Topic 16

Use of Elastic Constitutive Relations in Updated Lagrangian Formulation

Contents:

- **Use of updated Lagrangian (U.L.) formulation**
- **Detailed comparison of expressions used in total Lagrangian (T.L.) and U.L. formulations; strains, stresses, and constitutive relations**
- **Study of conditions to obtain in a general incremental analysis the same results as in the T.L. formulation, and vice versa**
- **The special case of elasticity**
- **The Almansi strain tensor**
- **One-dimensional example involving large strains**
- **Analysis of large displacement/small strain problems**
- **Example analysis: Large displacement solution of frame using updated and total Lagrangian formulations**

Textbook: 6.4, 6.4.1

Example: 6.19

SO FAR THE USE OF
THE T.L. FORMULATION
WAS IMPLIED

Now suppose that we wish to use the U.L. formulation in the analysis. We ask

- Is it possible to obtain, using the U.L. formulation, identically the same numerical results (for each iteration) as are obtained using the T.L. formulation?

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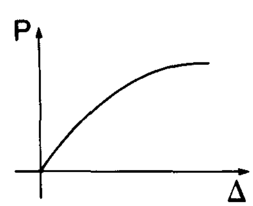
In other words, the situation is

Program 1

- Only T.L. formulation is implemented
  — Constitutive relations are

$$\boxed{\begin{array}{l} {}_0^tS_{ij} = \text{function of displacements} \\ d_0S_{ij} = {}_0C_{ijrs}\, d_0\varepsilon_{rs} \end{array}}$$

Information obtained from physical laboratory experiments.



Program 1 results

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Program 2
• Only U.L. formulation is implemented – Constitutive relations are ${}^{t}\tau_{ij} = \cdots \rightarrow$ ① ${}_{}d_tS_{ij} = \cdots \rightarrow$ ②

Question:

How can we obtain with program 2 identically the same results as are obtained from program 1?

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To answer, we consider the linearized equations of motion:

$$\left.\begin{aligned} &\int_{^0V} {}_0C_{ijrs}\,{}_0e_{rs}\,\delta_0 e_{ij}\,{}^0dV + \int_{^0V} {}_0^tS_{ij}\,\delta_0\eta_{ij}\,{}^0dV \\ &\quad = {}^{t+\Delta t}\mathcal{R} - \int_{^0V} {}_0^tS_{ij}\,\delta_0 e_{ij}\,{}^0dV \end{aligned}\right]\ \text{T.L.}$$

$$\left.\begin{aligned} &\int_{^tV} {}_tC_{ijrs}\,{}_te_{rs}\,\delta_t e_{ij}\,{}^tdV + \int_{^tV} {}^t\tau_{ij}\,\delta_t\eta_{ij}\,{}^tdV \\ &\quad = {}^{t+\Delta t}\mathcal{R} - \int_{^tV} {}^t\tau_{ij}\,\delta_t e_{ij}\,{}^tdV \end{aligned}\right]\ \text{U.L.}$$

Terms used in the formulations:

T.L. formulation	U.L. formulation	Transformation
$\int_{{}^0V} {}^0dV$	$\int_{{}^tV} {}^tdV$	${}^0dV = \frac{{}^t\rho}{{}^0\rho}\, {}^tdV$
${}_0e_{ij},\ {}_0\eta_{ij}$	${}_te_{ij},\ {}_t\eta_{ij}$	${}_0e_{ij} = {}^t_0x_{r,i}\ {}^t_0x_{s,j}\ {}_te_{rs}$ ${}_0\eta_{ij} = {}^t_0x_{r,i}\ {}^t_0x_{s,j}\ {}_t\eta_{rs}$
$\delta_0e_{ij},\ \delta_0\eta_{ij}$	$\delta_te_{ij},\ \delta_t\eta_{ij}$	$\delta_0e_{ij} = {}^t_0x_{r,i}\ {}^t_0x_{s,j}\ \delta_te_{rs}$ $\delta_0\eta_{ij} = {}^t_0x_{r,i}\ {}^t_0x_{s,j}\ \delta_t\eta_{rs}$

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Derivation of these kinematic relationships:

