Topic 14

Solution of Nonlinear Dynamic Response—Part II

Contents:

- **Mode superposition analysis in nonlinear dynamics**
- **Substructuring in nonlinear dynamics, a schematic example of a building on a flexible foundation**
- **Study of analyses to demonstrate characteristics of procedures for nonlinear dynamic solutions**
- **Example analysis: Wave propagation in a rod**
- **Example analysis: Dynamic response of a three degree of freedom system using the central difference method**
- **Example analysis: Ten-story tapered tower subjected to blast loading**
- **Example analysis: Simple pendulum undergoing large displacements**
- **Example analysis: Pipe whip solution**
- **Example analysis: Control rod drive housing with lower support**
- **Example analysis: Spherical cap under uniform pressure loading**
- **Example analysis: Solution of fluid-structure interaction problem**

Textbook: Sections 9.3.1, 9.3.2, 9.3.3, 9.5.3, 8.2.4

Examples: 9.6, 9.7, 9.8, 9.11

References:

The use of the nonlinear dynamic analysis techniques is described with example solutions in

Bathe, K. J., "Finite Element Formulation, Modeling and Solution of Nonlinear Dynamic Problems," Chapter in *Numerical Methods for Partial Differential Equations,* (Parter, S. V., ed.), Academic Press, 1979.

Bathe, K. J., and S. Gracewski, "On Nonlinear Dynamic Analysis Using Substructuring and Mode Superposition," *Computers & Structures, 13,* 699–707, 1981.

Ishizaki, T., and K. J. Bathe, "On Finite Element Large Displacement and Elastic-Plastic Dynamic Analysis of Shell Structures," *Computers & Structures, 12,* 309–318, 1980.

THE SOLUTION OF THE DYNAMIC EQUILIBRIUM EQUATIONS CAN BE ACHIEVED USING

- DIRECT INTEGRATION METHODS
  - EXPLICIT INTEGR.
  - IMPLICIT INTEGR.
- MODE SUPERPOSITION
- SUBSTRUCTURING

WE DISCUSS THESE TECHNIQUES BRIEFLY IN THIS LECTURE

EXAMPLES

EX. 1 WAVE PROPAGATION IN A ROD

EX. 2 RESPONSE OF A 3 D.O.F. SYSTEM

EX. 3 ANALYSIS OF TEN STORY TAPERED TOWER

EX. 4 ANALYSIS OF PENDULUM

EX. 5 PIPE WHIP RESPONSE SOLUTION

SLIDES REGARDING

- ANALYSIS OF CRD HOUSING
- SOLUTION OF RESPONSE OF SPHERICAL CAP
- ANALYSIS OF FLUID-STRUCTURE INTERACTION PROBLEM (PIPE TEST)

THE DETAILS OF THESE PROBLEM SOLUTIONS ARE GIVEN IN THE PAPERS, SEE STUDY GUIDE

Markerboard 14-1

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Mode superposition:

- The modes of vibration change due to the nonlinearities, however we can employ the modes at a particular time as basis vectors (generalized displacements) to express the response.
- This method is effective when, in nonlinear analysis,
  - the response lies in only a few vibration modes (displacement patterns)
  - the system has only local nonlinearities

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The governing equations in implicit time integration are (assuming no damping matrix)

$$\underline{M}\ {}^{t+\Delta t}\ddot{\underline{U}}^{(k)} + {}^{\tau}\underline{K}\ \Delta\underline{U}^{(k)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(k-1)}$$

Let now $\tau = 0$, hence the method of solution corresponds to the initial stress method.

Using

$${}^{t+\Delta t}\underline{U} = \sum_{i=r}^{s} \underline{\phi}_i\ {}^{t+\Delta t}x_i$$

$${}^{0}\underline{K}\ \underline{\phi}_i = \omega_i^2\ \underline{M}\ \underline{\phi}_i$$

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The modal transformation gives

$$ {}^{t+\Delta t}\ddot{\underline{X}}^{(k)} + \underline{\Omega}^2\, \Delta\underline{X}^{(k)} = \underline{\Phi}^T \underbrace{\left({}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(k-1)}\right)} $$

equations cannot be solved individually over the time span
Coupling!

where

$$ \underline{\Omega}^2 = \begin{bmatrix} \omega_r^2 & & \\ & \ddots & \\ & & \omega_s^2 \end{bmatrix} $$

$$ \underline{\Phi} = [\underline{\phi}_r \cdots \underline{\phi}_s] $$

$$ {}^{t+\Delta t}\underline{X}^T = [{}^{t+\Delta t}x_r \cdots {}^{t+\Delta t}x_s] $$

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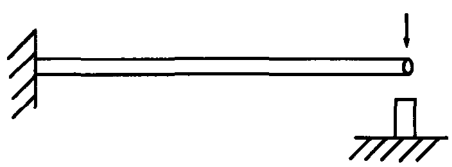
Typical problem:



Pipe whip: Elastic-plastic pipe
Elastic-plastic stop

- Nonlinearities in pipe and stop. But the displacements are reasonably well contained in a few modes of the linear (initial) system.

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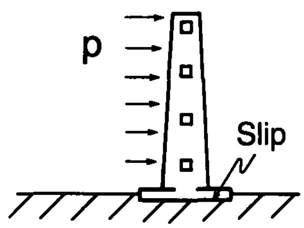
Substructuring

- Procedure is used with implicit time integration. All linear degrees of freedom can be condensed out prior to the incremental solution.
- Used for local nonlinearities:
  Contact problems
  Nonlinear support problems



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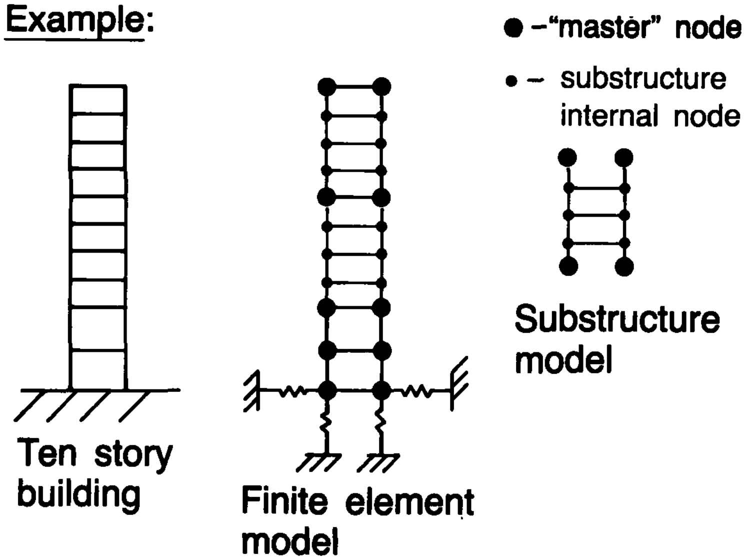
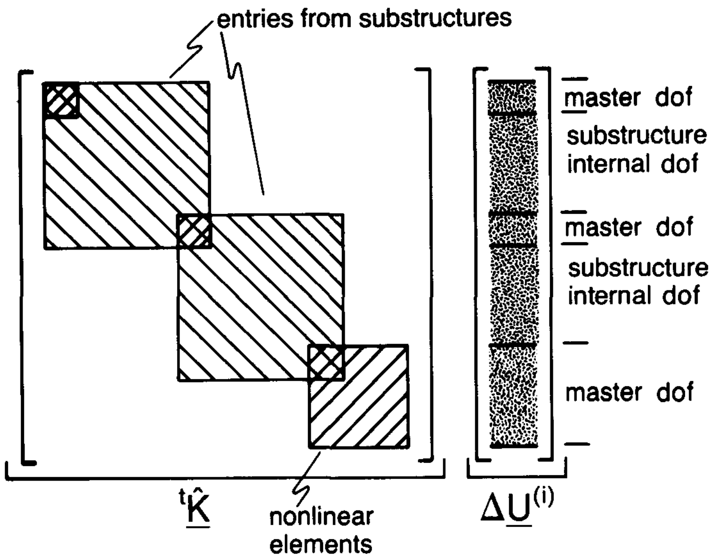
Example:





