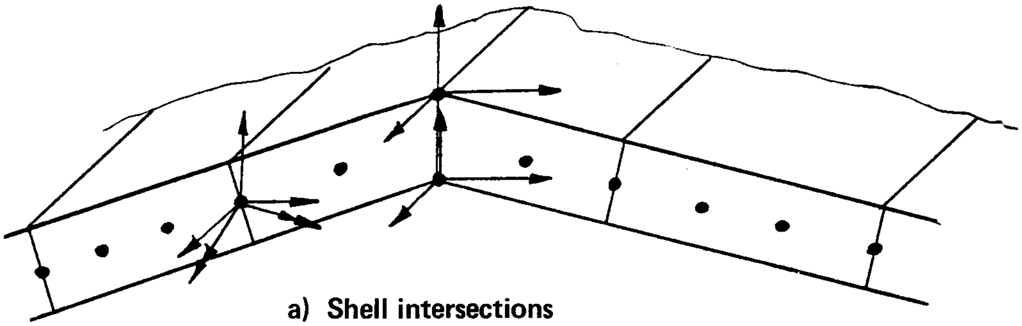
16 - node parent element with cubic interpolation



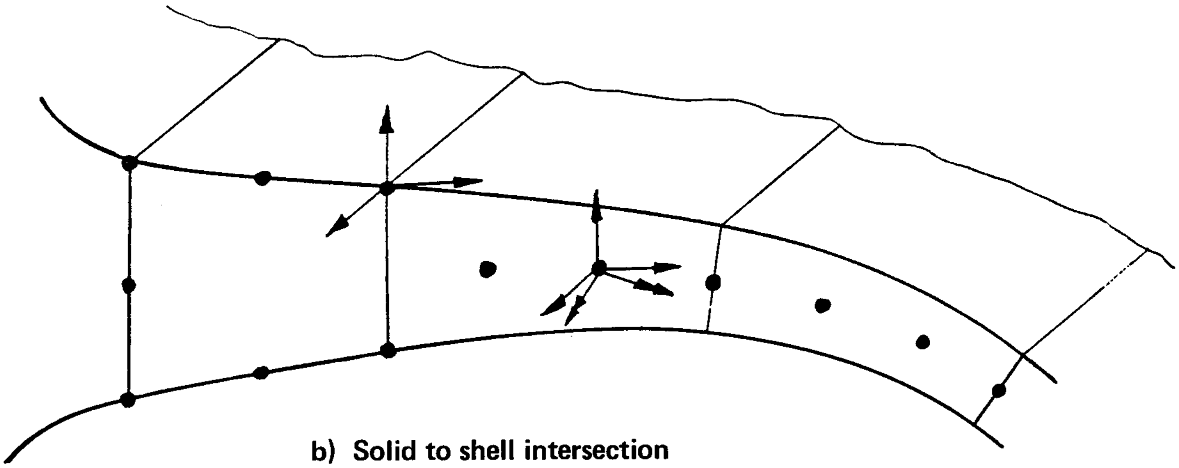
Some derived elements:



Variable - number - nodes shell element



a) Shell intersections



b) Solid to shell intersection

Fig. 5.39. Use of shell transition elements

NUMERICAL INTEGRATIONS, MODELING CONSIDERATIONS

LECTURE 8

47 MINUTES

LECTURE 8 Evaluation of isoparametric element matrices

Numercial integrations, Gauss, Newton-Cotes formulas

Basic concepts used and actual numerical operations performed

Practical considerations

Required order of integration, simple examples

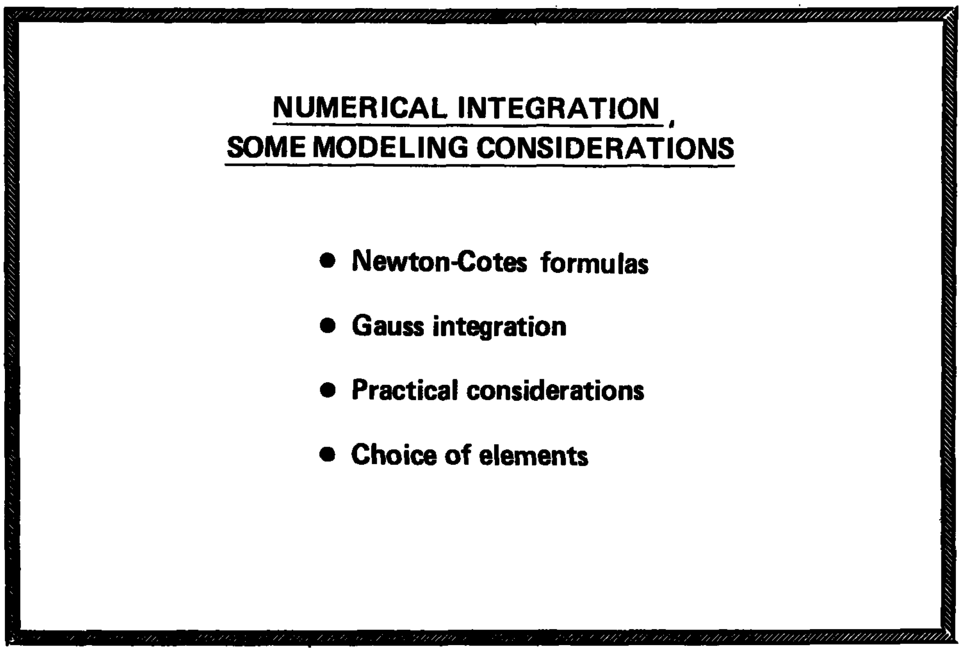
Calculation of stresses

Recommended elements and integration orders for one-, two-, three-dimensional analysis, and plate and shell structures

Modeling considerations using the elements

TEXTBOOK: Sections: 5.7.1, 5.7.2, 5.7.3, 5.7.4, 5.8.1, 5.8.2, 5.8.3

Examples: 5.28, 5.29, 5.30, 5.31, 5.32, 5.33, 5.34, 5.35, 5.36, 5.37, 5.38, 5.39



NUMERICAL INTEGRATION, SOME MODELING CONSIDERATIONS

- Newton-Cotes formulas
- Gauss integration
- Practical considerations
- Choice of elements

We had

$$\underline{K} = \int_V \underline{B}^T \, \underline{C} \, \underline{B} \, dV \qquad (4.29)$$

$$\underline{M} = \int_V \rho \, \underline{H}^T \, \underline{H} \, dV \qquad (4.30)$$

$$\underline{R}_B = \int_V \underline{H}^T \, \underline{f}^B \, dV \qquad (4.31)$$

$$\underline{R}_S = \int_S \underline{H}^{S^T} \, \underline{f}^S \, dS \qquad (4.32)$$

$$\underline{R}_I = \int_V \underline{B}^T \, \underline{\tau}^I \, dV \qquad (4.33)$$

In isoparametric finite element analysis we have:

- the displacement interpolation matrix $\underline{H}$ (r,s,t)
- the strain-displacement interpolation matrix $\underline{B}$ (r,s,t)

  Where r,s,t vary from −1 to +1.

Hence we need to use:

$$dV = \det \underline{J}\ dr\ ds\ dt$$

Hence, we now have, for example in two-dimensional analysis:

$$\underline{K} = \int_{-1}^{+1}\int_{-1}^{+1} \underline{B}^T\ \underline{C}\ \underline{B}\ \det \underline{J}\ dr\ ds$$

$$\underline{M} = \int_{-1}^{+1}\int_{-1}^{+1} \rho\ \underline{H}^T\ \underline{H}\ \det \underline{J}\ dr\ ds$$

etc...

The evaluation of the integrals is carried out effectively using numerical integration, e.g.:

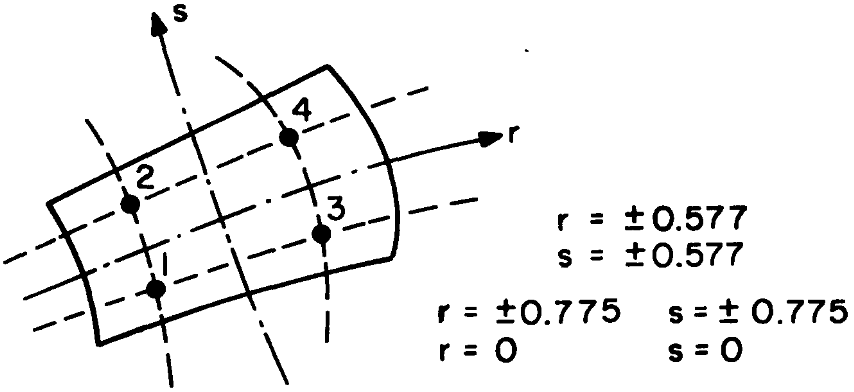
$$\underline{K} = \sum_i \sum_j \alpha_{ij} \underline{F}_{ij}$$

where

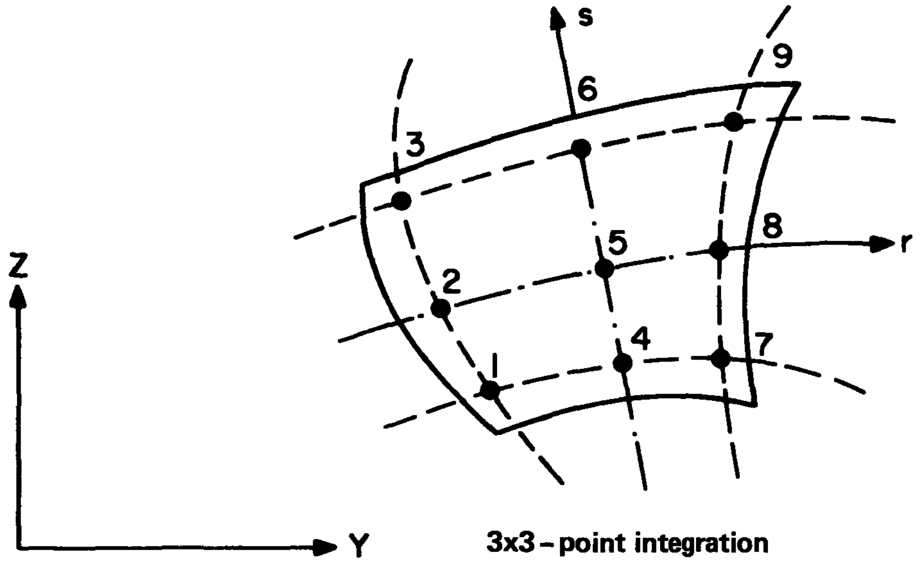
i, j denote the integration points

α_{ij} = weight coefficients

$\underline{F}_{ij} = \underline{B}_{ij}^T \underline{C} \underline{B}_{ij} \det \underline{J}_{ij}$

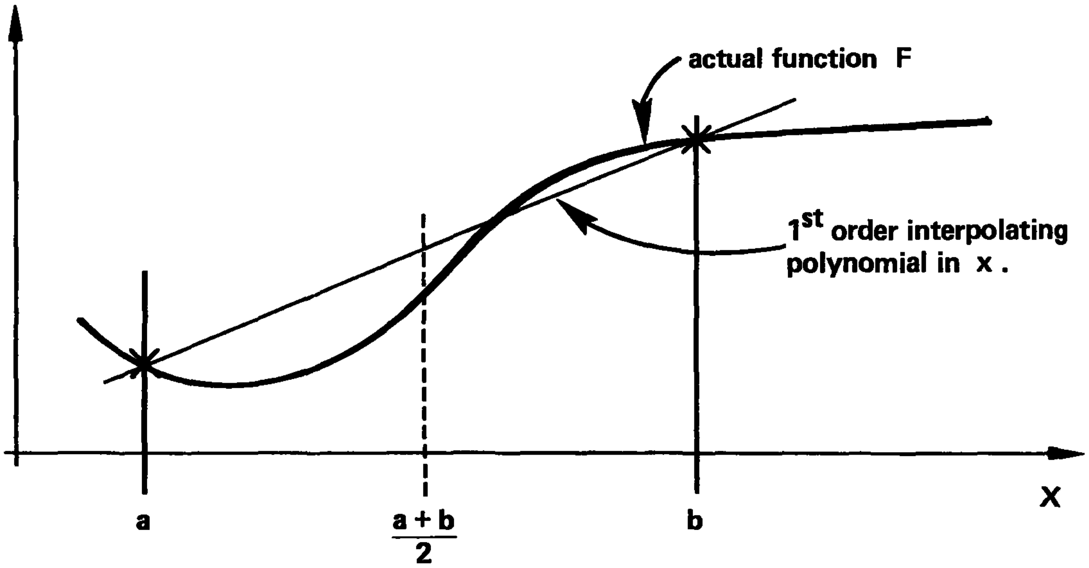


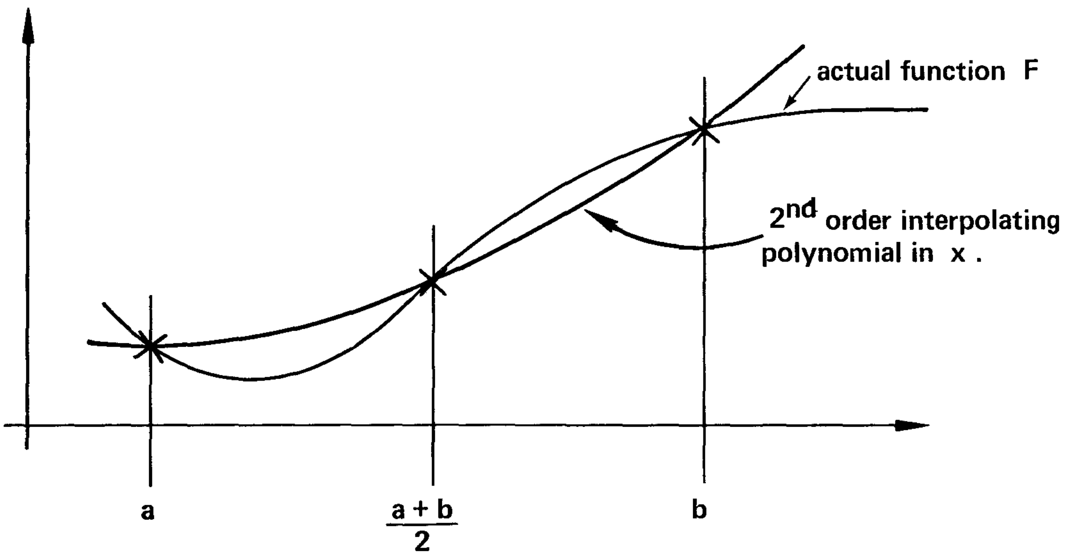
2x2-point integration



3x3 - point integration

Consider one-dimensional integration and the concept of an interpolating polynomial.





etc....