A fundamental property of ${}^t_0\varepsilon_{ij}$ is that

$${}^t_0\varepsilon_{ij}\, d^0x_i\, d^0x_j = \frac{1}{2}\left(({}^tds)^2 - ({}^0ds)^2\right)$$

Similarly,

$${}^{t+\Delta t}_{\ \ 0}\varepsilon_{ij}\, d^0x_i\, d^0x_j = \frac{1}{2}\left(({}^{t+\Delta t}ds)^2 - ({}^0ds)^2\right)$$

and

$${}_t\varepsilon_{rs}\, d^tx_r\, d^tx_s = \frac{1}{2}\left(({}^{t+\Delta t}ds)^2 - ({}^tds)^2\right)$$

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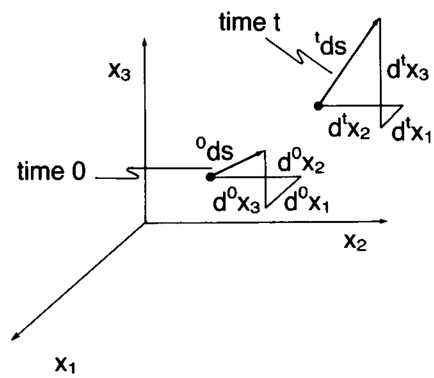

$$d^t\underline{x} = {}_0^t\underline{X}\, d^0\underline{x}$$

Fiber $d^0\underline{x}$ of length ${}^0ds$ moves to become $d^t\underline{x}$ of length ${}^tds$.

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Hence, by subtraction, we obtain

$$_0\varepsilon_{ij}\, d^0x_i\, d^0x_j = {}_t\varepsilon_{rs}\, d^tx_r\, d^tx_s$$

Using $d^t\underline{x} = {}_0^t\underline{X}\, d^0\underline{x}$, we obtain

$$_0\varepsilon_{ij}\, d^0x_i\, d^0x_j = {}_t\varepsilon_{rs}\, {}_0^tx_{r,i}\, {}_0^tx_{s,j}\, d^0x_i\, d^0x_j$$

Since this relationship holds for arbitrary material fibers, we have

$$_0\varepsilon_{ij} = {}_0^tx_{r,i}\, {}_0^tx_{s,j}\, {}_t\varepsilon_{rs}$$

Transparency 16-9

Now we see that

$$ {}_0e_{ij} + {}_0\eta_{ij} = {}^t_0x_{r,i}\, {}^t_0x_{s,j}\, {}_te_{rs} + {}^t_0x_{r,i}\, {}^t_0x_{s,j}\, {}_t\eta_{rs} $$

Since the factors ${}^t_0x_{r,i}\, {}^t_0x_{s,j}$ do not contain the incremental displacements u_i, we have

$$ {}_0e_{ij} = {}^t_0x_{r,i}\, {}^t_0x_{s,j}\, {}_te_{rs} \leftarrow \text{linear in } u_i $$

$$ {}_0\eta_{ij} = {}^t_0x_{r,i}\, {}^t_0x_{s,j}\, {}_t\eta_{rs} \leftarrow \text{quadratic in } u_i $$

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In addition, we have

$$ \delta\, {}_0e_{ij} = {}^t_0x_{r,i}\, {}^t_0x_{s,j}\, \delta\, {}_te_{rs} $$

$$ \delta\, {}_0\eta_{ij} = {}^t_0x_{r,i}\, {}^t_0x_{s,j}\, \delta\, {}_t\eta_{rs} $$

These follow because the variation is taken on the configuration $t+\Delta t$ and hence the factors ${}^t_0x_{r,i}\, {}^t_0x_{s,j}$ are taken as constant during the variation.