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Here

$$ {}^t\hat{\underline{K}} = \left(\underline{K} + \frac{4}{\Delta t^2}\underline{M}\right) + {}^t\underline{K}_{nonlinear} $$

- $\underline{K}$: all linear element contributions
- $\underline{M}$: total mass matrix
- ${}^t\underline{K}_{nonlinear}$: all nonlinear stiffness effects

$$ = \hat{\underline{K}} + {}^t\underline{K}_{nonlinear} $$

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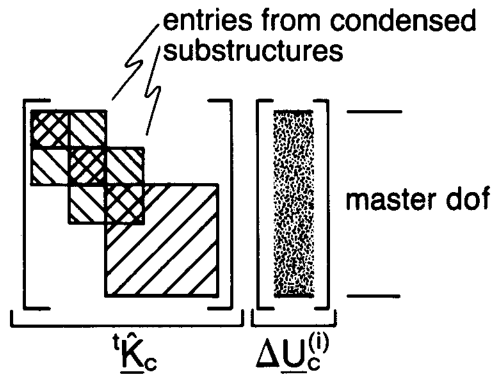
After condensing out all substructure internal degrees of freedom, we obtain a smaller system of equations:



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Major steps in solution:

- Prior to step-by-step solution, establish $\hat{\underline{K}}$ for all mass and constant stiffness contributions. Statically condense out internal substructure degrees of freedom to obtain $\hat{\underline{K}}_c$.

  We note that

$$\underbrace{{}^{t}\hat{\underline{K}}_c}_{\text{condensed}} = \hat{\underline{K}}_c + \underbrace{{}^{t}\underline{K}_{nonlinear}}_{\text{all nonlinear effects}}$$

($\hat{\underline{K}}_c$) from $\hat{\underline{K}} = \underline{K} + \frac{4}{\Delta t^2}\underline{M}$

$\underline{K}$: all linear element contributions

$\underline{M}$: total mass matrix

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- For each time step solution (and each equilibrium iteration):
  - — Update condensed matrix, $\hat{\underset{\sim}{K}}_c$, for nonlinearities.
  - — Establish complete load vector for all degrees of freedom and condense out substructure internal degrees of freedom.
  - — Solve for master dof displacements, velocities, accelerations and calculate all substructure dof disp., vel., acc.

    The substructure internal nodal disp., vel., acc. are needed to calculate the complete load vector (corresponding to all dof).

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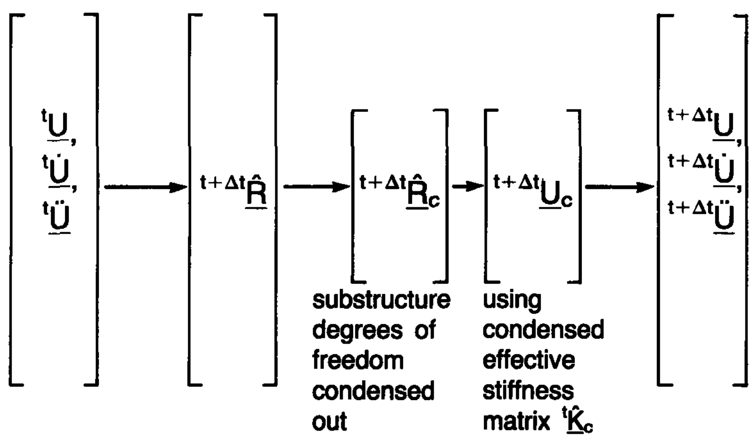
Solution procedure for each time step(and iteration):



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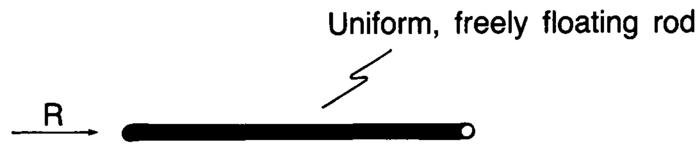
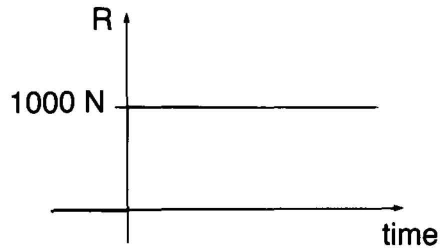
Example: Wave propagation in a rod





L = 1.0 m
A = 0.01 m^2
ρ = 1000 kg/m^3
E = 2.0 × 10^9 Pa

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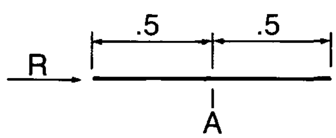
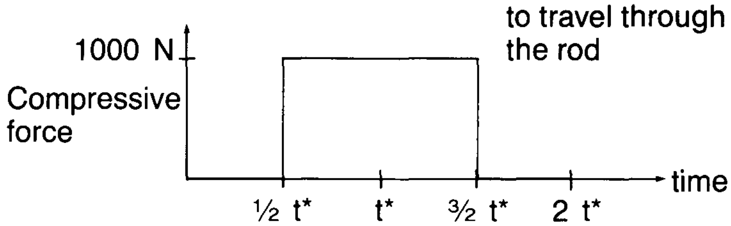
Consider the compressive force at a point at the center of the rod:



The exact solution for the force at point A is shown below.

t* = time for stress wave to travel through the rod



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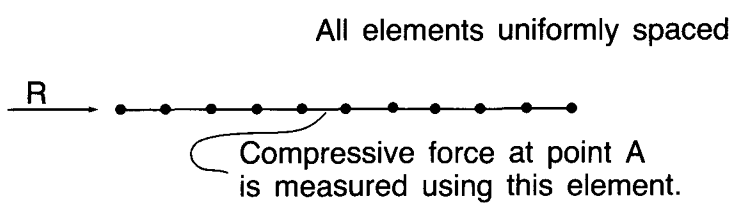
We now use a finite element mesh of ten 2-node truss elements to obtain the compressive force at point A.



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Central difference method:

- The critical time step for this problem is

$$\Delta t_{cr} = L_e / c = t^* \left(\frac{1}{\text{number of elements}} \right)$$

  $\Delta t > \Delta t_{cr}$ will produce an unstable solution

- We need to use the inital conditions as follows:

$$\underline{M}\, {}^0\underline{\ddot{U}} + \underline{K}\, {}^0\underline{U} = {}^0\underline{R} \qquad ({}^0\underline{U} = 0)$$

$$\downarrow$$

$${}^0\ddot{U}_i = \frac{{}^0R_i}{m_{ii}}$$

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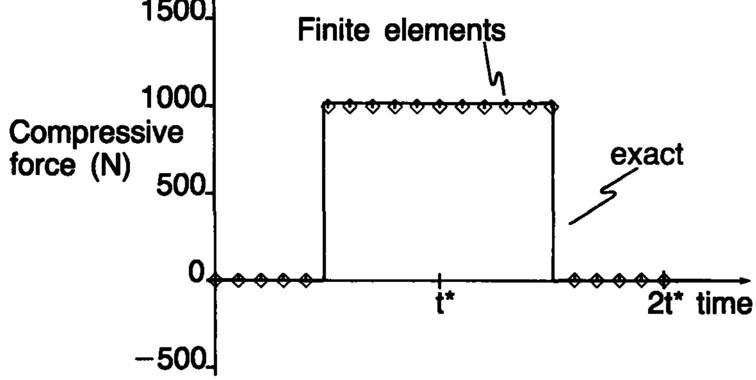
- Using a time step equal to Δt_{cr}, we obtain the correct result:
  - For this special case the exact solution is obtained



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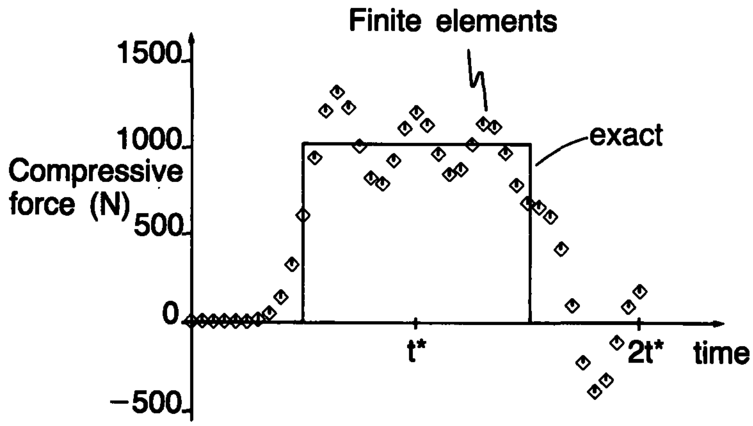
- Using a time step equal to $\frac{1}{2}\Delta t_{cr}$, the solution is stable, but highly inaccurate.