In <u>Newton - Cotes integration</u> we use sampling points at equal distances, and

$$\int_a^b F(r)dr = (b-a) \sum_{i=0}^{n} C_i^n F_i + R_n$$

(5.123)

n = number of intervals

C_i^n = Newton - Cotes constants

interpolating polynomial is of order n .

Number of Intervals n	C_0^n	C_1^n	C_2^n	C_3^n	C_4^n	C_5^n	C_6^n	Upper Bound on Error R_n as a Function of the Derivative of F
1	$\frac{1}{2}$	$\frac{1}{2}$						$10^{-1}(b-a)^3F^{\text{II}}(r)$
2	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$					$10^{-3}(b-a)^5F^{\text{IV}}(r)$
3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$				$10^{-3}(b-a)^5F^{\text{IV}}(r)$
4	$\frac{7}{90}$	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$			$10^{-6}(b-a)^7F^{\text{VI}}(r)$
5	$\frac{19}{288}$	$\frac{75}{288}$	$\frac{50}{288}$	$\frac{50}{288}$	$\frac{75}{288}$	$\frac{19}{288}$		$10^{-6}(b-a)^7F^{\text{VI}}(r)$
6	$\frac{41}{840}$	$\frac{216}{840}$	$\frac{27}{840}$	$\frac{272}{840}$	$\frac{27}{840}$	$\frac{216}{840}$	$\frac{41}{840}$	$10^{-9}(b-a)^9F^{\text{VIII}}(r)$

Table 5.1. Newton-Cotes numbers and error estimates.

In Gauss numerical integration we use

$$\int_a^b F(r)dr = \alpha_1 F(r_1) + \alpha_2 F(r_2) + \dots$$

$$+ \alpha_n F(r_n) + R_n \qquad (5.124)$$

where both the weights $\alpha_1, \dots, \alpha_n$ and the sampling points $r_1, \dots, r_n$ are variables.

The interpolating polynomial is now of order $2n-1$.

n	r_i	α_i
1	0. (15 zeros)	2. (15 zeros)
2	±0.57735 02691 89626	1.00000 00000 00000
3	±0.77459 66692 41483	0.55555 55555 55556
	0.00000 00000 00000	0.88888 88888 88889
4	±0.86113 63115 94053	0.34785 48451 37454
	±0.33998 10435 84856	0.65214 51548 62546
5	±0.90617 98459 38664	0.23692 68850 56189
	±0.53846 93101 05683	0.47862 86704 99366
	0.00000 00000 00000	0.56888 88888 88889
6	±0.93246 95142 03152	0.17132 44923 79170
	±0.66120 93864 66265	0.36076 15730 48139
	±0.23861 91860 83197	0.46791 39345 72691

Table 5.2. Sampling points and weights in Gauss-Legendre numerical integration.

Now let,

r_i be a sampling point and

α_i be the corresponding weight

for the interval -1 to +1.

Then the actual sampling point and weight for the interval a to b are

$$\frac{a+b}{2} + \frac{b-a}{2} r_i \text{ and } \frac{b-a}{2}\alpha_i$$

and the r_i and α_i can be tabulated as in Table 5.2.

In two- and three-dimensional analysis we use

$$\int_{-1}^{+1}\int_{-1}^{+1} F(r,s)\,dr\,ds = \sum_{i} \alpha_i \int_{-1}^{+1} F(r_i,s)\,ds \qquad (5.131)$$

or

$$\int_{-1}^{+1}\int_{-1}^{+1} F(r,s)\,dr\,ds = \sum_{i,j} \alpha_i\alpha_j F(r_i,s_j) \qquad (5.132)$$

---

and corresponding to (5.113), $\alpha_{ij} = \alpha_i\alpha_j$, where α_i and α_j are the integration weights for one-dimensional integration. Similarly,

$$\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} F(r,s,t)\,dr\,ds\,dt$$

$$= \sum_{i,j,k} \alpha_i\alpha_j\alpha_k F(r_i,s_j,t_k) \qquad (5.133)$$

and $\alpha_{ijk} = \alpha_i\alpha_j\alpha_k$.

Practical use of numerical integration

- The integration order required to evaluate a specific element matrix exactly can be evaluated by studying the function $\underline{F}$ to be integrated.
- In practice, the integration is frequently not performed exactly, but the integration order must be high enough.

Considering the evaluation of the element matrices, we note the following requirements:

a) stiffness matrix evaluation:

(1) the element matrix does not contain any spurious zero energy modes (i.e., the rank of the element stiffness matrix is not smaller than evaluated exactly); and

(2) the element contains the required constant strain states.

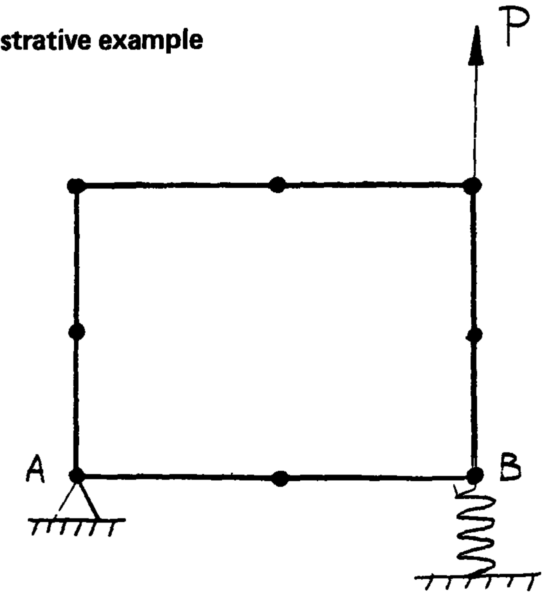
b) mass matrix evaluation:

the total element mass must be included.

c) force vector evaluations:

the total loads must be included.

Demonstrative example



2x2 Gauss integration
"absurd" results

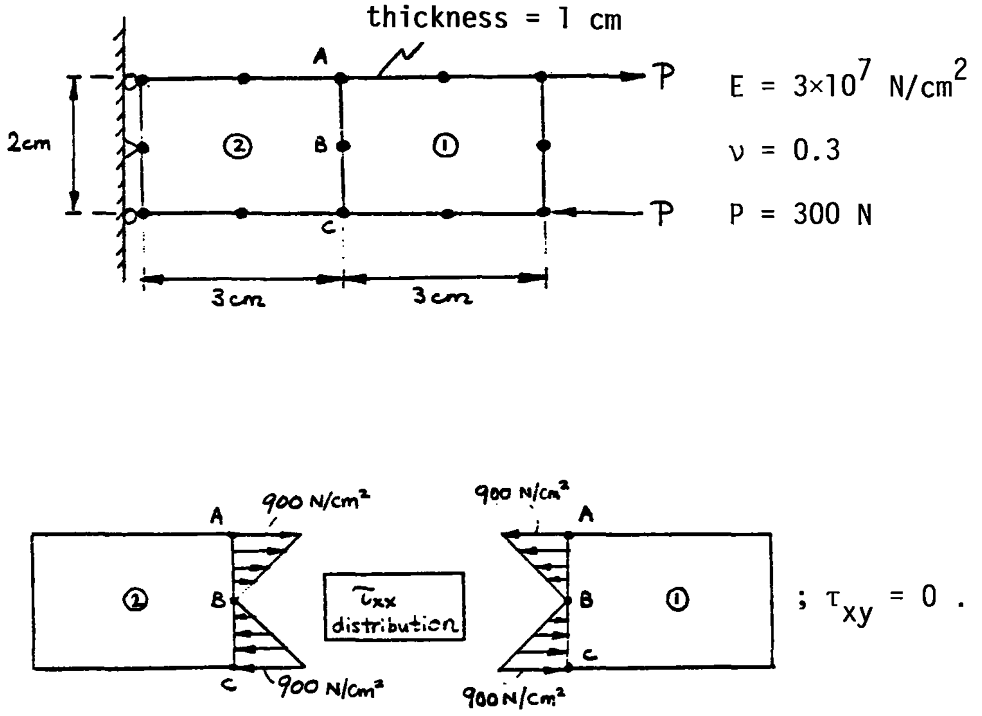
3x3 Gauss integration
correct results

Fig. 5.46. 8 - node plane stress element supported at B by a spring.

Stress calculations

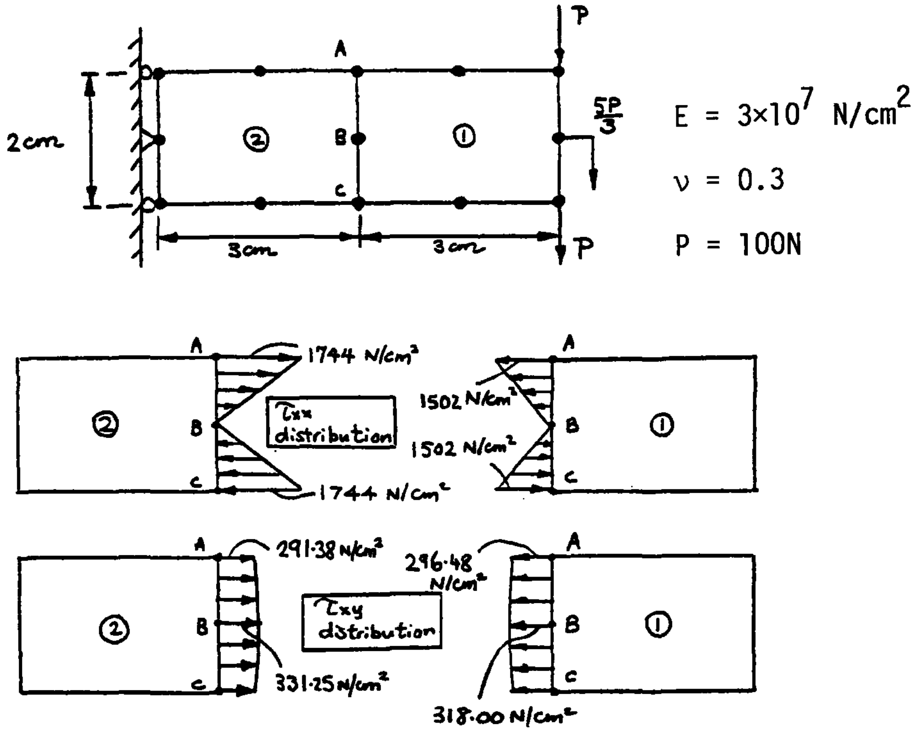
$$\underline{\tau} = \underline{C}\ \underline{B}\ \underline{U} + \underline{\tau}^{I} \qquad (5.136)$$

- **stresses can be calculated at any point of the element.**
- **stresses are, in general, discontinuous across element boundaries.**



(a) Cantilever subjected to bending moment and finite element solutions.

Fig. 5.47. Predicted longitudinal stress distributions in analysis of cantilever.



(b) Cantilever subjected to tip-shear force and finite element solutions

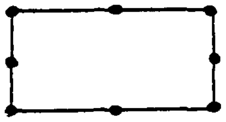
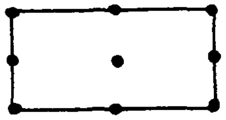
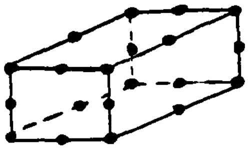
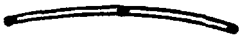
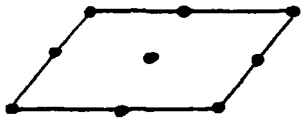
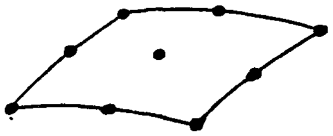
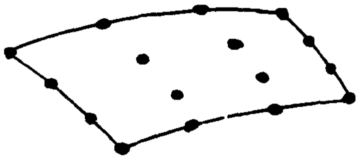
Fig. 5.47. Predicted longitudinal stress distributions in analysis of cantilever.

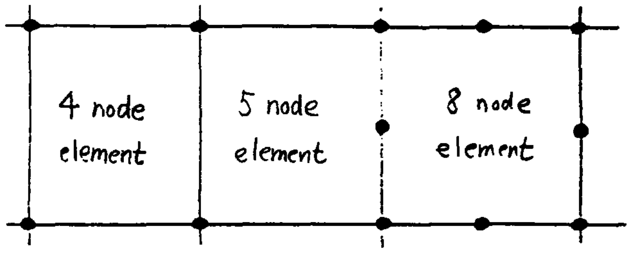
Some modeling considerations

We need

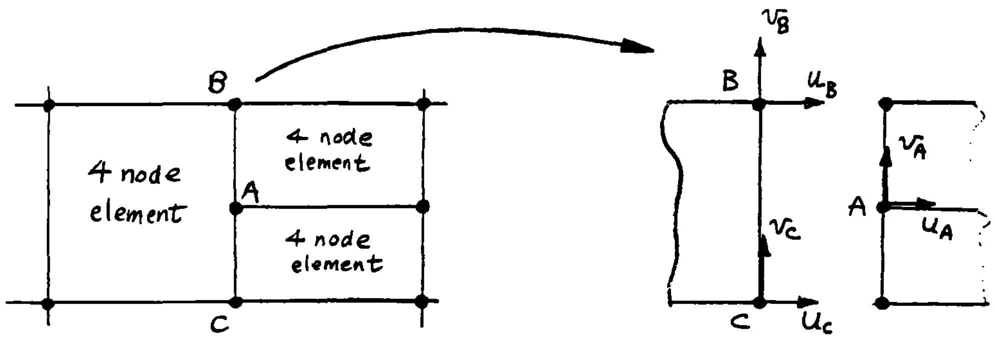
- a qualitative knowledge of the response to be predicted
- a thorough knowledge of the principles of mechanics and the finite element procedures available
- parabolic/undistorted elements usually most effective

Table 5.6 Elements usually effective in analysis.