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We also have

T.L. formulation	U.L. formulation	Transformation
${}_0^tS_{ij}$	${}^t\tau_{ij}$	${}_0^tS_{ij} = \frac{{}^0\rho}{{}^t\rho}\, {}_t^0x_{i,m}\, {}^t\tau_{mn}\, {}_t^0x_{j,n}$
${}_0C_{ijrs}$	${}_tC_{ijrs}$	${}_0C_{ijrs} = \frac{{}^0\rho}{{}^t\rho}\, {}_t^0x_{i,a}\, {}_t^0x_{j,b}\, {}_tC_{abpq}\, {}_t^0x_{r,p}\, {}_t^0x_{s,q}$ (To be derived below)

Transparency 16-12

Consider the tangent constitutive tensors ${}_0C_{ijrs}$ and ${}_tC_{ijrs}$:

Recall that

$$\underbrace{d_0S_{ij}} = {}_0C_{ijrs}\, \underbrace{d_0\varepsilon_{rs}}$$

$$\underbrace{d_tS_{ij}} = {}_tC_{ijrs}\, \underbrace{d_t\varepsilon_{rs}}$$

differential increments

Now we note that

$$d_0S_{ij} = \frac{{}^0\rho}{{}^t\rho}\, {}_t^0x_{i,a}\, {}_t^0x_{j,b}\, d_tS_{ab}$$

$$d_0\varepsilon_{rs} = {}_0^tx_{p,r}\, {}_0^tx_{q,s}\, d_t\varepsilon_{pq}$$

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Hence

$$\underbrace{\left(\frac{{}^{0}\rho}{{}^{t}\rho}\,{}^{0}_{t}x_{i,a}\,{}^{0}_{t}x_{j,b}\,d_{t}S_{ab}\right)}_{d_{0}S_{ij}} = {}_{0}C_{ijrs}\underbrace{\left({}^{t}_{0}x_{p,r}\,{}^{t}_{0}x_{q,s}\,d_{t}\varepsilon_{pq}\right)}_{d_{0}\varepsilon_{rs}}$$

Solving for d_tS_{ab} gives

$$d_{t}S_{ab} = \underbrace{\left(\frac{{}^{t}\rho}{{}^{0}\rho}\,{}^{t}_{0}x_{a,i}\,{}^{t}_{0}x_{b,j}\,{}_{0}C_{ijrs}\,{}^{t}_{0}x_{p,r}\,{}^{t}_{0}x_{q,s}\right)}_{{}_{t}C_{abpq}}\,d_{t}\varepsilon_{pq}$$

Transparency 16-14

And we therefore observe that the tangent material relationship to be used is

$${}_{t}C_{abpq} = \frac{{}^{t}\rho}{{}^{0}\rho}\,{}^{t}_{0}x_{a,i}\,{}^{t}_{0}x_{b,j}\,{}_{0}C_{ijrs}\,{}^{t}_{0}x_{p,r}\,{}^{t}_{0}x_{q,s}$$

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Now compare each of the integrals appearing in the T.L. and U.L. equations of motion:

$$1)\ \underbrace{\int_{^0V} {}^t_0S_{ij}\, \delta_0 e_{ij}\, {}^0dV = \int_{^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^t dV}\qquad ?$$

True, as we verify by substituting the established transformations:

$$\int_{^0V} \underbrace{\left(\frac{^0\rho}{^t\rho}\, {}^0_t x_{i,m}\, {}^t\tau_{mn}\, {}^0_t x_{j,n}\right)}_{^t_0S_{ij}} \underbrace{\left({}^t_0 x_{r,i}\, {}^t_0 x_{s,j}\, \delta_t e_{rs}\right)}_{\delta_0 e_{ij}} {}^0dV$$

$$= \int_{^0V} {}^t\tau_{mn}\, \delta_t e_{rs}\, \underbrace{\left({}^0_t x_{i,m}\, {}^t_0 x_{r,i}\right)}_{\delta_{mr}} \underbrace{\left({}^0_t x_{j,n}\, {}^t_0 x_{s,j}\right)}_{\delta_{ns}} \underbrace{\frac{^0\rho}{^t\rho}\, {}^0dV}_{^tdV}$$

$$= \int_{^tV} {}^t\tau_{mn}\, \delta_t e_{mn}\, {}^t dV$$

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$$2)\ \underbrace{\int_{^0V} {}^t_0S_{ij}\, \delta_0 \eta_{ij}\, {}^0dV = \int_{^tV} {}^t\tau_{ij}\, \delta_t \eta_{ij}\, {}^t dV}\ ?$$