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Now consider the use of the trapezoidal rule:

- A stable solution is obtained with any choice of Δt.
- Either a consistent or lumped mass matrix may be used. We employ a lumped mass matrix in this analysis.

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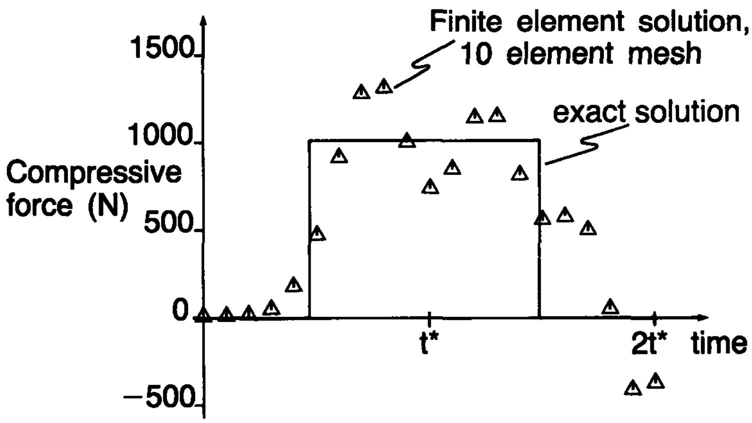
Trapezoidal rule, $\Delta t = \Delta t_{cr}|_{CDM}$, initial conditions computed using $\underline{M}\,{}^{0}\underline{\ddot{U}} = {}^{0}\underline{R}$.

— The solution is inaccurate.



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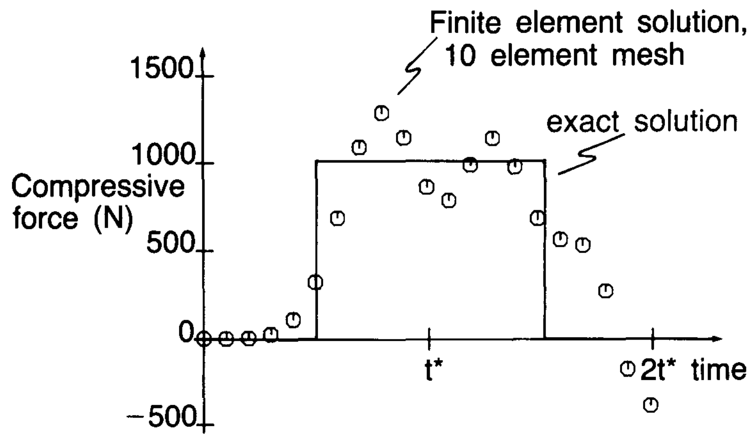
Trapezoidal rule, $\Delta t = \Delta t_{cr}|_{CDM}$, zero initial conditions.

— Almost same solution is obtained.



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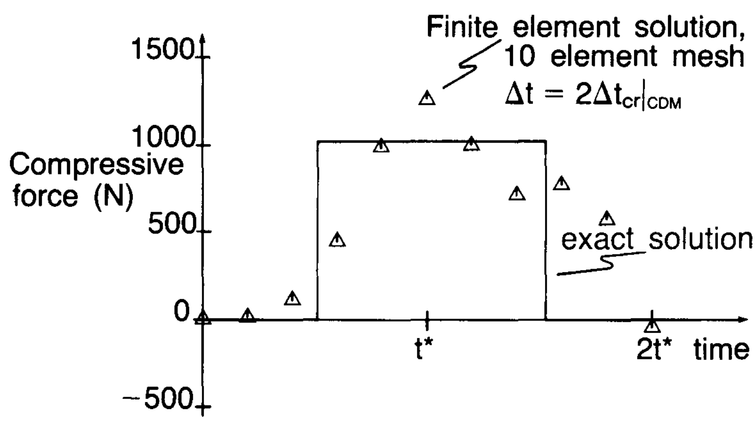
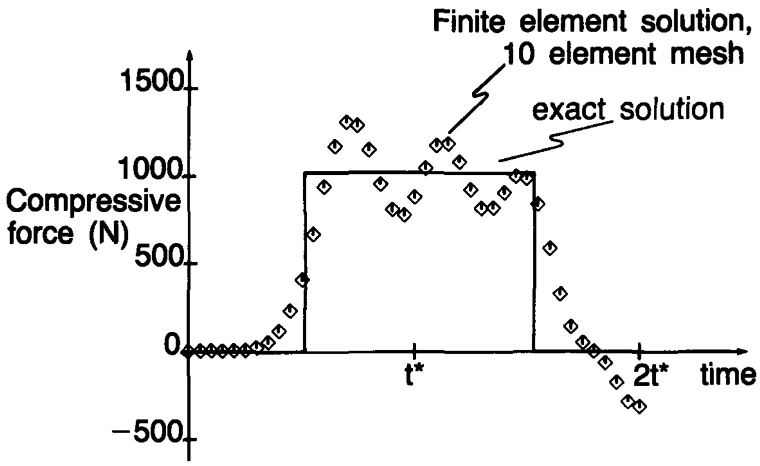
Trapezoidal rule, $\Delta t = 2\Delta t_{cr}|_{CDM}$

— The solution is stable, although inaccurate.



Trapezoidal rule, $\Delta t = \frac{1}{2}\Delta t_{cr}|_{CDM}$



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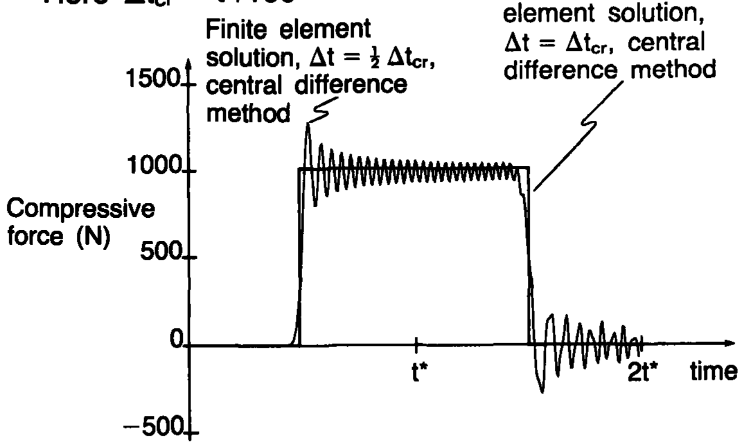
The same phenomena are observed when a mesh of one hundred 2-node truss elements is employed.