TYPE OF PROBLEM	ELEMENT	
TRUSS OR CABLE	2-node	
TWO-DIMENSIONAL PLANE STRESS PLANE STRAIN AXISYMMETRIC	8-node or 9-node	 
THREE-DIMENSIONAL	20-node	
3-D BEAM	3-node or 4-node	 
PLATE	9-node	
SHELL	9-node or 16-node	 



a) 4-node to 8-node element transition region

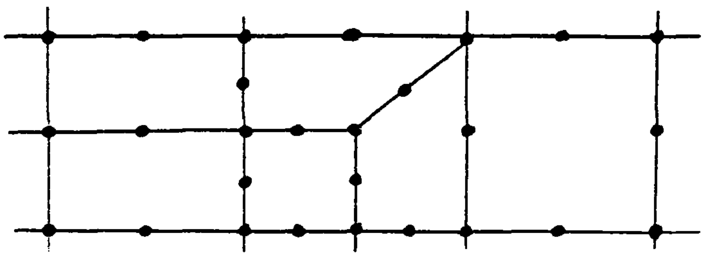


Constraint equations:

$$u_A = (u_C + u_B)/2$$

$$v_A = (v_C + v_B)/2$$

b) 4-node to 4-node element transition



c) 8-node to finer 8-node element layout transition region

Fig. 5.49. Some transitions with compatible element layouts

SOLUTION OF FINITE ELEMENT EQUILIBRIUM EQUATIONS IN STATIC ANALYSIS

LECTURE 9

60 MINUTES

LECTURE 9 Solution of finite element equations in static analysis

Basic Gauss elimination

Static condensation

Substructuring

Multi-level substructuring

Frontal solution

$\underline{L}\,\underline{D}\,\underline{L}^T$ - factorization (column reduction scheme) as used in SAP and ADINA

Cholesky factorization

Out-of-core solution of large systems

Demonstration of basic techniques using simple examples

Physical interpretation of the basic operations used

TEXTBOOK: Sections: 8.1, 8.2.1, 8.2.2, 8.2.3, 8.2.4,

Examples: 8.1, 8.2, 8.3, 8.4, 8.5, 8.6, 8.7, 8.8, 8.9, 8.10

SOLUTION OF EQUILIBRIUM EQUATIONS IN STATIC ANALYSIS

$$\underline{K}\;\underline{U} = \underline{R}$$

- Iterative methods, e.g. Gauss-Seidel
- Direct methods

  these are basically variations of Gauss elimination

– static condensation
– substructuring
– frontal solution
– $\underline{L}\,\underline{D}\,\underline{L}^T$ factorization
– Cholesky decomposition
– Crout
– column reduction (skyline) solver

THE BASIC GAUSS ELIMINATION PROCEDURE

Consider the Gauss elimination solution of

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad (8.2)$$

STEP 1: Subtract a multiple of equation 1 from equations 2 and 3 to obtain zero elements in the first column of K .

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & -\frac{16}{5} & \frac{29}{5} & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{8.3}$$

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & 0 & \frac{15}{7} & -\frac{20}{7} \\ 0 & 0 & -\frac{20}{7} & \frac{65}{14} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{8}{7} \\ -\frac{5}{14} \end{bmatrix} \tag{8.4}$$

STEP 3:

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & 0 & \frac{15}{7} & -\frac{20}{7} \\ 0 & 0 & 0 & \frac{5}{6} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{8}{7} \\ \frac{7}{6} \end{bmatrix} \qquad (8.5)$$

Now solve for the unknowns U_4, U_3, U_2 and U_1 :

$$U_4 = \frac{\frac{7}{6}}{\frac{5}{6}} = \frac{7}{5} \; ; \quad U_3 = \frac{\frac{8}{7} - (-\frac{20}{7})\, U_4}{\frac{15}{7}} = \frac{12}{5}$$

$$U_2 = \frac{1 - (-\frac{16}{5})\, U_3 - (1)\, U_4}{\frac{14}{5}} = \frac{13}{5} \qquad (8.6)$$

$$U_1 = \frac{0 - (-4)\frac{19}{35} - (1)\frac{36}{15} - (0)\,\frac{7}{5}}{5} = \frac{8}{5}$$

STATIC CONDENSATION

Partition matrices into

$$\begin{bmatrix} \underline{K}_{aa} & \underline{K}_{ac} \\ \underline{K}_{ca} & \underline{K}_{cc} \end{bmatrix} \begin{bmatrix} \underline{U}_{a} \\ \underline{U}_{c} \end{bmatrix} = \begin{bmatrix} \underline{R}_{a} \\ \underline{R}_{c} \end{bmatrix} \qquad (8.28)$$

Hence

$$\underline{U}_{c} = \underline{K}_{cc}^{-1} \, (\underline{R}_{c} - \underline{K}_{ca} \, \underline{U}_{a})$$

and

$$\underbrace{(\underline{K}_{aa} - \underline{K}_{ac} \, \underline{K}_{cc}^{-1} \, \underline{K}_{ca})}_{\overline{\underline{K}}_{aa}} \, \underline{U}_{a} = \underline{R}_{a} - \underline{K}_{ac} \, \underline{K}_{cc}^{-1} \, \underline{R}_{c}$$

Example

$$\left[\begin{array}{c|ccc} \overset{\underline{K}_{cc}}{5} & -4 & \overset{\underline{K}_{ca}}{1} & 0 \\ \hline -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ \underset{\underline{K}_{ac}}{0} & \multicolumn{3}{c}{\underbrace{1 \qquad -4 \qquad 5}_{\underline{K}_{aa}}} \end{array}\right] \left[\begin{array}{c} U_1 \\ \hline U_2 \\ U_3 \\ U_4 \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline 1 \\ 0 \\ 0 \end{array}\right]$$

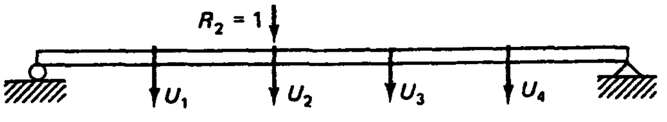
Hence (8.30) gives

$$\overline{\underline{K}}_{aa} = \begin{bmatrix} 6 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 5 \end{bmatrix} - \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} [1/5] \; [-4 \quad 1 \quad 0]$$

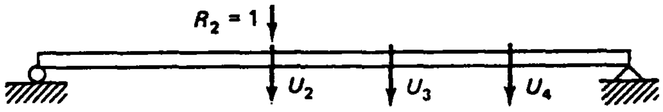
so that

$$\overline{\underline{K}}_{aa} = \begin{bmatrix} \frac{14}{5} & -\frac{16}{5} & 1 \\ -\frac{16}{5} & \frac{29}{5} & -4 \\ 1 & -4 & 5 \end{bmatrix}$$

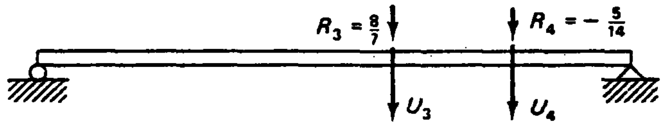
and we have obtained the 3x3 unreduced matrix in (8.3)



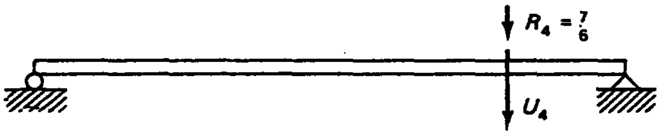
$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} \frac{14}{5} & -\frac{16}{5} & 1 \\ -\frac{16}{5} & \frac{29}{5} & -4 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} U_2 \\ U_3 \\ U_4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} \frac{15}{7} & -\frac{20}{7} \\ -\frac{20}{7} & \frac{65}{14} \end{bmatrix} \begin{bmatrix} U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} \frac{8}{7} \\ -\frac{5}{14} \end{bmatrix}$$

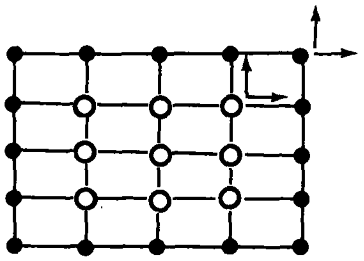


$$\begin{bmatrix} \frac{5}{6} \end{bmatrix} \begin{bmatrix} U_4 \end{bmatrix} = \begin{bmatrix} \frac{7}{6} \end{bmatrix}$$

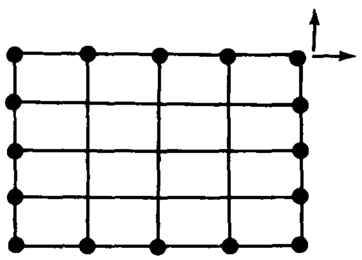
Fig. 8.1 Physical systems considered in the Gauss elimination solution of the simply supported beam.

SUBSTRUCTURING

- We use static condensation on the internal degrees of freedom of a substructure
- the result is a new stiffness matrix of the substructure involving boundary degrees of freedom only



50 x 50



32 x 32

Example

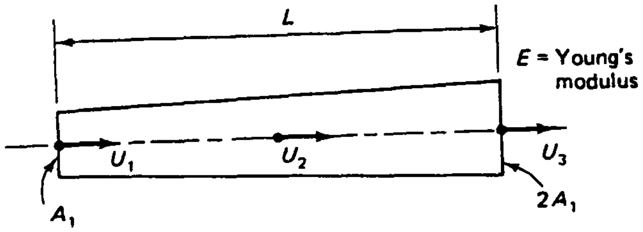


Fig. 8.3. Truss element with linearly varying area.

We have for the element,

$$\frac{EA_1}{6L}\begin{bmatrix} 17 & -20 & 3 \\ -20 & 48 & -28 \\ 3 & -28 & 25 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$$

First rearrange the equations

$$\frac{EA_1}{6L}\begin{bmatrix} 17 & 3 & -20 \\ 3 & 25 & -28 \\ -20 & -28 & 48 \end{bmatrix}\begin{bmatrix} U_1 \\ U_3 \\ U_2 \end{bmatrix}=\begin{bmatrix} R_1 \\ R_3 \\ R_2 \end{bmatrix}$$

Static condensation of U_2 gives

$$\frac{EA_1}{6L}\left\{\begin{bmatrix} 17 & 3 \\ 3 & 25 \end{bmatrix}-\begin{bmatrix} -20 \\ -28 \end{bmatrix}[\tfrac{1}{48}][-20 \quad -28]\right\}\begin{bmatrix} U_1 \\ U_3 \end{bmatrix}$$

$$=\begin{bmatrix} R_1 + \frac{20}{48} R_2 \\ R_3 + \frac{28}{48} R_2 \end{bmatrix}$$

---

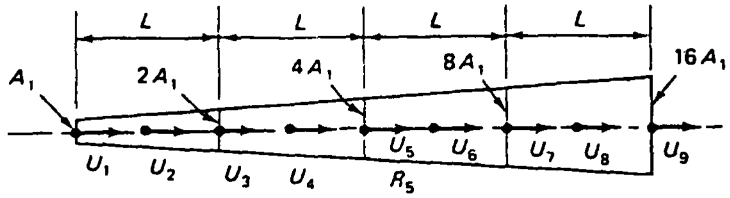
or

$$\frac{13}{9}\frac{EA_1}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} U_1 \\ U_3 \end{bmatrix}=\begin{bmatrix} R_1 + \frac{5}{12} R_2 \\ R_3 + \frac{7}{12} R_2 \end{bmatrix}$$

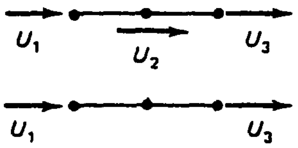
and

$$U_2 = \frac{1}{24}\left(\frac{3L}{EA_1} R_2 + 10\,U_1 + 14\,U_3\right)$$

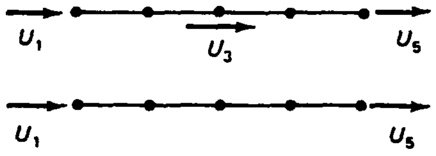
Multi-level Substructuring



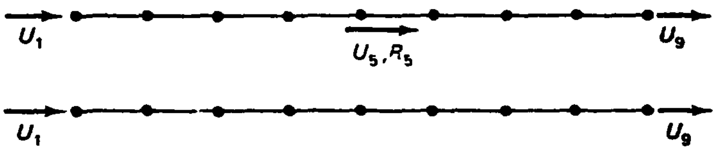
Bar with linearly varying area



(a) First-level substructure



(b) Second-level substructure



(c) Third-level substructure and actual structure.

Fig. 8.5. Analysis of bar using substructuring.

Frontal Solution

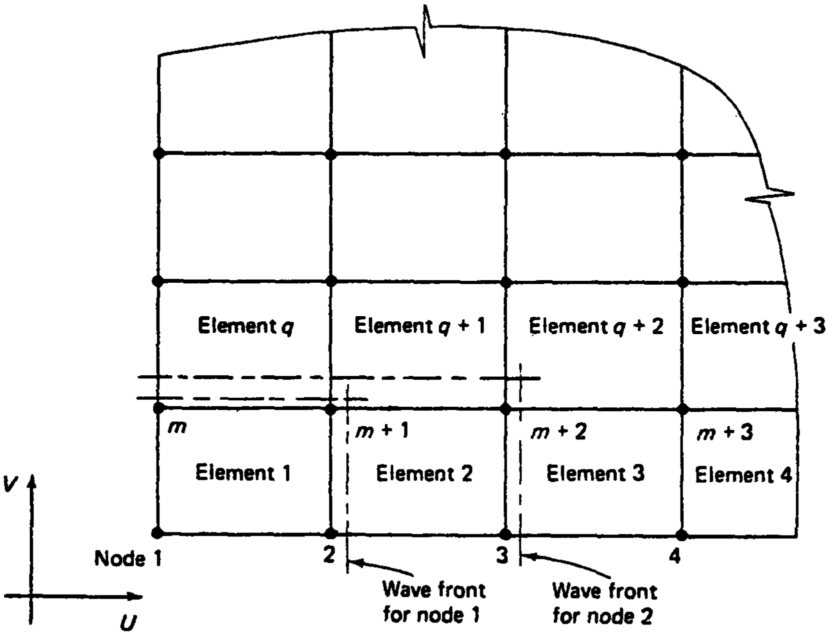


Fig. 8.6. Frontal solution of plane stress finite element idealization.