True, as we verify by substituting the established transformations:

$$\int_{^0V} \underbrace{\left(\frac{^0\rho}{^t\rho}\, {}^0_t x_{i,m}\, {}^t\tau_{mn}\, {}^0_t x_{j,n}\right)}_{^t_0S_{ij}} \underbrace{\left({}^t_0 x_{r,i}\, {}^t_0 x_{s,j}\, \delta_t \eta_{rs}\right)}_{\delta_0 \eta_{ij}} {}^0dV$$

$$= \int_{^0V} {}^t\tau_{mn}\, \delta_t \eta_{rs}\, \underbrace{\left({}^0_t x_{i,m}\, {}^t_0 x_{r,i}\right)}_{\delta_{mr}} \underbrace{\left({}^0_t x_{j,n}\, {}^t_0 x_{s,j}\right)}_{\delta_{ns}} \underbrace{\frac{^0\rho}{^t\rho}\, {}^0dV}_{^tdV}$$

$$= \int_{^tV} {}^t\tau_{mn}\, \delta_t \eta_{mn}\, {}^t dV$$

$$3)\ \int_{^0V} {}_0C_{ijrs}\ {}_0e_{rs}\ \delta_0 e_{ij}\ {}^0dV = \int_{^tV} {}_tC_{ijrs}\ {}_te_{rs}\ \delta_t e_{ij}\ {}^tdV\ ?$$

True, as we verify by substituting the established transformations:

$$\int_{^0V} \underbrace{\left(\frac{{}^0\rho}{{}^t\rho}\ {}^0_tx_{i,a}\ {}^0_tx_{j,b}\ {}_tC_{abpq}\ {}^0_tx_{r,p}\ {}^0_tx_{s,q}\right)}_{{}_0C_{ijrs}} \times$$

$$\underbrace{({}^t_0x_{k,r}\ {}^t_0x_{\ell,s}\ {}_te_{k\ell})}_{{}_0e_{rs}}\ \underbrace{({}^t_0x_{m,i}\ {}^t_0x_{n,j}\ \delta_t e_{mn})}_{\delta_0 e_{ij}}\ {}^0dV$$

$$= \int_{^tV} {}_tC_{abpq}\ {}_te_{pq}\ \delta_t e_{ab}\ {}^tdV$$

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Provided the established transformations are used, the three integrals are identical. Therefore the resulting finite element discretizations will also be identical.

$$({}^t_0\underline{K}_L + {}^t_0\underline{K}_{NL})\ \Delta\underline{U} = {}^{t+\Delta t}\underline{R} - {}^t_0\underline{F}$$

$$({}^t_t\underline{K}_L + {}^t_t\underline{K}_{NL})\ \Delta\underline{U} = {}^{t+\Delta t}\underline{R} - {}^t_t\underline{F}$$

$$\boxed{\begin{aligned} {}^t_0\underline{K}_L &= {}^t_t\underline{K}_L \\ {}^t_0\underline{K}_{NL} &= {}^t_t\underline{K}_{NL} \\ {}^t_0\underline{F} &= {}^t_t\underline{F} \end{aligned}}$$

The same holds for each equilibrium iteration.

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Hence, to summarize once more, program 2 gives the same results as program 1, provided

① → The Cauchy stresses are calculated from

$$ {}^{t}\tau_{ij} = \frac{{}^{t}\rho}{{}^{0}\rho}\, {}^{t}_{0}x_{i,m}\, {}^{t}_{0}S_{mn}\, {}^{t}_{0}x_{j,n} $$

② → The tangent stress-strain law is calculated from

$$ {}_{t}C_{ijrs} = \frac{{}^{t}\rho}{{}^{0}\rho}\, {}^{t}_{0}x_{i,a}\, {}^{t}_{0}x_{j,b}\, {}_{0}C_{abpq}\, {}^{t}_{0}x_{r,p}\, {}^{t}_{0}x_{s,q} $$

Transparency 16-20

Conversely, assume that the material relationships for program 2 are given, hence, from laboratory experimental information, ${}^{t}\tau_{ij}$ and ${}_{t}C_{ijrs}$ for the U.L. formulation are given.
Then we can show that, provided the appropriate transformations

$$ {}^{t}_{0}S_{ij} = \frac{{}^{0}\rho}{{}^{t}\rho}\, {}^{0}_{t}x_{i,m}\, {}^{t}\tau_{mn}\, {}^{0}_{t}x_{j,n} $$

$$ {}_{0}C_{ijrs} = \frac{{}^{0}\rho}{{}^{t}\rho}\, {}^{0}_{t}x_{i,a}\, {}^{0}_{t}x_{j,b}\, {}_{t}C_{abpq}\, {}^{0}_{t}x_{r,p}\, {}^{0}_{t}x_{s,q} $$

are used in program 1 with the T.L. formulation, again the same numerical results are generated.