— Here $\Delta t_{cr} = t^*/100$



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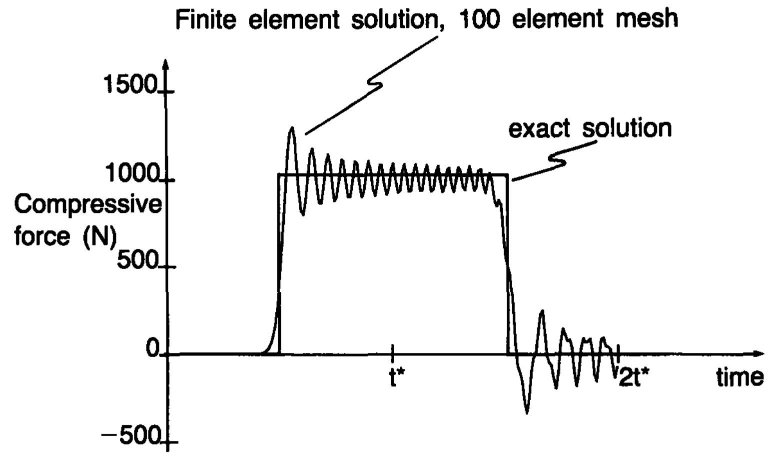
Trapezoidal rule, $\Delta t = \Delta t_{cr}|_{CDM}$



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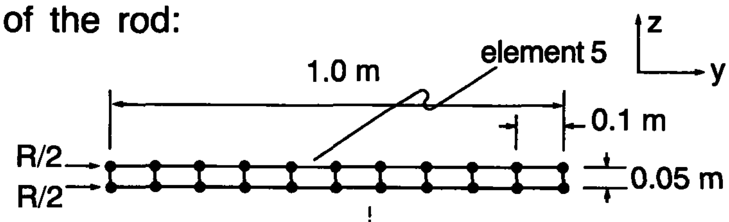
Now consider a two-dimensional model of the rod:



thickness = 0.2 m
$E = 2 \times 10^9$ Pa
$\nu = 0$
$\rho = 1000$ kg/m^3

Central difference method is employed

For this mesh, $\Delta t_{cr} \neq t^*/(10 \text{ elements})$ because the element width is less than the element length.

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If $\Delta t = t^*/(10 \text{ elements})$ is used, the solution diverges

— In element 5,

$$|\tau_{zz}| > \left(\frac{1000\ \text{N}}{0.01\ \text{m}^2}\right)$$

at $t = 1.9\,t^*$

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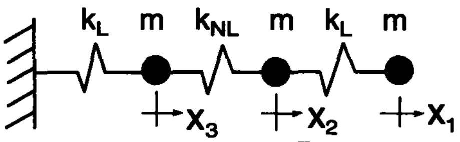
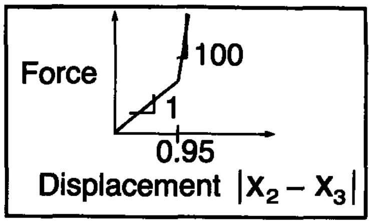
Example: Dynamic response of three degree-of-freedom system using central difference method



k_L = 1 lbf/ft
m = 1 slug

$^0x_1 = {}^0x_2 = {}^0x_3 = 0$

$^0\dot{x}_1 = 0.555$ ft/sec
$^0\dot{x}_2 = 1.000$ ft/sec
$^0\dot{x}_3 = 1.247$ ft/sec



$(\Delta t_{crit})_{linear} = 1.11$ sec
$(\Delta t_{crit})_{nonlinear} = 0.14$ sec

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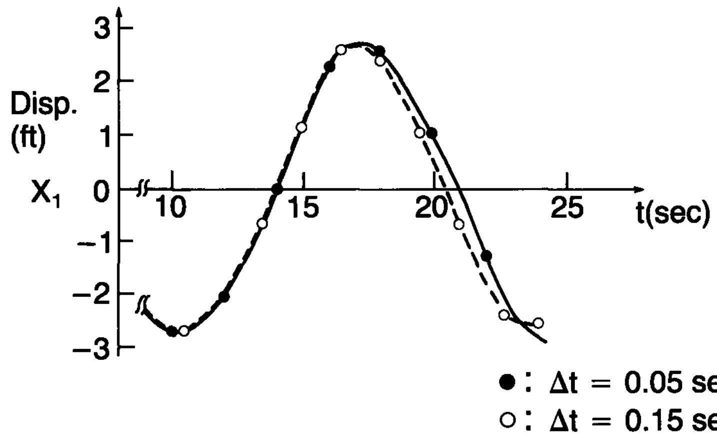
Results: Response of right mass



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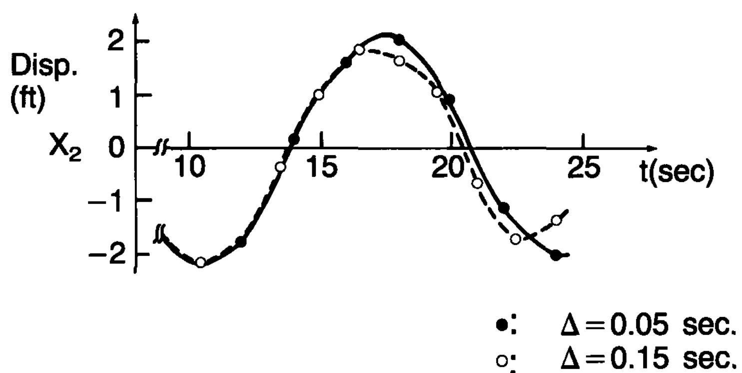
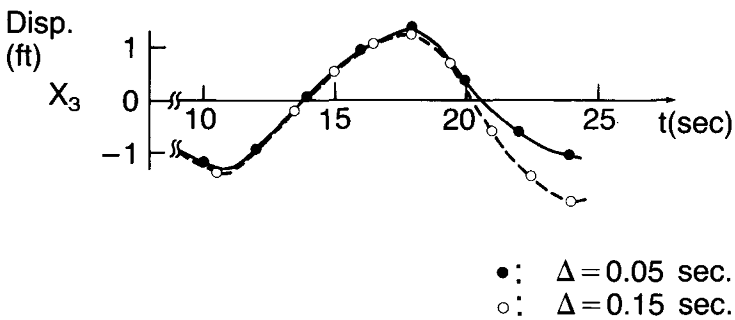
Response of center mass:



Response of left mass:



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14-31

Force (lbf) in center truss:

TIME	Δt = 0.05	Δt = 0.15
9.0	−0.666	−0.700
12.0	−0.804	−0.877
15.0	0.504	0.503
18.0	0.648	−0.100
21.0	−0.132	−0.059
24.0	−0.922	0.550

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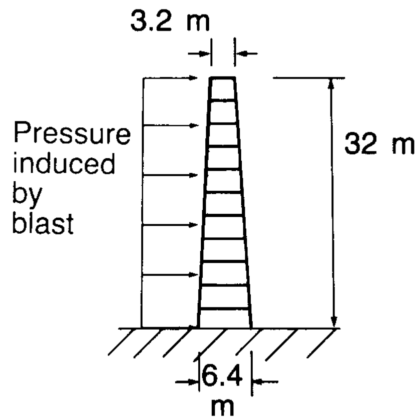
Example: 10 story tapered tower



Girder properties:
$E = 2.07 \times 10^{11}$ Pa
$\nu = 0.3$
$A = 0.01\ m^2$
$A_s = 0.009\ m^2$
$I = 8.33 \times 10^{-5}\ m^4$
$\rho = 7800\ kg/m^3$

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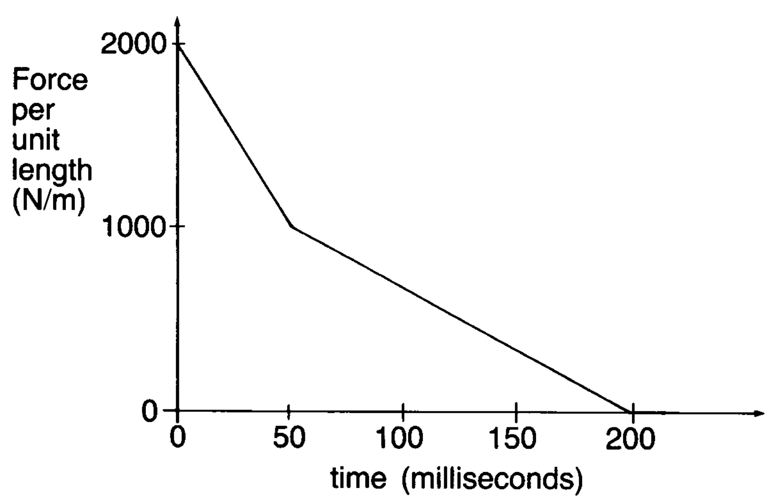
Applied load (blast):



Purpose of analysis:

- Determine displacements, velocities at top of tower.
- Determine moments at base of tower.

We use the trapezoidal rule and a lumped mass matrix in the following analysis.

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We must make two decisions:

- Choose mesh (specifically the number of elements employed).
- Choose time step Δt.

These two choices are closely related:

The mesh and time step to be used depend on the loading applied.