- **The frontal solution consists of successive static condensation of nodal degrees of freedom.**
- **Solution is performed in the order of the element numbering.**
- **Same number of operations are performed in the frontal solution as in the skyline solution, if the element numbering in the wave front solution corresponds to the nodal point numbering in the skyline solution.**

$\underline{L}\ \underline{D}\ \underline{L}^T$ FACTORIZATION

- is the basis of the skyline solution (column reduction scheme)

- Basic Step

$$\underline{L}_1^{-1}\ \underline{K} = \underline{K}_1$$

Example:

$$\begin{bmatrix} 1 & & & \\ \frac{4}{5} & 1 & & \\ -\frac{1}{5} & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & -\frac{16}{5} & \frac{29}{5} & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

We note

$$\underline{L}_1^{-1} = \begin{bmatrix} 1 & & & \\ \frac{4}{5} & 1 & & \\ -\frac{1}{5} & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad \underline{L}_1 = \begin{bmatrix} 1 & & & \\ -\frac{4}{5} & 1 & & \\ \frac{1}{5} & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Proceeding in the same way

$$\underline{L}_{n-1}^{-1}\,\underline{L}_{n-2}^{-1}\cdots\underline{L}_{2}^{-1}\,\underline{L}_{1}^{-1}\,\underline{K}=\underline{S}$$

$$\underline{S} = \left[\begin{array}{cccccc} x & x & x & x & \cdots & x \\ & x & x & x & \cdots & x \\ & & x & \cdots & \cdots & x \\ & & & x & \cdots & x \\ & & & & x & \\ & & & & & \ddots\ x \end{array}\right]\left.\vphantom{\begin{array}{c}x\\x\\x\\x\\x\end{array}}\right\}\begin{array}{l}\textbf{upper}\\\textbf{triangular}\\\textbf{matrix}\end{array}$$

Hence

$$\underline{K}=(\underline{L}_1\,\underline{L}_2\cdots\underline{L}_{n-2}\,\underline{L}_{n-1})\,\underline{S}$$

or

$$\underline{K}=\underline{L}\,\underline{S}\ ;\ \underline{L}=\underline{L}_1\,\underline{L}_2\cdots\underline{L}_{n-2}\,\underline{L}_{n-1}$$

Also, because $\underline{K}$ is symmetric

$$\underline{K}=\underline{L}\,\underline{D}\,\underline{L}^T\ ;$$

where

$$\underline{D}=\ \text{diagonal matrix}\ ;\ d_{ii}=s_{ii}$$

In the Cholesky factorization, we use

$$\underline{K} = \underline{\tilde{L}}\ \underline{\tilde{L}}^T$$

where

$$\underline{\tilde{L}} = \underline{L}\ \underline{D}^{\frac{1}{2}}$$

SOLUTION OF EQUATIONS

Using

$$\underline{K} = \underline{L}\ \underline{D}\ \underline{L}^T \qquad (8.16)$$

we have

$$\underline{L}\ \underline{V} = \underline{R} \qquad (8.17)$$

$$\underline{D}\ \underline{L}^T\ \underline{U} = \underline{V} \qquad (8.18)$$

where

$$\underline{V} = \underline{L}_{n-1}^{-1} \cdots \underline{L}_2^{-1}\ \underline{L}_1^{-1}\ \underline{R} \qquad (8.19)$$

and

$$\underline{L}^T\ \underline{U} = \underline{D}^{-1}\underline{V} \qquad (8.20)$$

COLUMN REDUCTION SCHEME

$$\begin{bmatrix} 5 & -4 & 1 & \\ & 6 & -4 & 1 \\ & & 6 & -4 \\ & & & 5 \end{bmatrix}$$

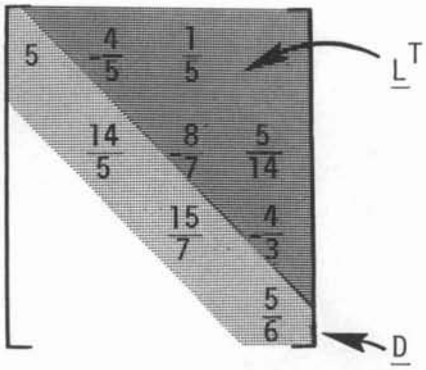
$$\downarrow$$

$$\left[\begin{array}{cc|cc} 5 & -\frac{4}{5} & 1 & \\ & \frac{14}{5} & -4 & 1 \\ & & 6 & -4 \\ & & & 5 \end{array}\right] \quad \left[\begin{array}{cc|cc} 5 & -\frac{4}{5} & 1 & \\ & \frac{14}{5} & -4 & 1 \\ & & 6 & -4 \\ & & & 5 \end{array}\right]$$

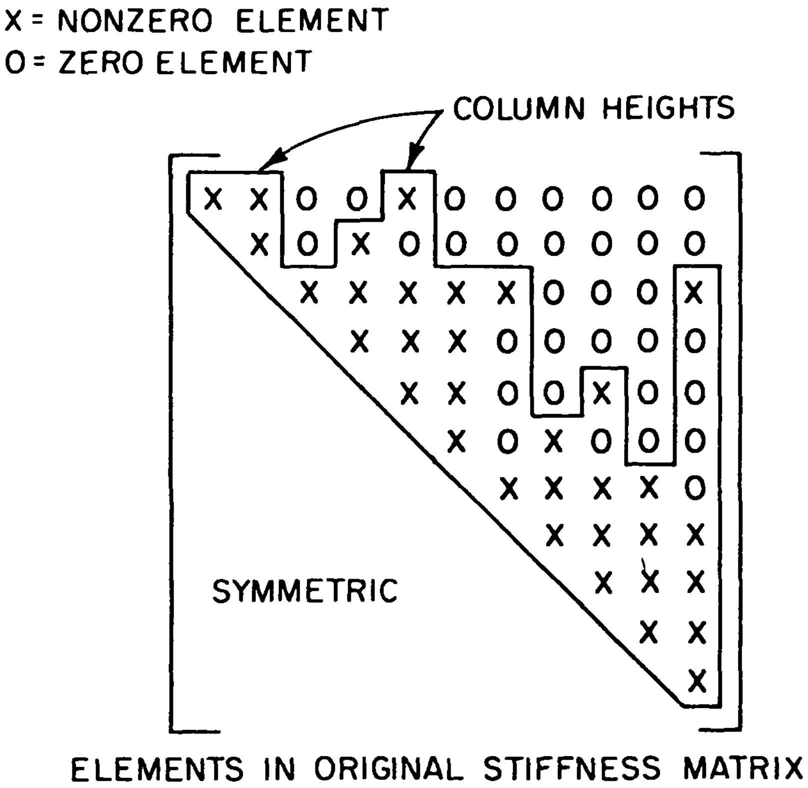
$$\downarrow$$

$$\left[\begin{array}{ccc|c} 5 & -\frac{4}{5} & \frac{1}{5} & \\ & \frac{14}{5} & -\frac{8}{7} & 1 \\ & & \frac{15}{7} & -4 \\ & & & 5 \end{array}\right] \quad \left[\begin{array}{ccc|c} 5 & -\frac{4}{5} & \frac{1}{5} & \\ & \frac{14}{5} & -\frac{8}{7} & 1 \\ & & \frac{15}{7} & -4 \\ & & & 5 \end{array}\right]$$

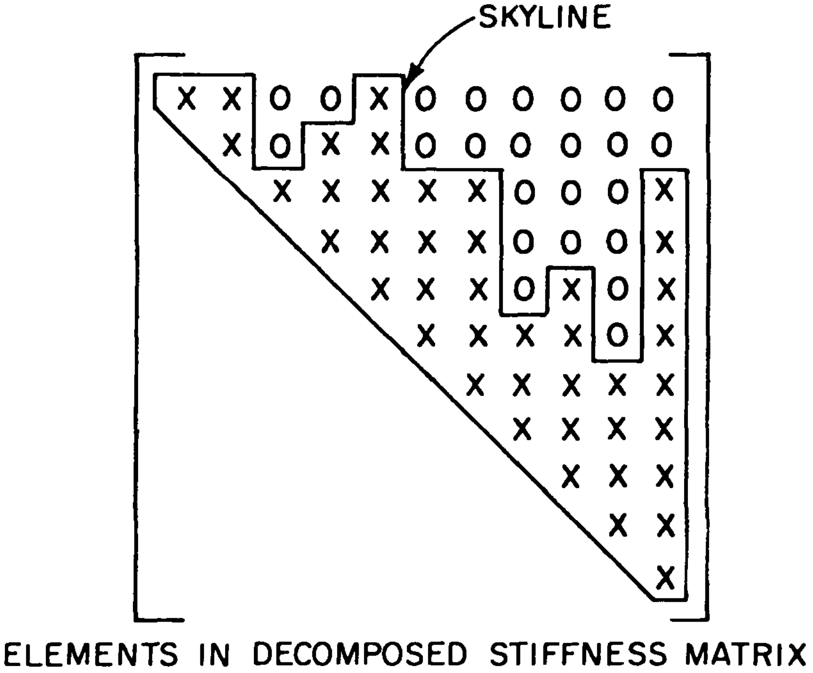
$$\downarrow$$



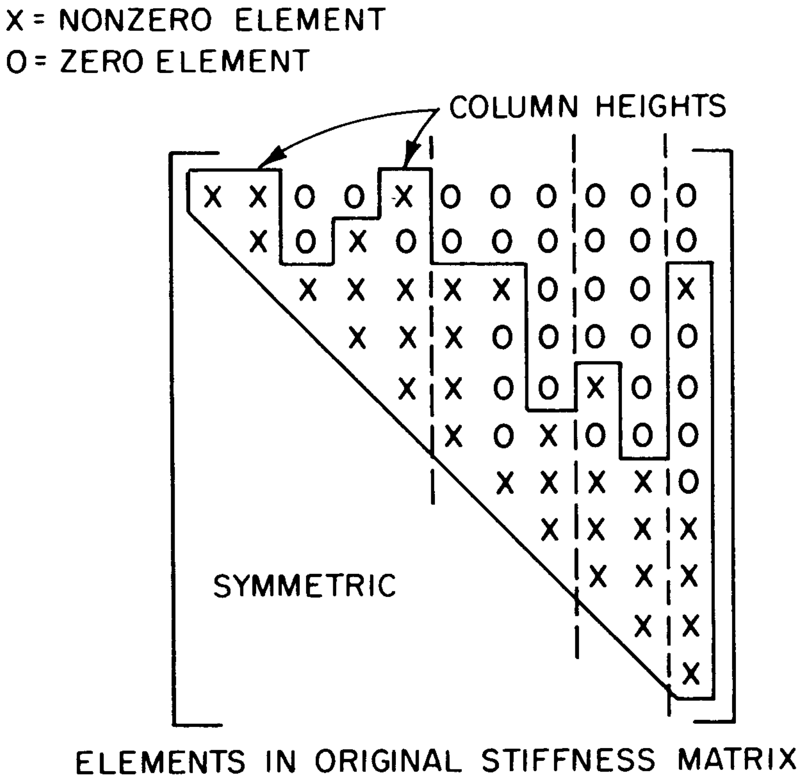
$$\begin{bmatrix} 5 & -\frac{4}{5} & \frac{1}{5} & \\ & \frac{14}{5} & -\frac{8}{7} & \frac{5}{14} \\ & & \frac{15}{7} & -\frac{4}{3} \\ & & & \frac{5}{6} \end{bmatrix}$$



Typical element pattern in
a stiffness matrix



Typical element pattern in
a stiffness matrix



Typical element pattern in a stiffness matrix using block storage.

SOLUTION OF FINITE ELEMENT EQUILIBRIUM EQUATIONS IN DYNAMIC ANALYSIS

LECTURE 10

56 MINUTES

LECTURE 10 Solution of dynamic response by direct integration

Basic concepts used

Explicit and implicit techniques

Implementation of methods

Detailed discussion of central difference and Newmark methods

Stability and accuracy considerations

Integration errors

Modeling of structural vibration and wave propagation problems

Selection of element and time step sizes

Recommendations on the use of the methods in practice

TEXTBOOK: Sections: 9.1, 9.2.1, 9.2.2, 9.2.3, 9.2.4, 9.2.5, 9.4.1, 9.4.2, 9.4.3, 9.4.4

Examples: 9.1, 9.2, 9.3, 9.4, 9.5, 9.12

DIRECT INTEGRATION SOLUTION OF EQUILIBRIUM EQUATIONS IN DYNAMIC ANALYSIS

$$\underline{M}\,\ddot{\underline{U}} + \underline{C}\,\dot{\underline{U}} + \underline{K}\,\underline{U} = \underline{R}$$

- explicit, implicit integration
- computational considerations
- selection of solution time step (Δt)
- some modeling considerations

Equilibrium equations in dynamic analysis

$$\underline{M}\,\ddot{\underline{U}} + \underline{C}\,\dot{\underline{U}} + \underline{K}\,\underline{U} = \underline{R} \qquad (9.1)$$

or

$$\underline{F}_I(t) + \underline{F}_D(t) + \underline{F}_E(t) = \underline{R}(t) \qquad (9.2)$$

Load description

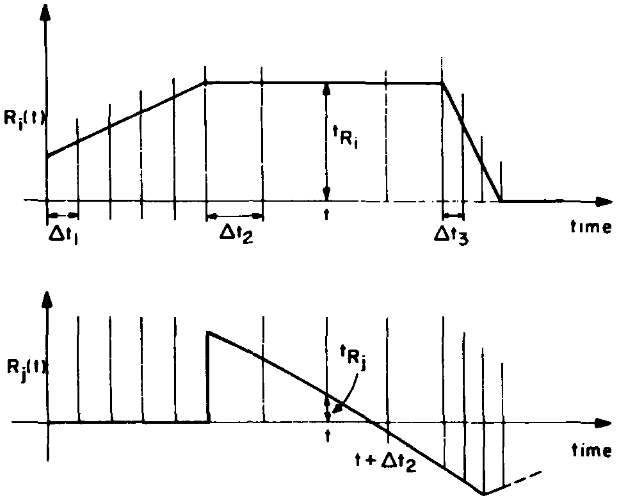


Fig. 1. Evaluation of externally applied nodal point load vector ^{t}R at time t.

THE CENTRAL DIFFERENCE METHOD (CDM)

$$^{t}\ddot{\underline{U}} = \frac{1}{\Delta t^{2}}\{^{t-\Delta t}\underline{U} - 2\,^{t}\underline{U} + {}^{t+\Delta t}\underline{U}\} \qquad (9.3)$$

$$^{t}\dot{\underline{U}} = \frac{1}{2\Delta t}(-{}^{t-\Delta t}\underline{U} + {}^{t+\Delta t}\underline{U}) \qquad (9.4)$$

$$\underline{M}\,^{t}\ddot{\underline{U}} + \underline{C}\,^{t}\dot{\underline{U}} + \underline{K}\,^{t}\underline{U} = {}^{t}\underline{R} \qquad (9.5)$$

an explicit integration scheme

Combining (9.3) to (9.5) we obtain

$$\left(\frac{1}{\Delta t^2}\underline{M} + \frac{1}{2\Delta t}\underline{C}\right){}^{t+\Delta t}\underline{U} = {}^{t}\underline{R} - \left(\underline{K} - \frac{2}{\Delta t^2}\underline{M}\right){}^{t}\underline{U}$$

$$-\left(\frac{1}{\Delta t^2}\underline{M} - \frac{1}{2\Delta t}\underline{C}\right){}^{t-\Delta t}\underline{U}$$

$$(9.6)$$

where we note

$$\underline{K}\;{}^{t}\underline{U} = \left(\sum_{m}\underline{K}^{(m)}\right){}^{t}\underline{U}$$

$$= \sum_{m}\left(\underline{K}^{(m)}\;{}^{t}\underline{U}\right) = \sum_{m}{}^{t}\underline{F}^{(m)}$$

Computational considerations

- to start the solution, use

$$-{}^{\Delta t}U(i) = {}^{0}U(i) - \Delta t\,{}^{0}\dot{U}(i) + \frac{\Delta t^2}{2}\,{}^{0}\ddot{U}(i)$$

$$(9.7)$$

- in practice, mostly used with lumped mass matrix and low-order elements.