Hence the choice of formulation (T.L. vs. U.L.) is based solely on the numerical effectiveness of the methods:

- The ${}_{t}^{t}\underline{B}_{L}$ matrix (U.L. formulation) contains less entries than the ${}_{0}^{t}\underline{B}_{L}$ matrix (T.L. formulation).
- The matrix product $\underline{B}^{T}\,\underline{C}\,\underline{B}$ is less expensive using the U.L. formulation.

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- If the stress-strain law is available in terms of ${}_{0}^{t}\underline{S}$, then the T.L. formulation will be in general most effective.
  - — Mooney-Rivlin material law
  - — Inelastic analysis allowing for large displacements / large rotations, but small strains

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THE SPECIAL CASE OF ELASTICITY

Consider that the components ${}_0^tC_{ijrs}$ are given:

$${}_0^tS_{ij} = {}_0^tC_{ijrs}\ {}_0^t\varepsilon_{rs}$$

From the above discussion, to obtain the same numerical results with the U.L. formulation, we would employ

$${}^t\tau_{ij} = \frac{{}^t\rho}{{}^0\rho}\ {}_0^tx_{i,m}\ ({}_0^tC_{mnrs}\ {}_0^t\varepsilon_{rs})\ {}_0^tx_{j,n}$$

$${}_tC_{ijrs} = \frac{{}^t\rho}{{}^0\rho}\ {}_0^tx_{i,a}\ {}_0^tx_{j,b}\ {}_0C_{abpq}\ {}_0^tx_{r,p}\ {}_0^tx_{s,q}$$

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We see that in the above equation, the Cauchy stresses are related to the Green-Lagrange strains by a transformation acting only on the m and n components of ${}_0^tC_{mnrs}$.

However, we can write the total stress-strain law using a tensor, ${}_t^tC^a_{ijrs}$, by introducing another strain measure, namely the Almansi strain tensor,

$${}^t\tau_{ij} = {}_t^tC^a_{ijrs}\ \underbrace{{}_t^t\varepsilon^a_{rs}}\ \text{Almansi strain tensor}$$

$${}_t^tC^a_{ijrs} = \frac{{}^t\rho}{{}^0\rho}\ {}_0^tx_{i,a}\ {}_0^tx_{j,b}\ {}_0^tC_{abpq}\ {}_0^tx_{r,p}\ {}_0^tx_{s,q}$$

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Definitions of the Almansi strain tensor:

$${}^t_t\varepsilon^a_{mn} = {}^0_t x_{i,m}\, {}^0_t x_{j,n}\, {}^t_0\varepsilon_{ij}$$

$${}^t_t\underline{\varepsilon}^a = \frac{1}{2}\left(\underline{I} - {}^0_t\underline{X}^T\, {}^0_t\underline{X}\right)$$

$${}^t_t\varepsilon^a_{ij} = \frac{1}{2}\left({}^t_t u_{i,j} + {}^t_t u_{j,i} - {}^t_t u_{k,i}\, \overbrace{{}^t_t u_{k,j}}^{\frac{\partial\, {}^t u_k}{\partial\, {}^t x_j}}\right)$$

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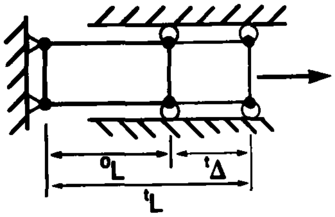
- A symmetric strain tensor, ${}^t_t\varepsilon^a_{ij} = {}^t_t\varepsilon^a_{ji}$
- The components of ${}^t_t\underline{\varepsilon}^a$ are not invariant under a rigid body rotation of the material.
- Hence, ${}^t_t\underline{\varepsilon}^a$ is not a very useful strain measure, but we wanted to introduce it here briefly.