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Transparency 14-37

Some observations:

- The choice of mesh determines the highest natural frequency (and corresponding mode shape) that is accurately represented in the finite element analysis.
- The choice of time step determines the highest frequency of the finite element mesh in which the response is accurately integrated during the time integration.

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- Hence, it is most effective to choose the mesh and time step such that the highest frequency accurately "integrated" is equal to the highest frequency accurately represented by the mesh.
- The applied loading can be represented as a Fourier series which displays the important frequencies to be accurately represented by the mesh.

Consider the Fourier representation of the load function:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(2\pi f_n t) + b_n \sin(2\pi f_n t))$$

Including terms up to

case 1: $f_n = 17$ Hz

case 2: $f_n = 30$ Hz

The loading function is represented as shown next.

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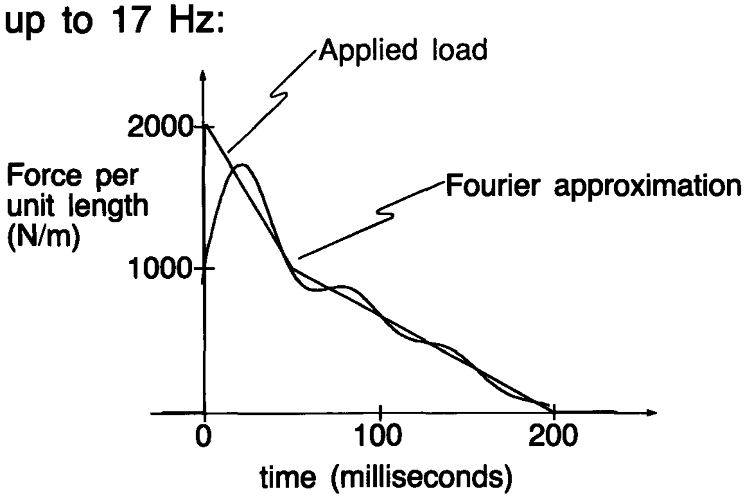
Fourier approximation including terms up to 17 Hz:



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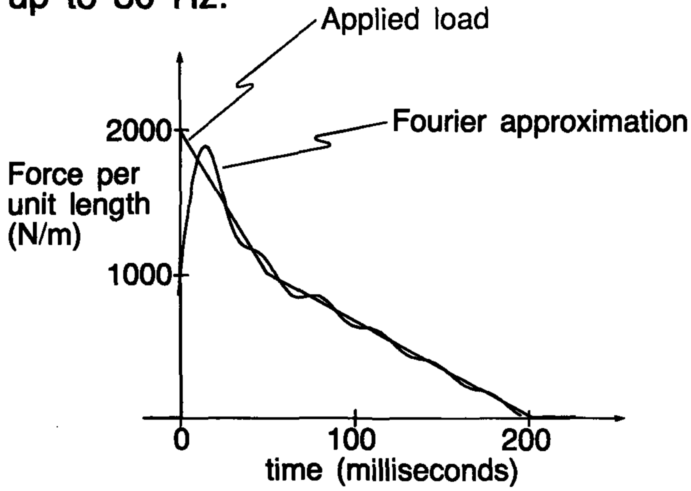
Fourier approximation including terms up to 30 Hz:



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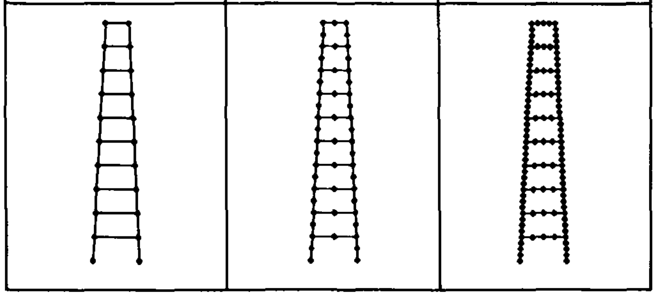
- We choose a 30 element mesh, a 60 element mesh and a 120 element mesh. All elements are 2-node Hermitian beam elements.

30 elements	60 elements	120 elements



Determine "accurate" natural frequencies represented by 30 element mesh:

From eigenvalue solutions of the 30 and 60 element meshes, we find

mode number	natural frequencies (Hz) 30 element mesh	natural frequencies (Hz) 60 element mesh	
1	1.914	1.914	accurate
2	4.815	4.828	accurate
3	8.416	8.480	accurate
4	12.38	12.58	accurate
5	16.79	17.27	accurate
6	21.45	22.47	inaccurate
7	26.18	28.08	inaccurate
8	30.56	29.80	inaccurate

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Calculate time step:

$$T_{co} = \frac{1}{17}\,\text{Hz} = .059 \text{ sec}$$

$$\Delta t \doteq \frac{1}{20} T_{co} = .003 \text{ sec}$$

- A smaller time step would accurately "integrate" frequencies, which are not accurately represented by the mesh.
- A larger time time step would not accurately "integrate" all frequencies which are accurately represented by the mesh.

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Transparency 14-45

Determine "accurate" natural frequencies represented by 60 element mesh:

From eigenvalue solutions of the 60 and 120 element meshes, we find

mode number	natural frequencies (Hz)	
	60 element mesh	120 element mesh
5	17.27	17.28
6	22.47	22.49
7	28.08	28.14
8	29.80	29.75
9	32.73	33.85
10	33.73	35.06
11	36.30	38.96

accurate (modes 5–8) ↕ inaccurate (modes 9–11)

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Calculate time step:

$$T_{co} = \frac{1}{30} \text{Hz} = .033 \text{ sec}$$

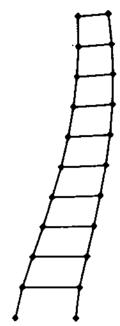
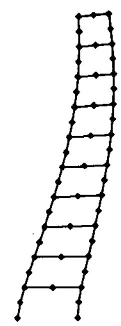
$$\Delta t \doteq \frac{1}{20} T_{co} = .0017 \text{ sec}$$

- The meshes chosen correspond to the Fourier approximations discussed earlier:

30 element mesh ⟷ Fourier approximation including terms up to 17 Hz.

60 element mesh ⟷ Fourier approximation including terms up to 30 Hz.

Pictorially, at time 200 milliseconds, we have (note that the displacements are amplified for visibility):

30 elements	60 elements
	

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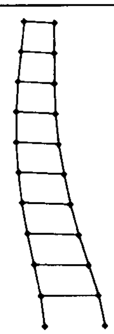
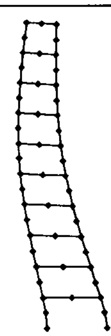
Pictorially, at time 400 milliseconds, we have (note that the displacements are amplified for visibility):

30 elements	60 elements
	

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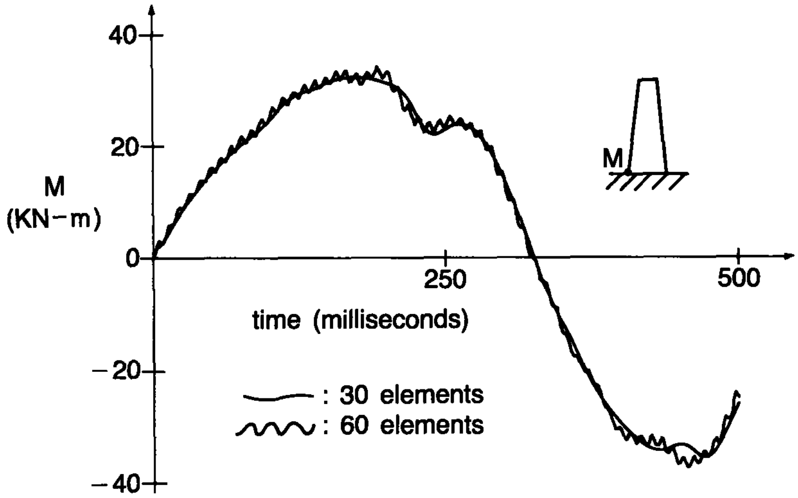
Consider the moment reaction at the base of the tower:



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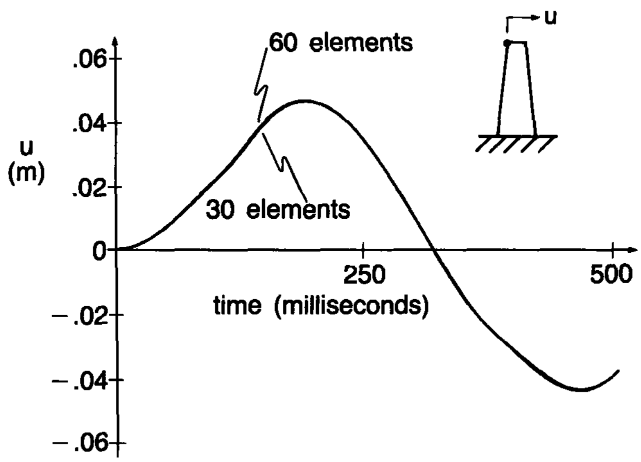
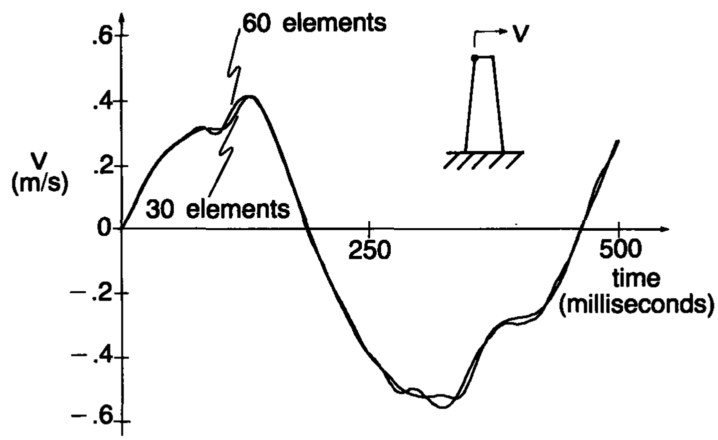
Consider the horizontal displacement at the top of the tower:



Consider the horizontal velocity at the top of the tower:



Transparency 14-51

Comments:

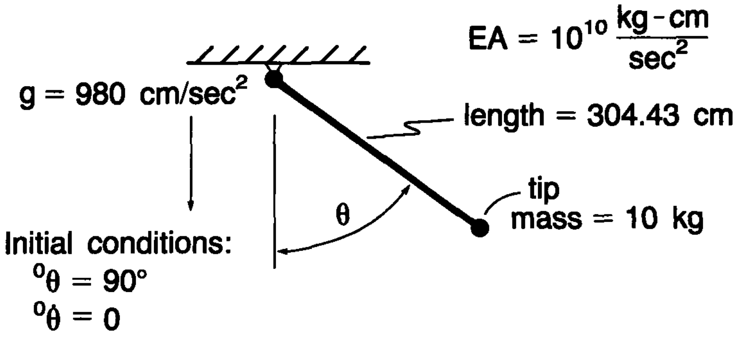
- The high-frequency oscillation observed in the moment reaction from the 60 element mesh is probably inaccurate. We note that the frequency of the oscillation is about 110 Hz (this can be seen directly from the graph).

- The obtained solutions for the horizontal displacement at the top of the tower are virtually identical.

Transparency 14-52

Transparency 14-53

Example: Simple pendulum undergoing large displacements



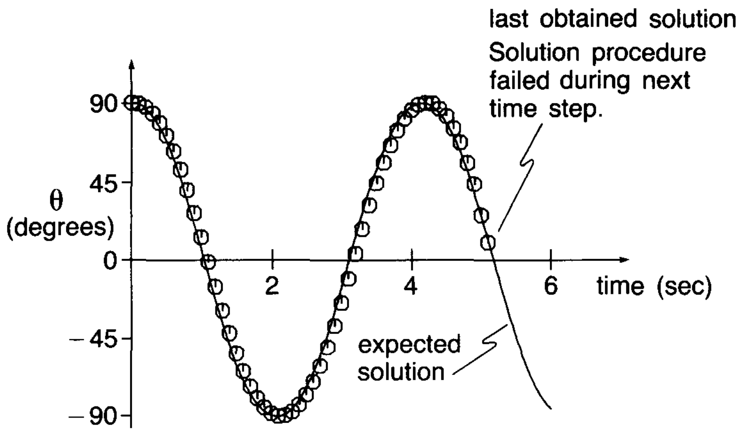
One truss element with tip concentrated mass is employed.

Transparency 14-54

Calculation of dynamic response:

- The trapezoidal rule is used to integrate the time response.
- Full Newton iterations are used to reestablish equilibrium during every time step.
- Convergence tolerance:
  ETOL = 10^{-7}
  (a tight tolerance)

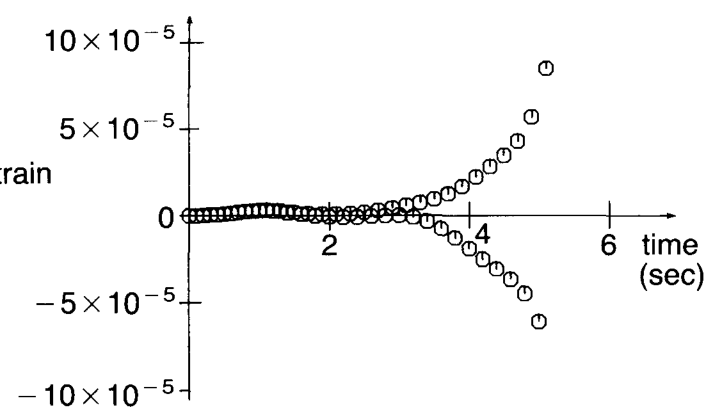
Choose $\Delta t = 0.1$ sec. The following response is obtained:



Transparency 14-55

The strain in the truss is plotted:

- An instability is observed.



Transparency 14-56

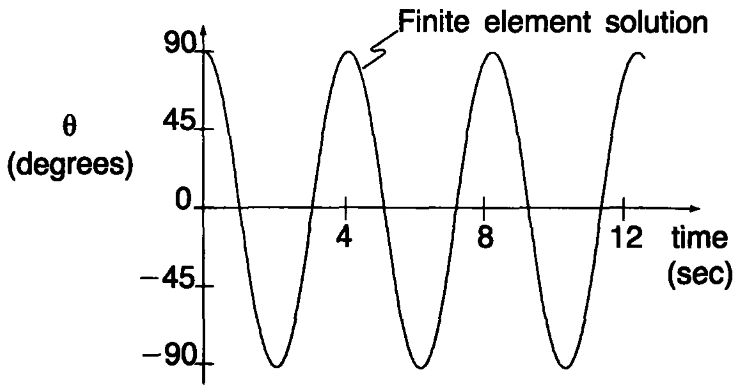
Transparency 14-57

- The instability is unchanged when we tighten our convergence tolerances.
- The instability is also observed when the BFGS algorithm is employed.
- Recall that the trapezoidal rule is unconditionally stable only in linear analysis.

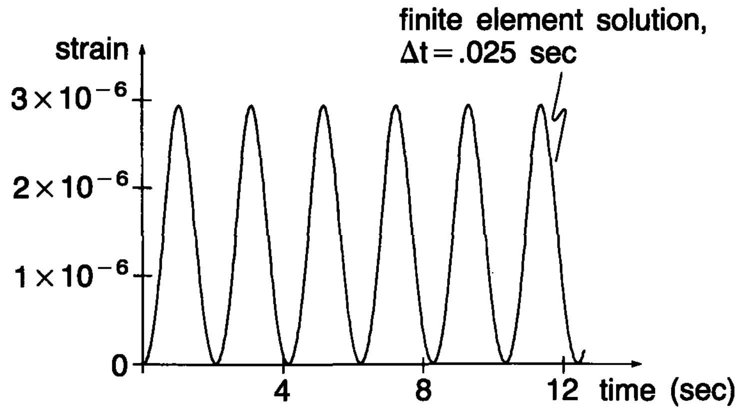
Transparency 14-58

Choose $\Delta t = 0.025$ sec, using the original tolerance and the full Newton algorithm (without line searches).

- The analysis runs to completion.