Stability and Accuracy of CDM

- Δt must be smaller than Δt_{cr}

$\Delta t_{cr} = \frac{T_n}{\pi}$; T_n = smallest natural period in the system

hence method is conditionally stable

- in practice, use for continuum elements,

$$\Delta t \leq \frac{\Delta L}{c} \quad ; \quad c = \sqrt{\frac{E}{\rho}}$$

---

for lower-order elements

ΔL = smallest distance between nodes

for high-order elements

ΔL = (smallest distance between nodes)/(rel. stiffness factor)

- method used mainly for wave propagation analysis
- number of operations $\propto$ no. of elements and no. of time steps

THE NEWMARK METHOD

$$ {}^{t+\Delta t}\dot{\underline{U}} = {}^{t}\dot{\underline{U}} + [(1-\delta)\,{}^{t}\ddot{\underline{U}} + \delta\,{}^{t+\Delta t}\ddot{\underline{U}}]\,\Delta t \qquad (9.27) $$

$$ {}^{t+\Delta t}\underline{U} = {}^{t}\underline{U} + {}^{t}\dot{\underline{U}}\,\Delta t + [(\tfrac{1}{2}-\alpha)\,{}^{t}\ddot{\underline{U}} + \alpha\,{}^{t+\Delta t}\ddot{\underline{U}}]\,\Delta t^2 \qquad (9.28) $$

$$ \underline{M}\,{}^{t+\Delta t}\ddot{\underline{U}} + \underline{C}\,{}^{t+\Delta t}\dot{\underline{U}} + \underline{K}\,{}^{t+\Delta t}\underline{U} = {}^{t+\Delta t}\underline{R} \qquad (9.29) $$

an implicit integration scheme solution is obtained using

$$ \hat{\underline{K}}\,{}^{t+\Delta t}\underline{U} = {}^{t+\Delta t}\hat{\underline{R}} $$

- **In practice, we use mostly**

  $\alpha = \frac{1}{4}\ ,\ \delta = \frac{1}{2}$

  which is the

  constant-average-acceleration method (Newmark's method)

- **method is unconditionally stable**
- **method is used primarily for analysis of structural dynamics problems**
- **number of operations**

  $\doteq \frac{1}{2}nm^2 + 2nmt$

Accuracy considerations

- time step Δt is chosen based on accuracy considerations only
- Consider the equations

$$\underline{M}\,\ddot{\underline{U}} + \underline{K}\,\underline{U} = \underline{R}$$

and

$$\underline{U} = \sum_{i=1}^{n} \underline{\phi}_i\, x_i(t)$$

where

$$\underline{K}\,\underline{\phi}_i = \omega_i^2\, \underline{M}\,\underline{\phi}_i$$

Using

$$\underline{\Phi}^T\, \underline{K}\, \underline{\Phi} = \underline{\Omega}^2 \; ; \; \underline{\Phi}^T\, \underline{M}\, \underline{\Phi} = \underline{I}$$

where

$$\underline{\Phi} = [\underline{\phi}_1, \ldots, \underline{\phi}_n] \; ; \; \underline{\Omega}^2 = \begin{bmatrix} \omega_1^2 & & \\ & \ddots & \\ & & \omega_n^2 \end{bmatrix}$$

we obtain n equations from which to solve for $x_i(t)$ (see Lecture 11)

$$\ddot{x}_i + \omega_i^2\, x_i = \underline{\phi}_i^T\, \underline{R} \qquad i = 1, \ldots, n$$

Hence, the direct step-by-step solution of

$$\underline{M}\,\underline{\ddot{U}} + \underline{K}\,\underline{U} = \underline{R}$$

corresponds to the direct step-by-step solution of

$$\ddot{x}_i + \omega_i^2\,x_i = \underline{\phi}_i^T\,\underline{R} \qquad i = 1,\ldots,n$$

with

$$\underline{U} = \sum_{i=1}^{n} \underline{\phi}_i\,x_i$$

Therefore, to study the accuracy of the Newmark method, we can study the solution of the single degree of freedom equation

$$\ddot{x} + \omega^2\,x = r$$

Consider the case

$$\ddot{x} + \omega^2\,x = 0$$

$${}^{0}x = 1.0 \quad ; \quad {}^{0}\dot{x} = 0 \quad ; \quad {}^{0}\ddot{x} = -\omega^2$$

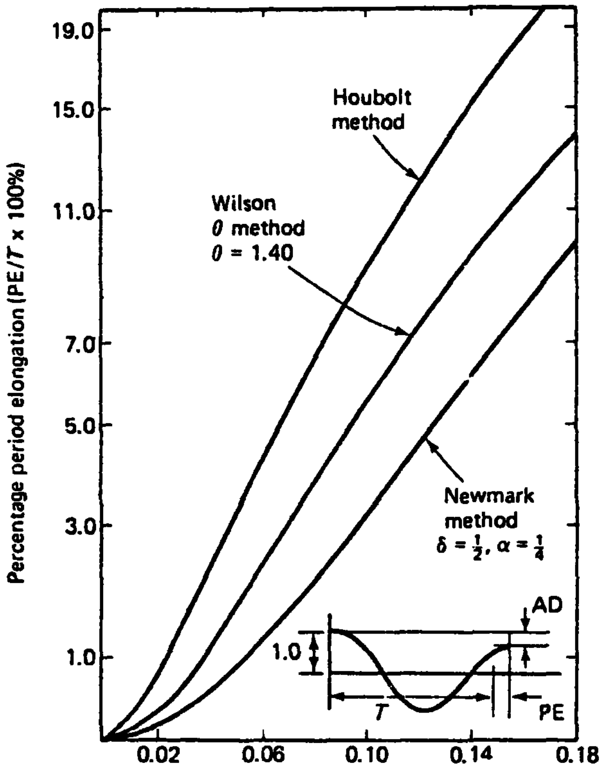


Fig. 9.8 (a) Percentage period elongations and amplitude decays.

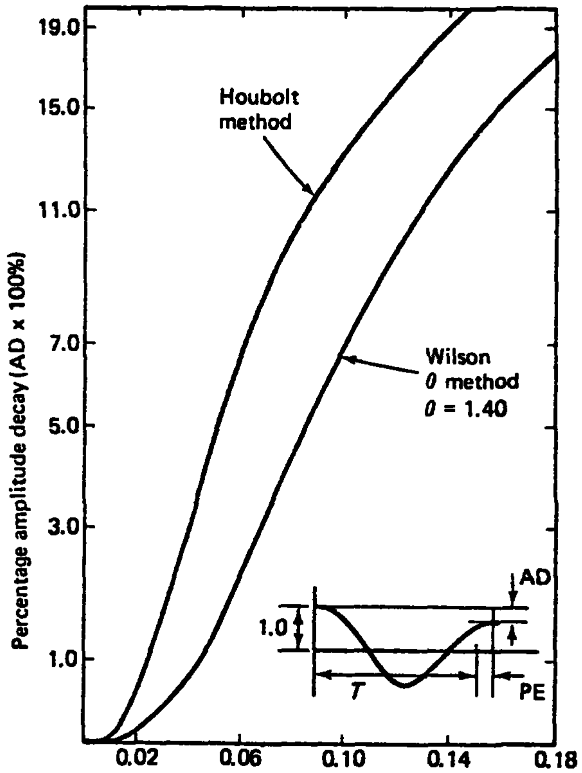


Fig. 9.8 (b) Percentage period elongations and amplitude decays.

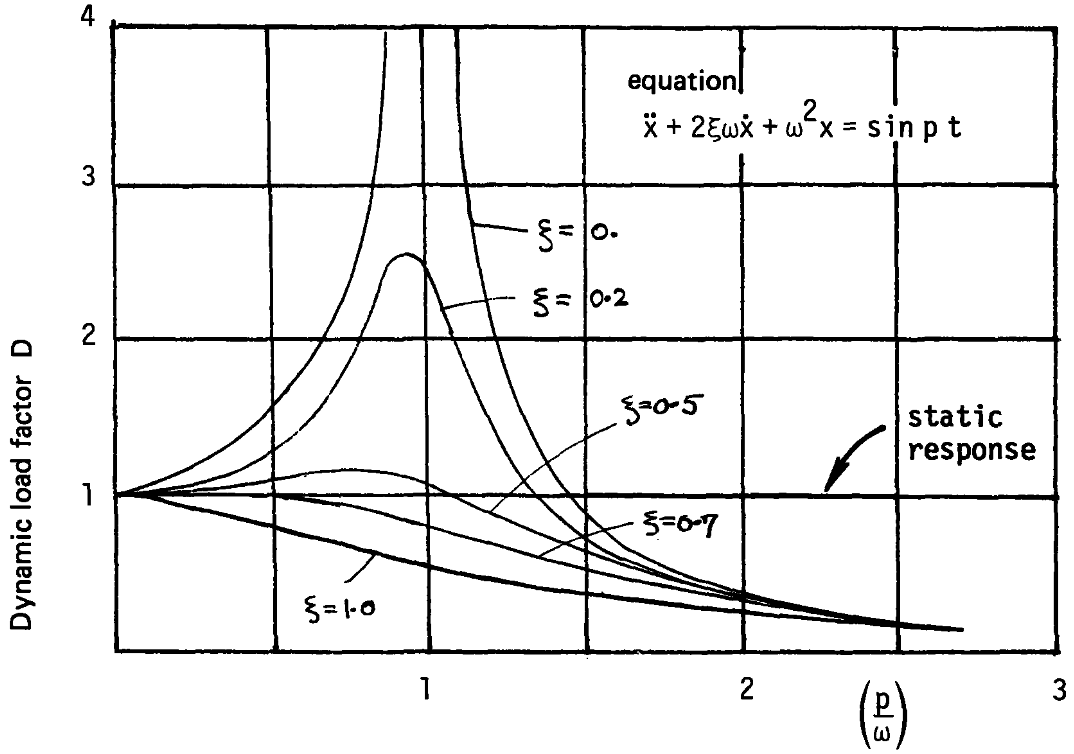
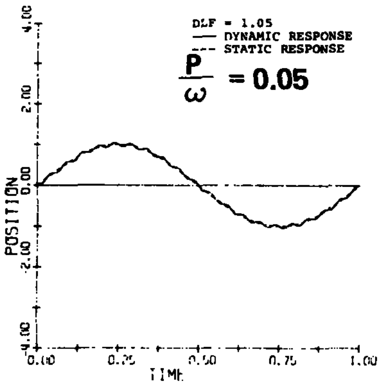
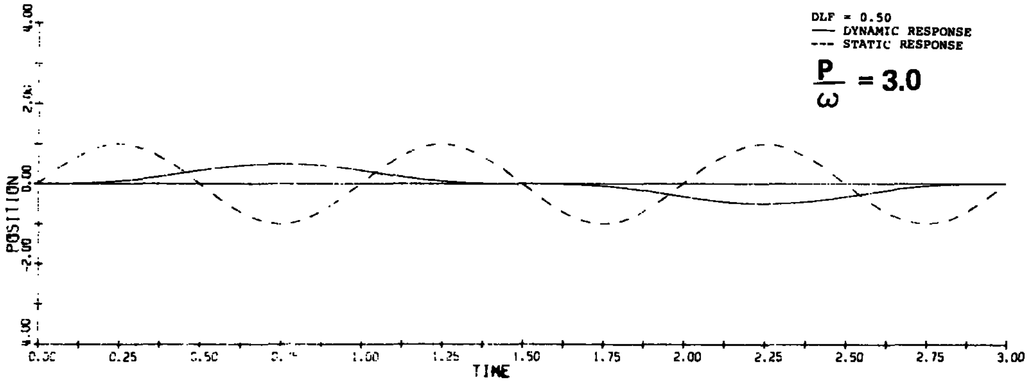


Fig. 9.4. The dynamic load factor



Response of a single degree
of freedom system.



Response of a single degree
of freedom system.

Modeling of a structural vibration problem

1) Identify the frequencies contained in the loading, using a Fourier analysis if necessary.

2) Choose a finite element mesh that accurately represents all frequencies up to about four times the highest frequency ω_u contained in the loading.

3) Perform the direct integration analysis. The time step Δt for this solution should equal about $\frac{1}{20} T_u$, where $T_u = 2\pi/\omega_u$, or be smaller for stability reasons.