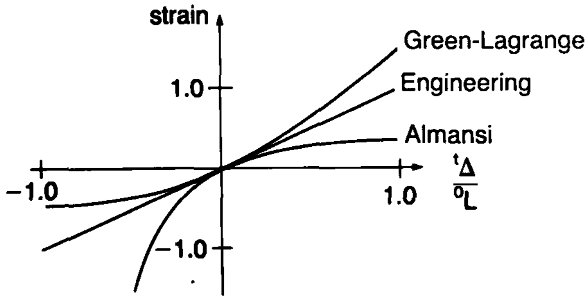
Transparency 16-27

Example: Uniaxial strain



$${}^{t}_{0}\varepsilon_{11} = \frac{{}^{t}\Delta}{{}^{0}L} + \frac{1}{2}\left(\frac{{}^{t}\Delta}{{}^{0}L}\right)^2$$

$${}^{t}_{t}\varepsilon^{a}_{11} = \frac{{}^{t}\Delta}{{}^{t}L} - \frac{1}{2}\left(\frac{{}^{t}\Delta}{{}^{t}L}\right)^2$$



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It turns out that the use of ${}^{t}_{t}C^{a}_{ijrs}$ with the Almansi strain tensor is effective when the U.L. formulation is used with a linear isotropic material law for large displacement / large rotation but small strain analysis.

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- In this case, ${}^{t}_{t}C^{a}_{ijrs}$ may be taken as

$$
\begin{aligned}
{}^{t}_{t}C^{a}_{ijrs} &= \underbrace{\lambda\delta_{ij}\,\delta_{rs} + \mu(\delta_{ir}\,\delta_{js} + \delta_{is}\,\delta_{jr})}_{\text{constants}} \\
&= {}_{t}C_{ijrs}
\end{aligned}
$$

Practically the same response is calculated using the T.L. formulation with

$$
\begin{aligned}
{}^{t}_{0}C_{ijrs} &= \underbrace{\lambda\,\delta_{ij}\,\delta_{rs} + \mu(\delta_{ir}\,\delta_{js} + \delta_{is}\,\delta_{jr})}_{\text{constants}} \\
&= {}_{0}C_{ijrs}
\end{aligned}
$$

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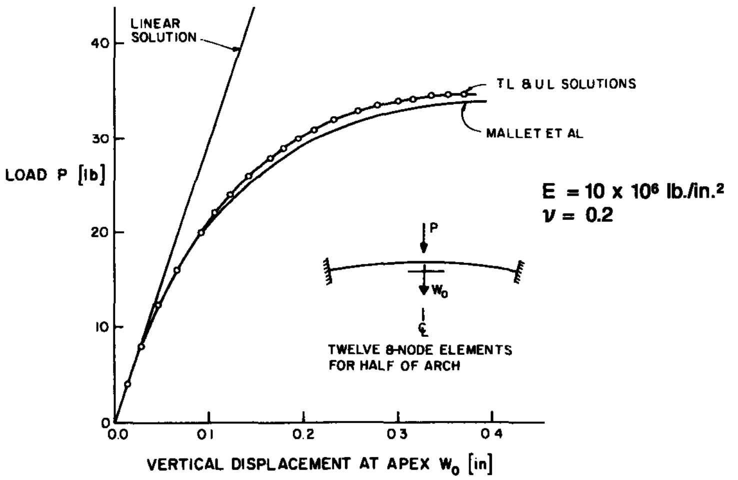

Load-deflection curve for a shallow arch under concentrated load

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The reason that practically the same response is calculated is that the required transformations to obtain exactly the same response reduce to mere rotations:

Namely, in the transformations from ${}_t^tC^a_{ijrs}$ to ${}_0^tC_{abpq}$, and in the relation between ${}_0C_{ijrs}$ and ${}_tC_{ijrs}$,

$$\frac{{}^0\rho}{{}^t\rho} \doteq 1, \quad [{}_0^tx_{i,j}] = {}_0^t\underline{X} = {}_0^t\underline{R}\;{}_0^t\underline{U} \doteq {}_0^t\underline{R}$$

