PROFESSOR: Ladies and gentlemen, welcome to this lecture on nonlinear finite element analysis of solids and structures. In this lecture, I would like to continue with our discussion of the use of constitutive relations used in nonlinear analysis. In the previous lecture, we already considered the modeling of elastic and hyper elastic materials, subjected to large displacements, large rotations, and large strains.

I'd like to not turn our attention to the use of the updated Lagrangian formulation. Of course, the kinematic relations for the total and updated Lagrangian formulations, we discussed in an earlier lecture already. In particular, I would like now, in very general terms, to ask the following question-- is it possible to obtain, using the updated Lagrangian formulation, identically the same numerical results for each load step and each iteration, as are obtained when we use a total Lagrangian formulation?

Let's look at this question here, now, in quite gentle terms. And I have rephrased it here once more, so as to really make very clear what I mean. Let's say that we have a Program 1 that uses only the total Lagrangian formulation. And the constitutive relations that we are employing in that total Lagrangian formulation are defined as shown here. So total stress at time, t, of course, second Piola-Kirchhoff stress, is given as a function of displacements. And the tangent material relationship, relating the increment in the second Piola-Kirchhoff stress and the increment in the Green-Lagrange strain is as shown here.

The program results for a particular physical problem might look systematically as shown here. Of course, these two relationships here have been obtained from a set of physical laboratory experiments. This is how Program 1 works.

Now we have also, say, a Program 2. And this Program 2 only uses the updated Lagrangian formulation. Now, remember our laboratory physical experimental results have given us the considered affiliations for the towards Lagrangian formulation. However, we now have a program that only is operating on the updated Lagrangian formulation.

The constitutive relations are, in this case, well, like that and like that, where I have given here three dots, because we want to now study what they should be looking like in order to obtain the same results as we are attaining with Program 1. In other words, how can we obtain, with this Program 2, identically the same results as are obtained using Program 1. This is a question that we really are asking.

To answer this question, we have to go back to some derivations that we discussed in an earlier lecture. And there, we discussed that this is here the governing continuum mechanics equation, corresponding to the total Lagrangian formulation. Here we have the incremental stress strain law going into the formulation. Notice this is a tangent material tensor. Here we have the second Piola-Kirchhoff stress, and here, as well. And of course, here on the right hand side, we have to total external loads entering the total external virtual work.

In the updated Lagrangian formulation, this is the equation that we derived earlier. Here, again, is a total external virtual work. And here, a constitutive relation enters, corresponding to this formulation. And here we have the Cauchy stress that is being used in the updated Lagrangian formulation.

The terms used in the formulations are summarized on this view graph. In the TL formulation, of course, we're integrating over the original volume. In the updated Lagrangian formulation, we're integrating over the current volume, time t. And there is this transformation, where the mass density's at time t, and the original mass density, enters. Notice here are listed the linear and the nonlinear increment in the Green-Lagrange strain. Here, there also is a linear and a nonlinear increment in the Green-Lagrange, but for the updated Lagrangian formulation, and here, for the total Lagrangian formulation.

There's a relationship between these increments, as shown here. And this relationship I will focus our attention upon further on the next view graph, because I'd like you to clearly understand how we obtain this relationship. But we notice is that the deformation gradient enters in this equation, as well as in this equation. The variations on these linear strain increments and nonlinear strain increments of the total Lagrangian formulation and the updated Lagrangian formulation are also related via the deformation gradient. In fact, twice the deformation gradient enters here.

Let's look at how we derive this kinematic relationships. Well, we use that there is a fundamental property of Green-Lagrange strain as expressed in this equation. If you know a Green-Lagrange strain at a point, then we can pick any material fiber with these components. And we know, if these are the origin of components of the material fiber, at that point, that after motion, in other words, in configuration t, corresponding to time t, that material fiber has this length, where the original length is given here.

This is a very basic equation that we want to study a bit more on the next view graph. But let me already point out that we can, directly, use this basic relation, and, of course, apply it also corresponding to time t, plus delta t. Notice here, t, here, t plus delta t. Here t, here, t plus delta t. Otherwise, same relationship.