Modeling of a wave propagation problem

If we assume that the wave length is L_w, the total time for the wave to travel past a point is

$$t_w = \frac{L_w}{c} \qquad (9.100)$$

where c is the wave speed. Assuming that n time steps are necessary to represent the wave, we use

$$\Delta t = \frac{t_w}{n} \qquad (9.101)$$

and the "effective length" of a finite element should be

$$L_e = c\ \Delta t \qquad (9.102)$$

SUMMARY OF STEP-BY-STEP INTEGRATIONS

— INITIAL CALCULATIONS —

1. Form linear stiffness matrix $\underline{K}$, mass matrix $\underline{M}$ and damping matrix $\underline{C}$, whichever applicable;

Calculate the following constants:

Newmark method: $\delta \geq 0.50, \ \alpha \geq 0.25(0.5+\delta)^2$

$a_0 = 1/(\alpha\Delta t^2) \qquad a_1 = \delta/(\alpha\Delta t) \qquad a_2 = 1/(\alpha\Delta t) \qquad a_3 = 1/(2\alpha)-1$

$a_4 = \delta/\alpha - 1 \qquad a_5 = \Delta t(\delta/\alpha - 2)/2 \qquad a_6 = a_0 \qquad a_7 = -a_2$

$a_8 = -a_3 \qquad a_9 = \Delta t(1-\delta) \qquad a_{10} = \delta\Delta t$

Central difference method:

$a_0 = 1/\Delta t^2 \qquad a_1 = 1/2\Delta t \qquad a_2 = 2a_0 \qquad a_3 = 1/a_2$

---

2. Initialize ${}^{0}\underline{U}, \ {}^{0}\dot{\underline{U}}, \ {}^{0}\ddot{\underline{U}}$;

For central difference method only, calculate ${}^{\Delta t}\underline{U}$ from initial conditions:

$${}^{\Delta t}\underline{U} = {}^{0}\underline{U} + \Delta t \ {}^{0}\dot{\underline{U}} + a_3 \ {}^{0}\ddot{\underline{U}}$$

3. Form effective linear coefficient matrix;

in implicit time integration:

$$\hat{\underline{K}} = \underline{K} + a_0\underline{M} + a_1\underline{C}$$

in explicit time integration:

$$\hat{\underline{M}} = a_0\underline{M} + a_1\underline{C}$$

4. In dynamic analysis using implicit time integration triangularize $\hat{\underline{K}}$.

--- FOR EACH STEP ---

(i) Form effective load vector;

in implicit time integration:

$$
{}^{t+\Delta t}\hat{\underline{R}} = {}^{t+\Delta t}\underline{R} + \underline{M}(a_0\,{}^{t}\underline{U} + a_2\,{}^{t}\dot{\underline{U}} + a_3\,{}^{t}\ddot{\underline{U}})
$$

$$
+ \underline{C}(a_1\,{}^{t}\underline{U} + a_4\,{}^{t}\dot{\underline{U}} + a_5\,{}^{t}\ddot{\underline{U}})
$$

in explicit time integration:

$$
{}^{t}\hat{\underline{R}} = {}^{t}\underline{R} + a_2\underline{M}({}^{t}\underline{U} - {}^{t-\Delta t}\underline{U}) + \hat{\underline{M}}\,{}^{t-\Delta t}\underline{U} - {}^{t}\underline{F}
$$

(ii) Solve for displacement increments;

in implicit time integration:

$$
\hat{\underline{K}}\,{}^{t+\Delta t}\underline{U} = {}^{t+\Delta t}\hat{\underline{R}}\ ;\ \underline{U} = {}^{t+\Delta t}\underline{U} - {}^{t}\underline{U}
$$

in explicit time integration:

$$
\hat{\underline{M}}\,{}^{t+\Delta t}\underline{U} = {}^{t}\hat{\underline{R}}
$$

Newmark Method:

$${}^{t+\Delta t}\ddot{\underline{U}} = a_6\, \underline{U} + a_7\, {}^{t}\dot{\underline{U}} + a_8\, {}^{t}\ddot{\underline{U}}$$

$${}^{t+\Delta t}\dot{\underline{U}} = {}^{t}\dot{\underline{U}} + a_9\, {}^{t}\ddot{\underline{U}} + a_{10}\, {}^{t+\Delta t}\ddot{\underline{U}}$$

$${}^{t+\Delta t}\underline{U} = {}^{t}\underline{U} + \underline{U}$$

Central Difference Method:

$${}^{t}\dot{\underline{U}} = a_1 ({}^{t+\Delta t}\underline{U} - {}^{t-\Delta t}\underline{U})$$

$${}^{t}\ddot{\underline{U}} = a_0 ({}^{t+\Delta t}\underline{U} - 2\,{}^{t}\underline{U} + {}^{t-\Delta t}\underline{U})$$

MODE SUPERPOSITION ANALYSIS; TIME HISTORY

LECTURE 11

48 MINUTES

LECTURE 11 Solution of dynamic response by mode superposition

The basic idea of mode superposition

Derivation of decoupled equations

Solution with and without damping

Caughey and Rayleigh damping

Calculation of damping matrix for given damping ratios

Selection of number of modal coordinates

Errors and use of static correction

Practical considerations

TEXTBOOK: Sections: 9.3.1, 9.3.2, 9.3.3

Examples: 9.6, 9.7, 9.8, 9.9, 9.10, 9.11

Mode Superposition Analysis

Basic idea is:

transform dynamic equilibrium equations into a more effective form for solution, using

$$\underset{n\times 1}{\underline{U}} = \underset{n\times n}{\underline{P}} \; \underset{n\times 1}{\underline{X}(t)}$$

$\underline{P}$ = transformation matrix

$\underline{X}(t)$ = generalized displacements

Using

$$\underline{U}(t) = \underline{P}\,\underline{X}(t) \qquad (9.30)$$

on

$$\underline{M}\,\ddot{\underline{U}} + \underline{C}\,\dot{\underline{U}} + \underline{K}\,\underline{U} = \underline{R} \qquad (9.1)$$

we obtain

$$\tilde{\underline{M}}\,\ddot{\underline{X}}(t) + \tilde{\underline{C}}\,\dot{\underline{X}}(t) + \tilde{\underline{K}}\,\underline{X}(t) = \tilde{\underline{R}}(t) \qquad (9.31)$$

where

$$\tilde{\underline{M}} = \underline{P}^T\,\underline{M}\,\underline{P} \;\; ; \;\; \tilde{\underline{C}} = \underline{P}^T\,\underline{C}\,\underline{P} \; ;$$

$$\tilde{\underline{K}} = \underline{P}^T\,\underline{K}\,\underline{P} \;\; ; \;\; \tilde{\underline{R}} = \underline{P}^T\,\underline{R} \qquad (9.32)$$

An effective transformation matrix $\underline{P}$ is established using the displacement solutions of the free vibration equilibrium equations with damping neglected,

$$\underline{M}\,\ddot{\underline{U}} + \underline{K}\,\underline{U} = \underline{0} \qquad (9.34)$$

Using

$$\underline{U} = \underline{\phi}\ \sin\omega(t - t_0) \qquad (9.35)$$

we obtain the generalized eigenproblem,

$$\underline{K}\,\underline{\phi} = \omega^2\,\underline{M}\,\underline{\phi} \qquad (9.36)$$

---

with the n eigensolutions $(\omega_1^2\,,\underline{\phi}_1)$, $(\omega_2^2\,,\underline{\phi}_2),\ldots,(\omega_n^2\,,\underline{\phi}_n)$, and

$$\underline{\phi}_i^T\,\underline{M}\,\underline{\phi}_j\ \begin{cases} = 1 & ;\quad i = j \\ = 0 & ;\quad i \neq j \end{cases} \qquad (9.37)$$

$$0 \leq \omega_1^2 \leq \omega_2^2 \leq \omega_3^2 \cdots \leq \omega_n^2 \qquad (9.38)$$

Defining

$$\underline{\Phi} = [\underline{\phi}_1, \underline{\phi}_2, \ldots, \underline{\phi}_n] \; ; \; \underline{\Omega}^2 = \begin{bmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_n^2 \end{bmatrix}$$

(9.39)

we can write

$$\underline{K}\,\underline{\Phi} = \underline{M}\,\underline{\Phi}\,\underline{\Omega}^2 \qquad (9.40)$$

and have

$$\underline{\Phi}^T \underline{K}\,\underline{\Phi} = \underline{\Omega}^2 \; ; \; \underline{\Phi}^T \underline{M}\,\underline{\Phi} = \underline{I} \qquad (9.41)$$

Now using

$$\underline{U}(t) = \underline{\Phi}\,\underline{X}(t) \qquad (9.42)$$

we obtain equilibrium equations that correspond to the modal generalized displacements

$$\ddot{\underline{X}}(t) + \underline{\Phi}^T \underline{C}\,\underline{\Phi}\,\dot{\underline{X}}(t) + \underline{\Omega}^2 \underline{X}(t) = \underline{\Phi}^T \underline{R}(t)$$

(9.43)

The initial conditions on $\underline{X}(t)$ are obtained using (9.42) and the $\underline{M}$ - orthonormality of $\underline{\Phi}$; i.e., at time 0 we have

$${}^0\underline{X} = \underline{\Phi}^T \underline{M}\,{}^0\underline{U} \; ; \; {}^0\dot{\underline{X}} = \underline{\Phi}^T \underline{M}\,{}^0\dot{\underline{U}}$$

(9.44)

Analysis with Damping Neglected

$$\ddot{\underline{X}}(t) + \underline{\Omega}^2 \underline{X}(t) = \underline{\Phi}^T \underline{R}(t) \qquad (9.45)$$

i.e., n individual equations of the form

$$\left.\begin{aligned} &\ddot{x}_i(t) + \omega_i^2 x_i(t) = r_i(t) \\ &\text{where} \\ &\qquad r_i(t) = \underline{\phi}_i^T \underline{R}(t) \end{aligned}\right\} i = 1,2,\ldots,n$$

$$(9.46)$$

with

$$\begin{aligned} x_i\Big|_{t=0} &= \underline{\phi}_i^T \underline{M}\, {}^0\underline{U} \\ \dot{x}_i\Big|_{t=0} &= \underline{\phi}_i^T \underline{M}\, {}^0\dot{\underline{U}} \end{aligned} \qquad (9.47)$$

Using the Duhamel integral we have

$$x_i(t) = \frac{1}{\omega_i}\int_0^t r_i(\tau)\,\sin\omega_i(t-\tau)\,d\tau \qquad (9.48)$$

$$+\ \alpha_i \sin\omega_i t + \beta_i \cos\omega_i t$$

where α_i and β_i are determined from the initial conditions in (9.47). And then

$$\underline{U}(t) = \sum_{i=1}^{n} \underline{\phi}_i\, x_i(t) \qquad (9.49)$$

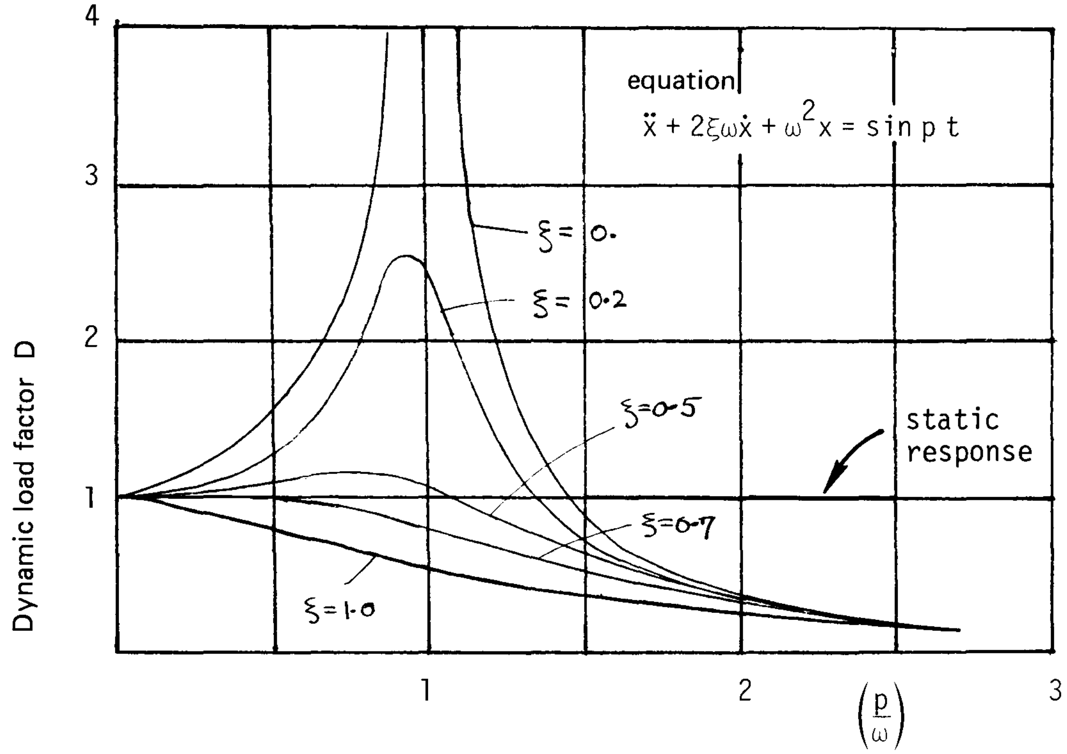


Fig. 9.4. The dynamic load factor

Hence we use

$$\underline{U}^p = \sum_{i=1}^{p} \underline{\phi}_i \, x_i(t)$$

where

$$\underline{U}^p \doteq \underline{U}$$

The error can be measured using

$$\epsilon^p(t) = \frac{\|\underline{R}(t) - \left(\underline{M}\,\ddot{\underline{U}}^p(t) + \underline{K}\,\underline{U}^p(t)\right)\|_2}{\|\underline{R}(t)\|_2} \qquad (9.50)$$

Static correction

Assume that we used p modes to obtain $\underline{U}^p$, then let

$$\underline{R} = \sum_{i=1}^{n} r_i (\underline{M}\,\underline{\phi}_i)$$

Hence

$$r_i = \underline{\phi}_i^T\,\underline{R}$$

Then

$$\Delta\underline{R} = \underline{R} - \sum_{i=1}^{p} r_i\,(\underline{M}\,\underline{\phi}_i)$$

and

$$\underline{K}\,\Delta\underline{U} = \Delta\underline{R}$$

Analysis with Damping Included

Recall, we have

$$\ddot{\underline{X}}(t) + \underline{\Phi}^T\,\underline{C}\,\underline{\Phi}\,\dot{\underline{X}}(t) + \underline{\Omega}^2\,\underline{X}(t) = \underline{\Phi}^T\,\underline{R}(t)$$

(9.43)

If the damping is proportional

$$\underline{\phi}_i^T\,\underline{C}\,\underline{\phi}_j = 2\omega_i\,\xi_i\,\delta_{ij} \qquad (9.51)$$

and we have

$$\ddot{x}_i(t) + 2\omega_i\,\xi_i\,\dot{x}_i(t) + \omega_i^2\,x_i(t) = r_i(t)$$

$$i = 1,\ldots,n$$

(9.52)

A damping matrix that satisfies the relation in (9.51) is obtained using the Caughey series,

$$\underline{C} = \underline{M} \sum_{k=0}^{p-1} a_k [\underline{M}^{-1} \underline{K}]^k \qquad (9.56)$$

where the coefficients a_k , $k = 1, \ldots, p$, are calculated from the p simultaneous equations

$$\xi_i = \frac{1}{2}\left(\frac{a_0}{\omega_i} + a_1\omega_i + a_2\omega_i^3 + \ldots + a_{p-1}\omega_i^{2\,p-3}\right) \qquad (9.57)$$

A special case is Rayleigh damping,

$$\underline{C} = \underline{\alpha\,M} + \underline{\beta\,K} \qquad (9.55)$$

example:

Assume $\xi_1 = 0.02$; $\xi_2 = 0.10$

$\omega_1 = 2$ $\quad \omega_2 = 3$

calculate α and β

We use

$$\underline{\phi}_i^T(\underline{\alpha\,M} + \underline{\beta\,K})\,\underline{\phi}_i = 2\omega_i\,\xi_i$$

or

$$\underline{\alpha} + \underline{\beta}\,\omega_i^2 = 2\omega_i\,\xi_i$$

Using this relation for ω_1 , ξ_1 and ω_2 , ξ_2 , we obtain two equations for α and β :

$$\underline{\alpha} + 4\underline{\beta} = 0.08$$

$$\underline{\alpha} + 9\underline{\beta} = 0.60$$

The solution is $\alpha = -0.336$ and $\beta = 0.104$. Thus the damping matrix to be used is

$$\underline{C} = -0.336\,\underline{M} + 0.104\,\underline{K}$$

Note that since

$$\alpha + \beta\omega_i^2 = 2\omega_i\,\xi_i$$

for any i , we have, once α and β have been established,

$$\xi_i = \frac{\alpha + \beta\omega_i^2}{2\omega_i}$$

$$= \frac{\alpha}{2\omega_i} + \frac{\beta}{2}\,\omega_i$$

Response solution

As in the case of no damping.
we solve p equations

$$\ddot{x}_i + 2\omega_i \xi_i x_i + \omega_i^2 x_i = r_i$$

with

$$r_i = \underline{\phi}_i^T \underline{R}$$

$$x_i \big|_{t=0} = \underline{\phi}_i^T \underline{M} \, {}^0\underline{U}$$

$$\dot{x}_i \big|_{t=0} = \underline{\phi}_i^T \underline{M} \, {}^0\dot{\underline{U}}$$

and then

$$\underline{U}^p = \sum_{i=1}^{p} \underline{\phi}_i \, x_i(t)$$

Practical considerations

mode superposition analysis
is effective

- when the response lies in a few modes only, $p << n$
- when the response is to be obtained over many time intervals (or the modal response can be obtained in closed form).

  e.g. earthquake engineering
  vibration excitation

- it may be important to calculate $\varepsilon_p(t)$ or the static correction.