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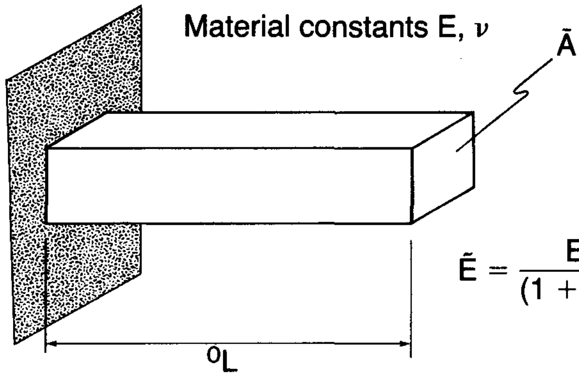
However, when using constant material moduli (E, ν) for large strain analysis, with

$$\left.\begin{array}{l} {}^t\tau_{ij} = \underbrace{{}_t^tC^a_{ijrs}}\;{}_t^t\varepsilon^a_{rs} \\ \text{and} \\ {}_0^tS_{ij} = \overbrace{{}_0^tC_{ijrs}}\;{}_0^t\varepsilon_{rs} \end{array}\right\} = \lambda\,\delta_{ij}\,\delta_{rs} + \mu\,(\delta_{ir}\,\delta_{js} + \delta_{is}\,\delta_{jr})$$

totally different results are obtained.

Consider the 1-D problem already solved earlier:



$$\tilde{E} = \frac{E\,(1-\nu)}{(1+\nu)\,(1-2\nu)}$$

Before, we used ${}^t_0S_{11} = \tilde{E}\;{}^t_0\varepsilon_{11}$.

Now, we consider ${}^t\tau_{11} = \tilde{E}\;{}^t_t\varepsilon^a_{11}$.

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Here, we have

$$ {}^t_t\varepsilon^a_{11} = \underbrace{{}^t_tu_{1,1}}_{\dfrac{{}^tL - {}^0L}{{}^tL}} - \frac{1}{2}\left({}^t_tu_{1,1}\right)^2 = \frac{1}{2}\left[1 - \left(\frac{{}^0L}{{}^tL}\right)^2\right] $$

$$ {}^t\tau_{11} = \frac{{}^tP}{\bar{A}} $$

Using ${}^tL = {}^0L + {}^t\Delta$, ${}^t\tau_{11} = \tilde{E}\;{}^t_t\varepsilon^a_{11}$, we obtain the force-displacement relationship.

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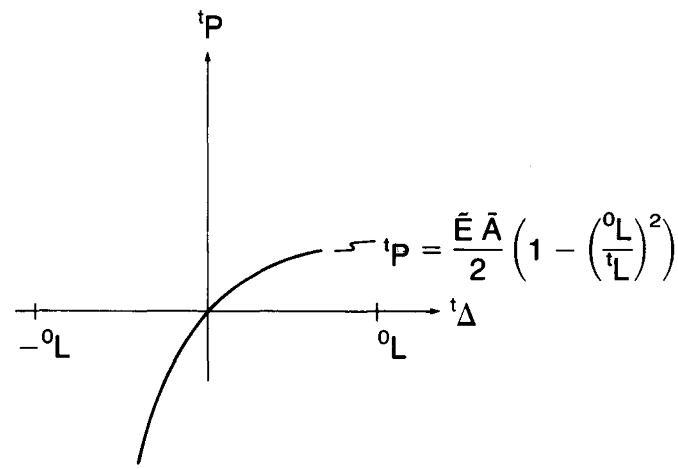

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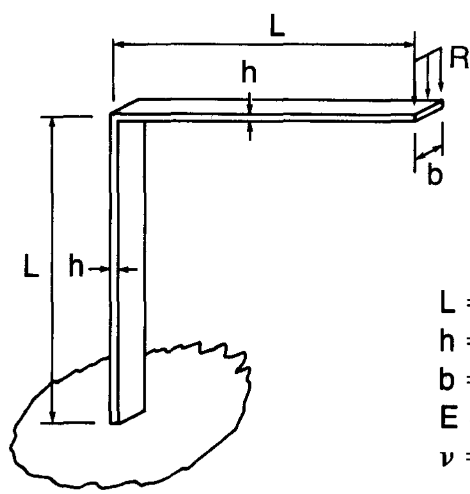
Example: Corner under tip load



$$\left.\begin{array}{l} L = 10.0 \text{ m} \\ h = 0.2 \text{ m} \end{array}\right\} \frac{h}{L} = \frac{1}{50}$$