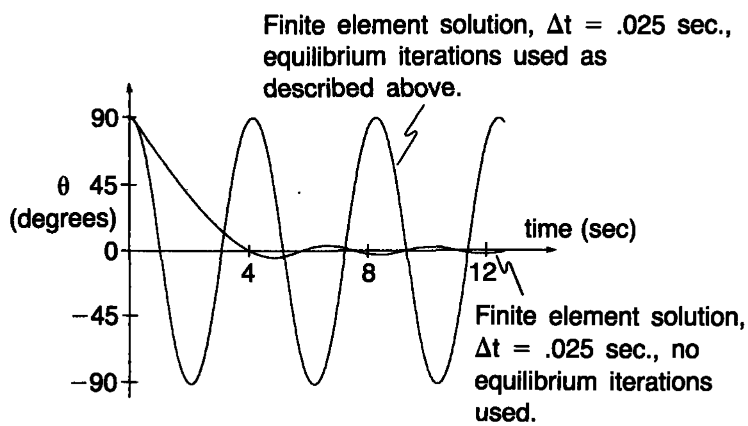


The strain in the truss is stable:



Transparency 14-59

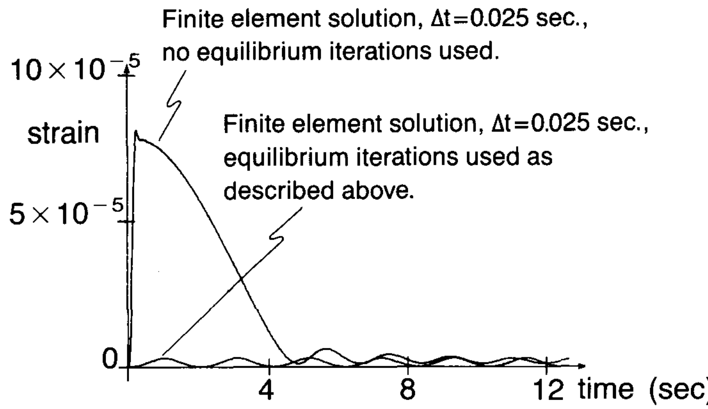
It is important that equilibrium be accurately satisfied at the end of each time step:



Transparency 14-60

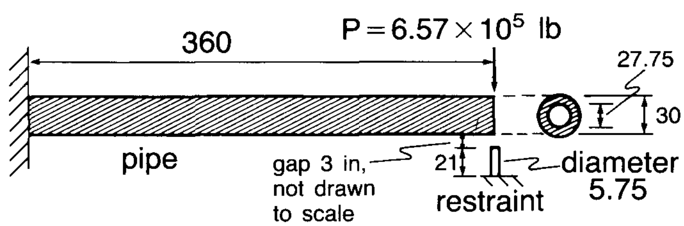
Transparency 14-61

Although the solution obtained without equilibrium iterations is highly inaccurate, the solution is stable:



Transparency 14-62

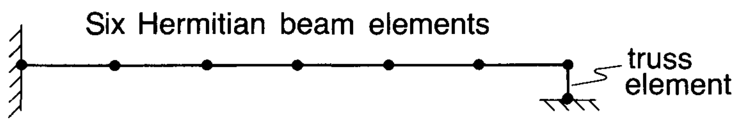
Example: Pipe whip analysis:



all dimensions in inches

- Determine the transient response when a step load P is suddenly applied.

Finite element model:



- The truss element incorporates a 3 inch gap.

Transparency 14-63

Material properties:

Pipe: $E = 2.698 \times 10^7$ psi
$\nu = 0.3$
$\sigma_y = 2.914 \times 10^4$ psi
$E_T = 0$
$\rho = 8.62 \times 10^{-3} \dfrac{\text{slug}}{\text{in}^3} = 7.18 \times 10^{-4} \dfrac{\text{lbf-sec}^2}{\text{in}^4}$

Restraint: $E = 2.99 \times 10^7$ psi
$\sigma_y = 3.80 \times 10^4$ psi
$E_T = 0$

Transparency 14-64

Transparency 14-65

The analysis is performed using

- Mode superposition (2 modes)
- Direct time integration

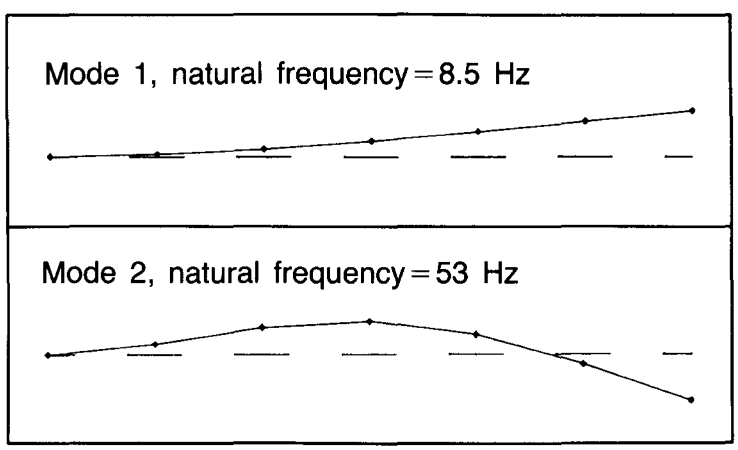
We use, for each analysis,

- Trapezoidal rule
- Consistent mass matrix

A convergence tolerance of ETOL $= 10^{-7}$ is employed.

Transparency 14-66

Eigenvalue solution:



Choice of time step:

We want to accurately integrate the first two modes:

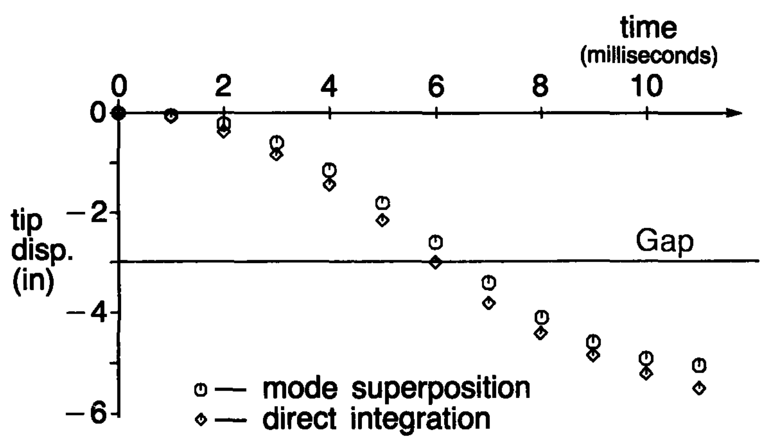
$$\Delta t \doteq \frac{1}{20}\, T_{co} = \frac{1}{20}\left(\frac{1}{\text{(frequency of mode 2)}}\right)$$

$= .001$ sec

Note: This estimate is based solely on a linear analysis (i.e, before the pipe hits the restraint and while the pipe is still elastic).