SOLUTION METHODS FOR CALCULATIONS OF FREQUENCIES AND MODE SHAPES

LECTURE 12

58 MINUTES

LECTURE 12 Solution methods for finite element eigenproblems

Standard and generalized eigenproblems

Basic concepts of vector iteration methods, polynomial iteration techniques, Sturm sequence methods, transformation methods

Large eigenproblems

Details of the determinant search and subspace iteration methods

Selection of appropriate technique, practical considerations

TEXTBOOK: Sections: 12.1, 12.2.1, 12.2.2, 12.2.3, 12.3.1, 12.3.2, 12.3.3, 12.3.4, 12.3.6 (the material in Chapter 11 is also referred to)

Examples: 12.1, 12.2, 12.3, 12.4

SOLUTION METHODS FOR EIGENPROBLEMS

Standard EVP:

$$\underset{n\times n}{\underline{K}}\,\underline{\phi} = \lambda\,\underline{\phi}$$

Generalized EVP:

$$\underline{K}\,\underline{\phi} = \lambda\,\underline{M}\,\underline{\phi} \quad \longleftarrow \quad (\lambda = \omega^2)$$

Quadratic EVP:

$$(\underline{K} + \lambda\,\underline{C} + \lambda^2\,\underline{M})\underline{\phi} = \underline{0}$$

Most emphasis on the generalized EVP e.g. earthquake engineering

"Large EVP" $n > 500$ $p = 1, \ldots, \frac{1}{3}n$
$m > 60$

In dynamic analysis, proportional damping

$$\underline{K}\,\underline{\phi} = \omega^2\,\underline{M}\,\underline{\phi}$$

If zero freq. are present we can use the following procedure

$$\underline{K}\,\underline{\phi} + \mu\,\underline{M}\,\underline{\phi} = (\omega^2 + \mu)\underline{M}\,\underline{\phi}$$

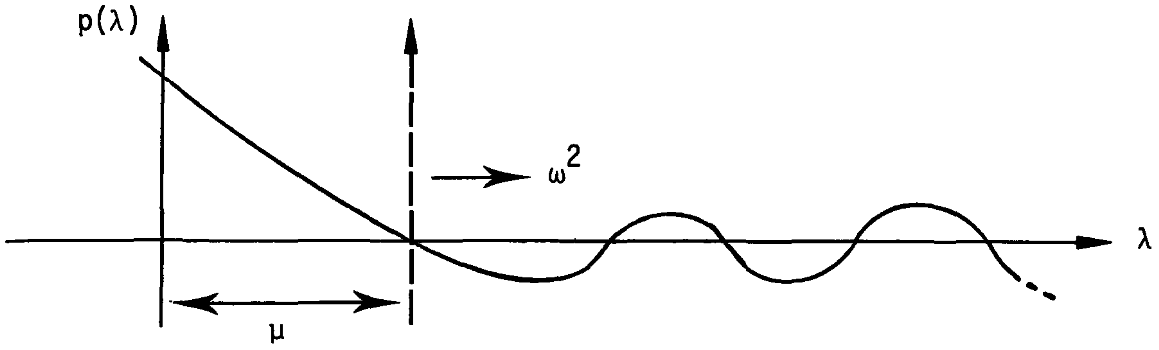
or

$$(\underline{K} + \mu\,\underline{M})\underline{\phi} = \lambda\,\underline{M}\,\underline{\phi}$$

$$\lambda = \omega^2 + \mu$$

or

$$\omega^2 = \lambda - \mu$$



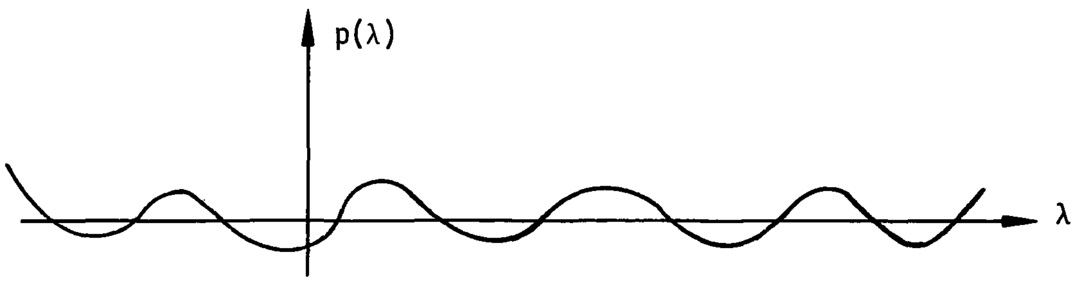
$$p(\lambda) = \det(\underline{\bar{K}} - \lambda\ \underline{M})\ ;\quad \underline{\bar{K}} = \underline{K} + \mu\ \underline{M}$$

In buckling analysis

$$\underline{K}\ \underline{\phi} = \lambda\ \underline{K}_G\ \underline{\phi}$$

where

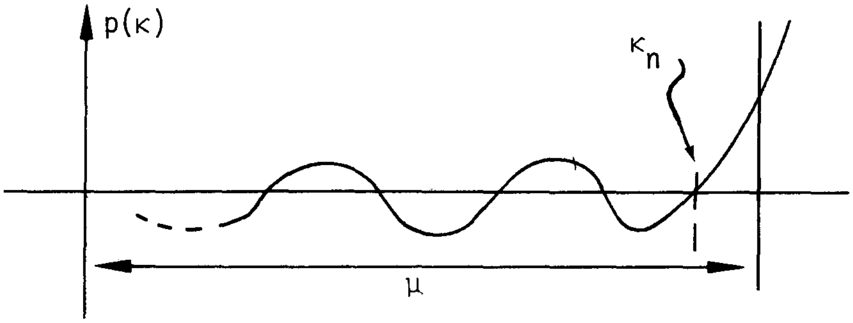
$$p(\lambda) = \det(\underline{K} - \lambda\ \underline{K}_G)$$



Rewrite problem as:

$$\underline{K}_G\,\underline{\phi} = \kappa\,\underline{K}\,\underline{\phi} \qquad \kappa = \frac{1}{\lambda}$$

and solve for largest κ:



$$(\underline{K}_G - \mu\,\underline{K})\,\underline{\phi} = \eta\,\underline{K}\,\underline{\phi}$$

Traditional Approach: Transform the generalized EVP or quadratic EVP into a standard form, then solve using one of the many techniques available
e.g.

$$\underline{K}\,\underline{\phi} = \lambda\,\underline{M}\,\underline{\phi}$$

$$\underline{M} = \underline{\tilde{L}}\,\underline{\tilde{L}}^T \;;\quad \underline{\tilde{\phi}} = \underline{\tilde{L}}^T\,\underline{\phi}$$

hence

$$\underline{\tilde{K}}\,\underline{\tilde{\phi}} = \lambda\,\underline{\tilde{\phi}}\;;\quad \underline{\tilde{K}} = \underline{\tilde{L}}^{-1}\,\underline{K}\,\underline{\tilde{L}}^{-T}$$

or

$$\underline{M} = \underline{W}\,\underline{D}^2\,\underline{W}^T \qquad \text{etc...}$$

Direct solution is more effective.
Consider the Gen. EVP $\underline{K}\ \underline{\phi} = \lambda\ \underline{M}\ \underline{\phi}$
with

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots \leq \lambda_n$$

$$\underline{\phi}_1 \quad \underline{\phi}_2 \quad \underline{\phi}_3 \cdots \quad \underline{\phi}_n$$

eigenpairs $(\lambda_i\ ,\ \underline{\phi}_i)$ $i = 1, \ldots, p$
are required or $i = r, \ldots, s$

The solution procedures in use operate on the basic equations that have to be satisfied.

---

1) VECTOR ITERATION TECHNIQUES

Equation: $\underline{K}\ \underline{\phi} = \lambda\ \underline{M}\ \underline{\phi}$

e.g. Inverse It. $\underline{K}\ \underline{\bar{x}}_{k+1} = M\ \underline{x}_k$

$$\underline{x}_{k+1} = \frac{\underline{\bar{x}}_{k+1}}{(\underline{\bar{x}}_{k+1}^T\ \underline{M}\ \underline{\bar{x}}_{k+1})^{\frac{1}{2}}} \longrightarrow \underline{\phi}_1$$

- Forward Iteration
- Rayleigh Quotient Iteration

can be employed to calculate one eigenvalue and vector, deflate then to calculate additional eigenpair

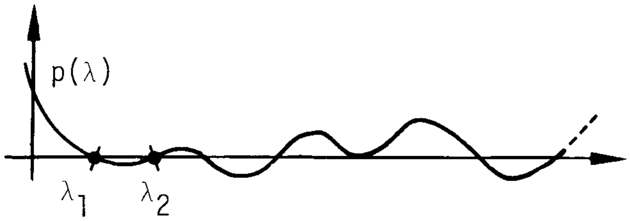
Convergence to "an eigenpair", which one is not guaranteed (convergence may also be slow)

2) POLYNOMIAL ITERATION METHODS

$$\underline{K}\,\underline{\phi} = \lambda\,\underline{M}\,\underline{\phi} \longrightarrow (\underline{K} - \lambda\,\underline{M})\,\underline{\phi} = \underline{0}$$

Hence

$$p(\lambda) = \det(\underline{K} - \lambda\underline{M}) = 0$$



Newton Iteration

$$\mu_{i+1} = \mu_i - \frac{p(\mu_i)}{p'(\mu_i)}$$

$$p(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \dots + a_n\lambda^n$$
$$= b_0\,(\lambda-\lambda_1)\,(\lambda-\lambda_2)\,\dots\,(\lambda-\lambda_n)$$

Explicit polynomial iteration:

- Expand the polynomial and iterate for zeros.
- Technique not suitable for larger problems
  - much work to obtain a_i's
  - unstable process

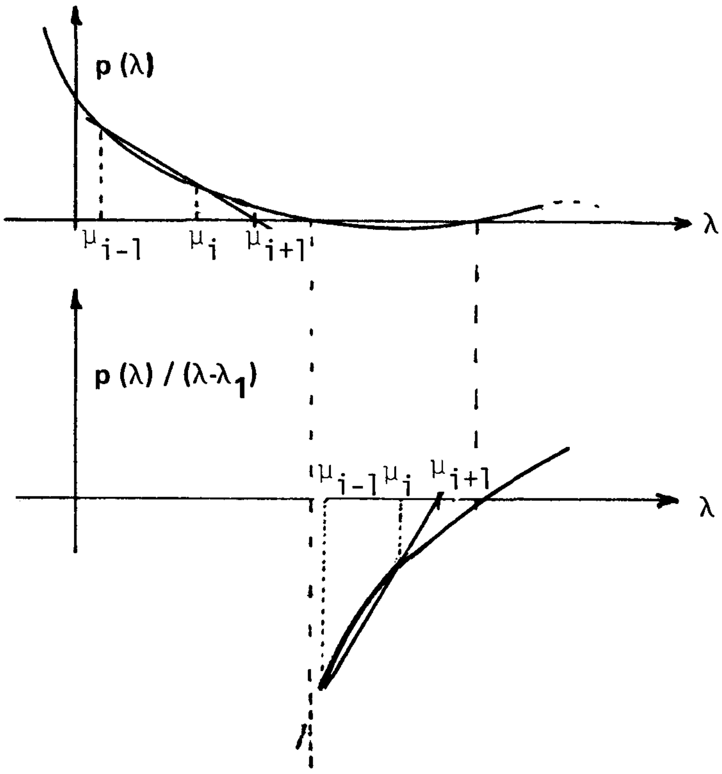
Implicit polynomial iteration:

$$p(\mu_i) = \det(\underline{K} - \mu_i\,\underline{M})$$
$$= \det \underline{L}\,\underline{D}\,\underline{L}^T = \prod_i d_{ii}$$

- accurate, provided we do not encounter large multipliers
- we directly solve for $\lambda_1, \dots$
- use SECANT ITERATION:

$$\mu_{i+1} = \mu_i - \frac{p(\mu_i)}{\left(\dfrac{p(\mu_i) - p(\mu_{i-1})}{\mu_i - \mu_{i-1}}\right)}$$

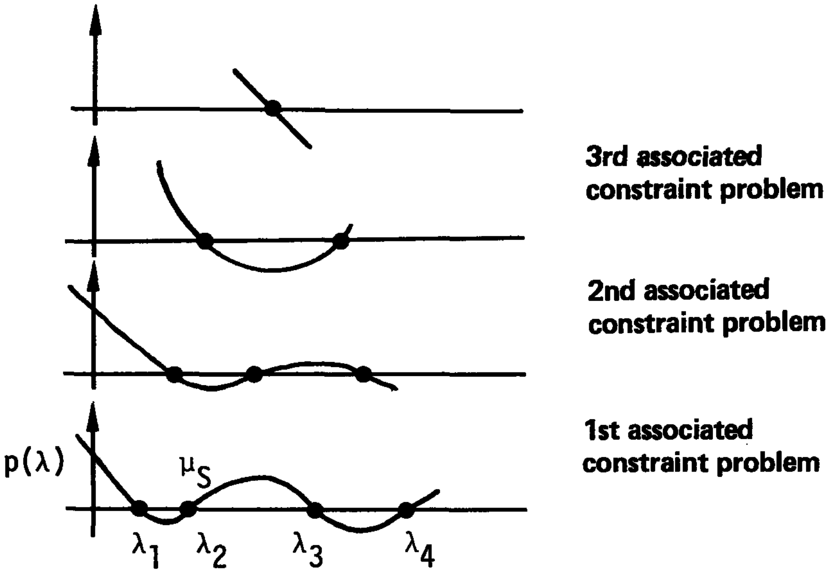
- deflate polynomial after convergence to λ_1



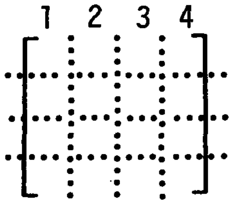
Convergence guaranteed to λ_1, then λ_2, etc. but can be slow when we calculate multiple roots.