And we also can apply incrementally. Notice this a Green-Lagrange strain increment, from time t, to time t plus delta t. But refer to time t. That's why we have the little t down here. Up here, we always referred the increments in strains, or the total strains, if it is a total strain, to the configuration at time zero. Now we are referring the quantity to time t. And, if we simply apply this same basic relationship here, we directly obtain this equation here for the increment in Green-Lagrange strain from time, t, to time, t plus delta t. Refer to time t configuration, this relationship holds.

Let's look at this relationship here. What does it really mean physically? Well, it means the following-- here we have our stationary coordinate frame, x1, x2, x3. And at time zero, we have a material fiber here, any material fiber. But let's just pick one. And that has a length of 0 ds. The components of this vector, the vector that is shown here, of length 0 ds, the components are shown here. And that vector we call a d 0 x. This material fiber moves from time 0 to time t, to this configuration. The new components are listed here-- dtx2, dtx1, dtx3. Of course, there's a relationship between the new components and the original components of the material fiber. And here, goes into the deformation gradient.

Now the equation that I just referred to really relates this length and that length. And the way we obtain it is to look at the definition of the Green-Lagrange strain. strain The Green-Lagrange strain is defined as follows-- I'm going to simply write it in here. You see here, we have-- oops. I should have not to use that delta t, because I'm looking at time t. This is the equation for the Green-Lagrange strain, from here to there.

And what I'm now going to do is to premultiply this relationship by d0x, transposed, and postmultiplied by d0x. Of course, I have to do the same thing here. d0x, transposed, goes in front of the bracket. And d0x goes into the back of the bracket.

Let me make clear what I'm doing here. I'm putting this in here. And I'm putting that on the back. Notice I'm using a transposed here and a transposed there.

The basic relation of the Green-Lagrange strain, once again, is this being equal to that part with out the d0x transposed in front, and without the d0x on the back. But now we notice, directly, that this part here and this part here is nothing else then dtx. If you recognize that this is dtx, that is dtx, you immediately get on the right hand side, one half, bracket open, d-- well, I should say t ds, squared, minus 0ds squared. That's what you have on the right hand side.

And on the left hand side, you have what I showed you earlier. Let me go back one view graph, just to make sure that we understand each other. On the right hand side, you immediately obtain this term minus that term. And on the left hand side, you end up with this term. So this is the relationship, a very basic relationship, that we want to use in our derivation. And it is obtained by simply looking at the Green-Lagrange strain, the way it's defined, and the way we have defined it in an earlier lecture, and premultiplying by d0x transposed, and postmultiplying by d0x. Of course, what you're seeing there, in that relationship I showed you, you see the components of this term here, written out in terms of i and j components.

Well, so what we are doing, then, is to follow a material fiber, original length 0 ds, through it's motion to t ds and to t plus delta t ds. And that's expressed by the three formulas that are on the previous view graph. Hence, if we subtract from the second formula, the first formula, we directly obtain this relationship here.

And we now use this term here, or this equation, linking up dtx to d0x and expand these two terms to obtain this equation. Now, notice that this equation has to hold for any material fiber. And if it does hold for any material fiber, it immediately follows that this equation here has to hold it, because we can apply it to a material fiber that is oriented along the x1 axis, then another one that is oriented along the x2 axis, then another one that is oriented along the x3 axis, and so on. And surely, this relationship here must then hold.

If we look at this relationship more closely, we see that we can write it out as a linear strain increment and as a nonlinear strain increment on the left hand side and on the right hand side of the equation. And this shows, then, directly that the 0 eij must be equal to twice a deformation gradient times ters. The reason being that this term here is a function of the incremental displacement only. Linear functional incremental displacement, so is this. And if you simply equate those terms that are linear, we get that this part must be equal to that part. And if you create all the nonlinear parts, we find that 0ei, e to ij, is equal to twice the deformation gradient times te to rs. And those two equations are given down here.

If we now take variations on the left hand side and on the right hand side, and if you recognize that's a variation, is respect to the configuration at time t plus delta t, it is clear that the deformation gradient terms don't change. And, therefore taking the variation on the left hand side, that variation is simply applied to the ters, with these being constant terms. Similarly here, and this then, completes the derivation of all the kinematic relationships that I had in the earlier table and that we want to use now.

In addition, we have, also, the important relationship between the second Piola-

Kirchhoff stress and the Cauchy stress. And that relationship is shown here. Of course, we introduced that one in an earlier lecture.