$b = 1.0$ m

$E = 207000$ MPa

$\nu = 0.3$

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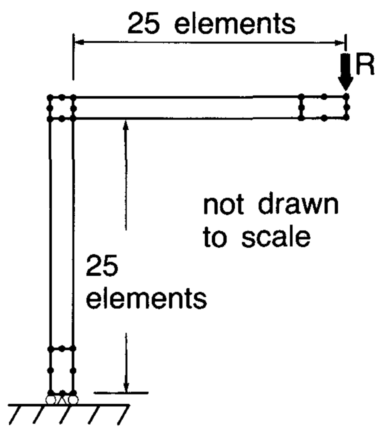
Finite element mesh: 51 two-dimensional 8-node elements



All elements are plane strain elements.

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Consider a nonlinear elastic analysis. For what loads will the T.L. and U.L. formulations give similar results?

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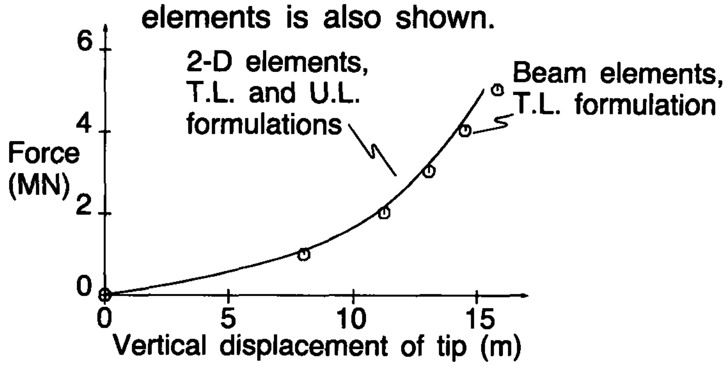
— For large displacement/large rotation, but small strain conditions, the T.L. and U.L. formulations will give similar results.

— For large displacement/large rotation and large strain conditions, the T.L. and U.L. formulations will give different results, because different constitutive relations are assumed.

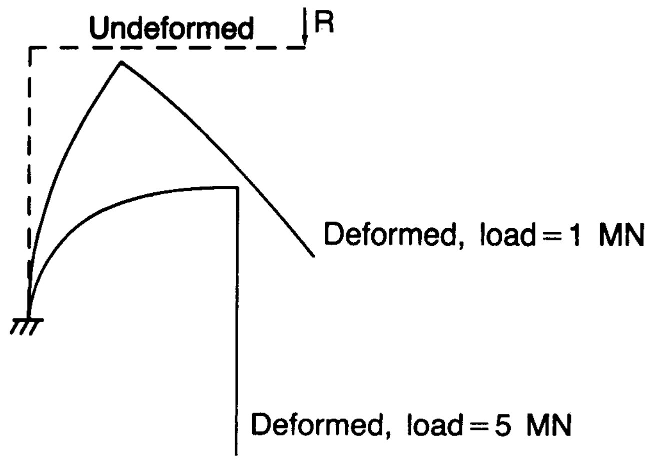
Transparency 16-39

Results: Force-deflection curve

- Over the range of loads shown, the T.L. and U.L. formulations give practically identical results
- The force-deflection curve obtained with two 4-node isoparametric beam elements is also shown.



Deformed configuration for a load of 5 MN (2-D elements are used):



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Numerically, for a load of 5 MN, we have, using the 2-D elements,

	T.L. formulation	U.L. formulation
vertical tip displacement	15.289 m	15.282 m

The displacements and rotations are large. However, the strains are small – they can be estimated using strength of materials formulas:

$$\varepsilon_{base} = \frac{M(h/2)}{E\,I} \quad \text{where } M \doteq (5\text{ MN})(7.5\text{ m})$$

$$\doteq 3\%$$

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