Care need be taken in $\underline{L}\,\underline{D}\,\underline{L}^T$ factorization.

3) STURM SEQUENCE METHODS



$\underline{K}\,\underline{\phi} = \lambda\,\underline{M}\,\underline{\phi} \Rightarrow$



$\underline{\phi} = \lambda\,\underline{M}\,\underline{\phi}$

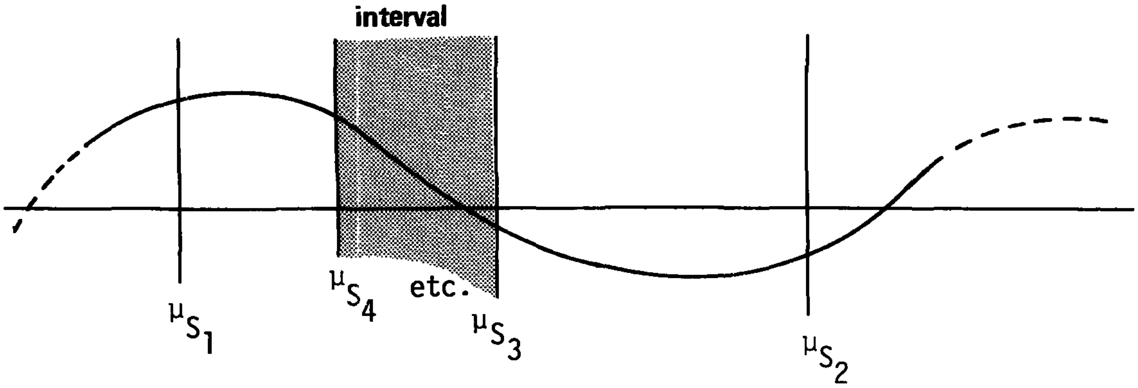
$$\underline{K} - \mu_S\,\underline{M} = \underline{L}\,\underline{D}\,\underline{L}^T$$

Number of negative elements in D is equal to the number of eigenvalues smaller than μ_S .

3) STURM SEQUENCE METHODS

Calculate $\underline{K} - \mu_{S_i} \underline{M} = \underline{L}\,\underline{D}\,\underline{L}^T$

Count number of negative elements in $\underline{D}$ and use a strategy to isolate eigenvalue(s).



- Need to take care in $\underline{L}\,\underline{D}\,\underline{L}^T$ factorization
- Convergence can be very slow

4) TRANSFORMATION METHODS

$$\underline{K}\,\underline{\phi} = \lambda\,\underline{M}\,\underline{\phi} \longrightarrow \begin{cases} \underline{\Phi}^T \underline{K}\,\underline{\Phi} = \underline{\Lambda} \\ \underline{\Phi}^T \underline{M}\,\underline{\Phi} = \underline{I} \end{cases}$$

Construct $\underline{\Phi}$ iteratively:

$$\underline{\Phi} = [\underline{\phi},\ldots\,\underline{\phi}_n]\ ; \quad \underline{\Lambda} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$\underline{P}_k^T \cdots \underline{P}_2^T \, \underline{P}_1^T \, \underline{K} \, \underline{P}_1 \, \underline{P}_2 \cdots \underline{P}_k \longrightarrow \underline{\Lambda}$$

$$\underline{P}_k^T \cdots \underline{P}_2^T \, \underline{P}_1^T \, \underline{M} \, \underline{P}_1 \, \underline{P}_2 \cdots \underline{P}_k \longrightarrow \underline{I}$$

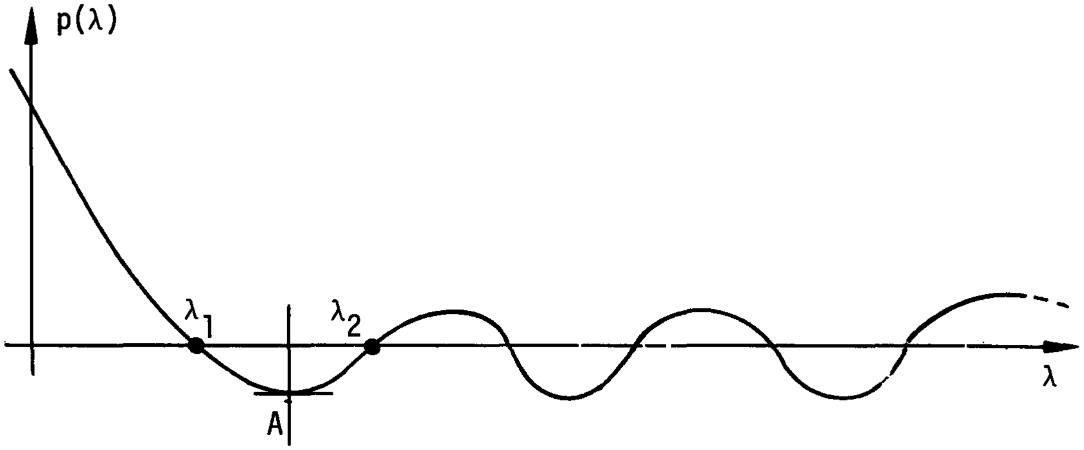
e.g. generalized Jacobi method

- Here we calculate all eigenpairs simultaneously
- Expensive and ineffective (impossible) or large problems.

For large eigenproblems it is best to use <u>combinations</u> <u>of the above basic techniques</u>:

- Determinant search to get near a root
- Vector iteration to obtain eigenvector and eigenvalue
- Transformation method for orthogonalization of iteration vectors.
- Sturm sequence method to ensure that required eigenvalue(s) has (or have) been calculated

THE DETERMINANT SEARCH METHOD



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1) Iterate on polynomial to obtain shifts close to λ_1

$$p(\mu_i) = \det(\underline{K} - \mu_i \underline{M})$$

$$= \det \underline{L}\,\underline{D}\,\underline{L}^T = \prod_i d_{ii}$$

$$\mu_{i+1} = \mu_i - \eta \frac{p(\mu_i)}{\dfrac{p(\mu_i) - p(\mu_{i-1})}{\mu_i - \mu_{i-1}}}$$

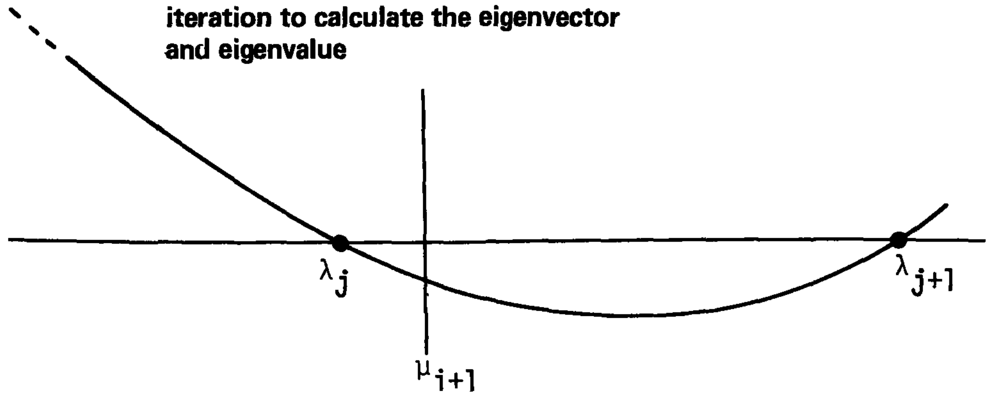
η is normally = 1.0

η = 2., 4., 8.,... when convergence is slow

Same procedure can be employed to obtain shift near λ_i, provided $p(\lambda)$ is deflated of $\lambda_1, \ldots, \lambda_{i-1}$

2) Use Sturm sequence property to check whether μ_{i+1} is larger than an unknown eigenvalue.

3) Once μ_{i+1} is larger than an unknown eigenvalue, use inverse iteration to calculate the eigenvector and eigenvalue



$$(\underline{K} - \mu_{i+1}\underline{M})\,\bar{\underline{x}}_{k+1} = \underline{M}\,\underline{x}_k \qquad k = 1,2,\ldots$$

$$\underline{x}_{k+1} = \frac{\bar{\underline{x}}_{k+1}}{(\bar{\underline{x}}_{k+1}^T\,\underline{M}\,\bar{\underline{x}}_{k+1})^{\frac{1}{2}}}$$

$$\rho(\bar{\underline{x}}_{k+1}) = \frac{\bar{\underline{x}}_{k+1}^T\,\underline{M}\,\underline{x}_k}{\bar{\underline{x}}_{k+1}^T\,\underline{M}\,\bar{\underline{x}}_{k+1}}$$

---

4) Iteration vector must be deflated of the previously calculated eigenvectors using, e.g. Gram-Schmidt orthogonalization.

If convergence is slow use Rayleigh quotient iteration

Advantage:
Calculates only eigenpairs actually required; no prior transformation of eigenproblem

Disadvantage:
Many triangular factorizations

- **Effective only for small banded systems**

We need an algorithm with less factorizations and more vector iterations when the bandwidth of the system is large.

SUBSPACE ITERATION METHOD

Iterate with q vectors when the lowest p eigenvalues and eigenvectors are required.

$$\begin{array}{l}\text{inverse} \\ \text{iteration}\end{array}\left\{ \underline{K}\ \underline{\overline{X}}_{k+1} = \underline{M}\ \underline{X}_k \right. \qquad k = 1,2,\ldots$$

$$\underline{K}_{k+1} = \underline{\overline{X}}_{k+1}^T\ \underline{K}\ \underline{\overline{X}}_{k+1}$$

$$\underline{M}_{k+1} = \underline{\overline{X}}_{k+1}^T\ \underline{M}\ \underline{\overline{X}}_{k+1}$$

$$\underline{K}_{k+1}\ \underline{Q}_{k+1} = \underline{M}_{k+1}\ \underline{Q}_{k+1}\ \underline{\Lambda}_{k+1}$$

$$\underline{X}_{k+1} = \underline{\overline{X}}_{k+1}\ \underline{Q}_{k+1}$$

"Under conditions" we have

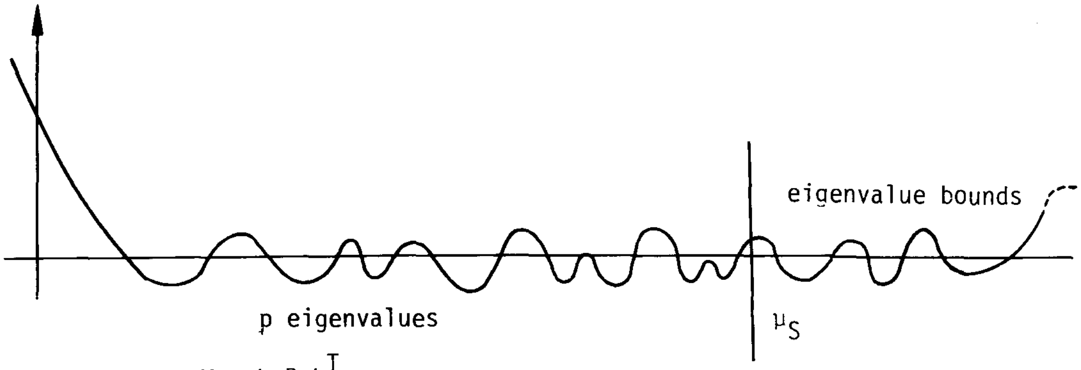
$$\underline{X}_{k+1} \longrightarrow \underline{\Phi} \; ; \quad \underline{\Lambda}_{k+1} \longrightarrow \underline{\Lambda}$$

$$\underline{\Phi} = [\underline{\phi}_1, \ldots, \underline{\phi}_q] \; ; \quad \underline{\Lambda} = \mathrm{diag}\ (\lambda_i)$$

CONDITION:

starting subspace spanned by $\underline{X}_1$ must not be orthogonal to least dominant subspace required.

Use Sturm sequence check



$$\underline{K} - \mu_S \underline{M} = \underline{L}\ \underline{D}\ \underline{L}^T$$

no. of −ve elements in $\underline{D}$ must be equal to p .

<u>Convergence rate</u>:

$$\underline{\phi}_i \Rightarrow \underline{\underline{\lambda_i / \lambda_{q+1}}} \qquad \lambda_i \Rightarrow \underline{\underline{\left(\lambda_i / \lambda_{q+1}\right)^2}}$$

convergence reached when $\left| \frac{\lambda_i^{(k)} - \lambda_i^{(k-1)}}{\lambda_i^{(k)}} \right| \leq \mathrm{tol}$

Starting Vectors

Two choices

1) $\underline{x}_1 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$; $\underline{x}_j = \underline{e}_k$

$j = 2, \ldots, q-1$

$\underline{x}_q$ = random vector

2) **Lanczos method**
   Here we need to use q much larger than p .

Checks on eigenpairs

1. **Sturm sequence checks**

2. $$\varepsilon_i = \frac{\| \underline{K}\, \underline{\phi}_i^{(\ell+1)} - \lambda_i^{(\ell+1)}\, \underline{M}\, \underline{\phi}_i^{(\ell+1)} \|_2}{\| \underline{K}\, \underline{\phi}_i^{(\ell+1)} \|_2}$$

important in all solutions.

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