Transparency 14-67

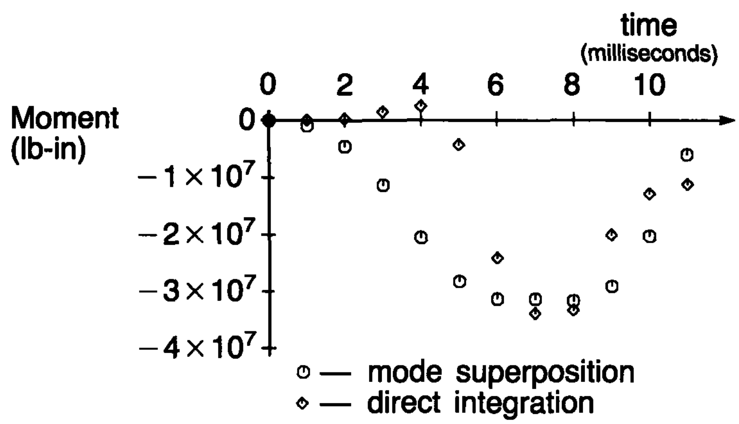
Determine the tip displacement:

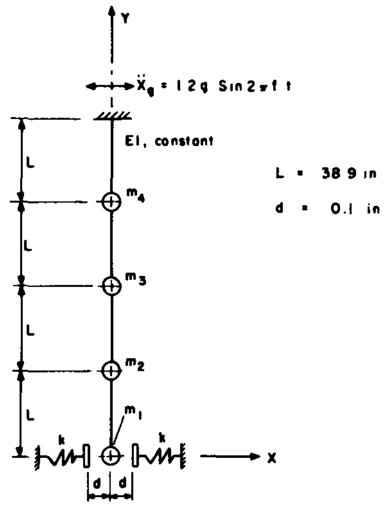


Transparency 14-68

Transparency 14-69

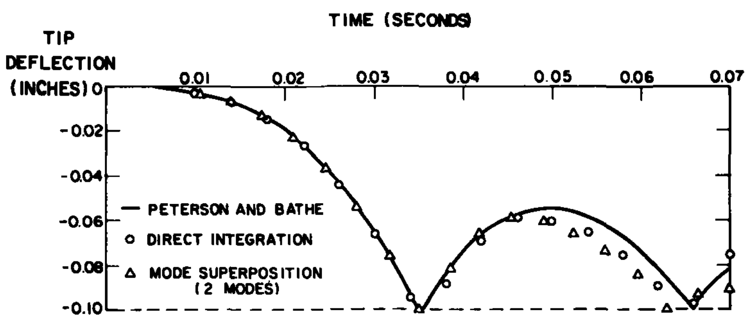
Determine the moment at the built-in end of the beam:





Analysis of CRD housing with lower support

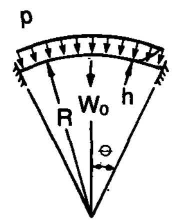
Slide 14-1



CRD housing tip deflection

Slide 14-2

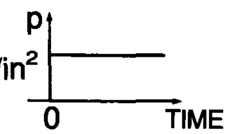
Slide 14-3



R = 22.27 in.
h = 0.41 in.
θ = 26.67°

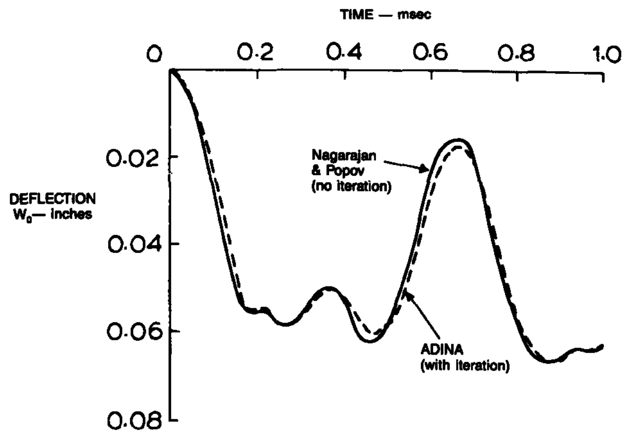
$E = 1.05 \times 10^7$ lb/in²
$\nu = 0.3$
$\sigma_y = 2.4 \times 10^4$ lb/in²
$E_T = 2.1 \times 10^5$ lb/in²
$\rho = 9.8 \times 10^{-2}$ lb/in³

Ten 8-node axisymmetric els.
Newmark inte ($\delta = 0.55$, $\alpha = 0.276$)
2×2 Gauss integration
consistent mass
$\Delta t = 10\mu$sec, T.L.

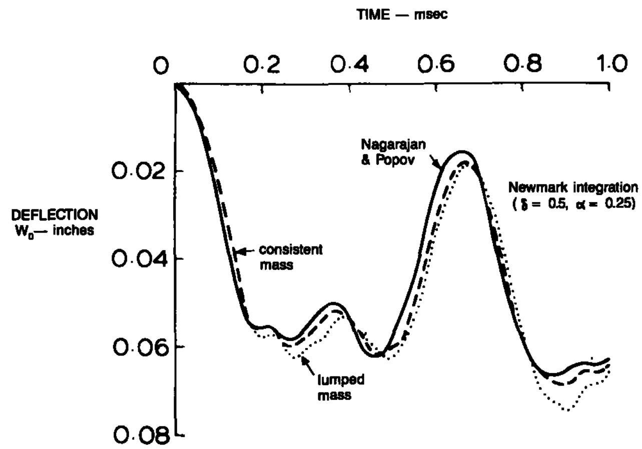


Spherical cap nodes under uniform pressure loading

Slide 14-4

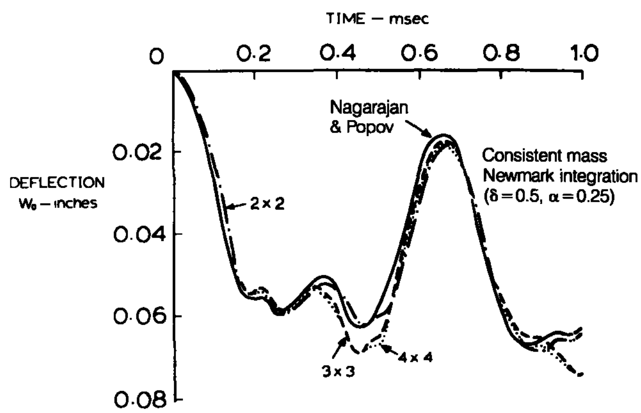


Dynamic elastic-plastic response of a spherical cap, *p* deformation independent



Response of the cap using consistent and lumped mass idealization

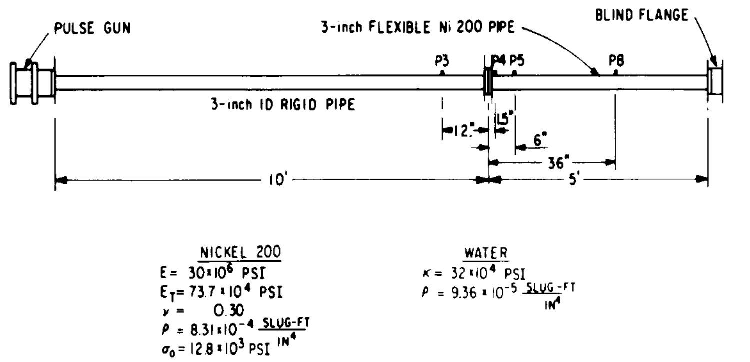
Slide 14-5



Effect of numbers of Gauss integration points on the cap response predicted

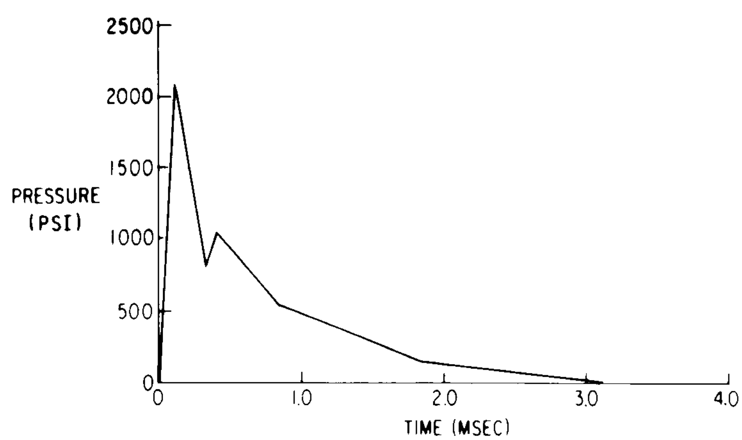
Slide 14-6

Slide 14-7



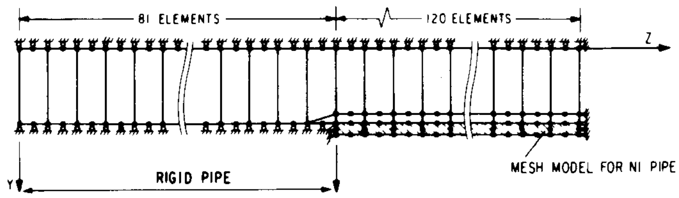
Analysis of fluid—structure interaction problem (pipe test)

Slide 14-8



PRESSURE PULSE INPUT

Slide 14-9



Finite element model

Slide 14-10