Finally, we have a relationship between the tangent material tensor, corresponding to the total Lagrangian formulation and corresponding to the updated Lagrangian formulation. Notice zero here, t there. And that relationship is given here. It's given by a fourth order tensor transformation, which I'd like to now discuss with you further. This is an important relationship that we want to spend a bit off time on.

The relationship is derived as follows-- the increment in the second Piola-Kirchhoff stress is clearly given by the tangent material relation times increment in the Green-Lagrange strain. And we're looking at differential increments here. The increment in the second Piola-Kirchhoff stress, refer to time t, is given by the right hand side here, where this increment in Green-Lagrange strain is refer to time t.

We also have this relationship between the stress increments. This is the same relationship that holds between they total second Piola-Kirchhoff stress at time t and the Cauchy stress at time t. And we have this relationship, which we just derived.

Now, using these four equations, we directly obtain by substitution, that this left hand side, substituted for d0 Sij, must be equal to this right hand side, where we have substituted this relationship for d0 epsilon rs. And this equation, then, directly yields this equation, where the coefficient here must be what we are looking for, namely tCabpq. This is the tangent material tensor to be used in the updated Lagrangian formulation, referred to time t.

The tangent material relation is once more summarized here. And this is what we need to use in the updated Lagrangian formulation. Notice it's a fourth order tensor transformation on this value here, on this tensor here.

Now we can compare the updated and total Lagrangian formulations. And the question that we ask is under what condition is this left hand side, which we use in the total Lagrangian formulation equal to this term here, which we use in the updated Lagrangian formulation. Well, we simply substitute here for the stress, and for the strain, our earlier relationships. We multiply out, and immediately, we obtain the right hand side. Notice ij, of course, are dummy indices, which can be substituted by m and n.

It's interesting to upset here that this transformation is a kinematic transformation, which is actually buried in the proper use of the updated Lagrangian formulation. This transformation is a stress transformation that has to be enforced for any Cauchy stress that we would be calculating in the computer program.

Let us look at the second term. Here, we have, on the left hand side, the term of the total Lagrangian formulation. And we ask whether this is equal to the term that I showed you on the right hand side, corresponding to the updated Lagrangian formulation.

Well, if we substitute our definitions for the second Piola-Kirchhoff stress here, and what we know to be a true kinematic relations, here, and we multiply out, and also use, of course, this relationship here, we directly obtain what we know we have to obtain for the updated Lagrangian formulation. In other words, the left hand side is equal to the right hand side. Notice, again, the kinematic transformation here and the stress transformation here.

Finally, we also have to look at this term, the term in which the material tensor enters. We ask again if this left hand side equal to the right hand side using the proper transformations. And if we substitute, as shown here, we directly observe yes, if the term of the total Lagrangian formulation is equal to the term of the updated Lagrangian formulation, provided, of course, we use the proper transformations. And here, in particular, the constitutive transformation that we discussed just a bit earlier.

So we conclude then that the updated Lagrangian terms are identically equal to the total Lagrangian terms, provided we follow the transformation rules that I have summarized. This means that in the finite element analysis, if we use the same finite element interpolation functions in the updated Lagrangian formulation, as in the total Lagrangian formulation-- of course we have to use the same interpolation functions and the same finite element assumptions-- then, we can see directly that, for the total Lagrangian formulation with this equivalent equation and the updated Lagrangian formulation with this equivalent equation, that these matrices are identically equal. And that's a very important observation, a very important observation.

It means, then, going back to the question that I asked earlier, regarding the use of Program 1 and Program 2, it means that, to summarize, if we use Program 2, then Program 2 gives the same results as Program 1, provided the Cauchy stresses are calculated from this relationship. What does that mean? It means that the second Piola-Kirchhoff stresses of course are given by the relationships that were determined by physical laboratory results.

Remember, we know how old the second Piola-Kirchhoff stress is defined as a function of deformations, of the deformations in the material. That was our assumption because that went into the use of Program 1. And we are asking now only how can we use Program 2, which only contains the updated Lagrangian formulation.

So we know how St0Smn is defined. And if we know this quantity, we have to make this transformation to calculate the Cauchy stress in the Program 2. Similarly, we also remember, postulated that we know this tensor from the physical laboratory test results. We know this tensor. And what I'm saying now is that we have to transform this tensor, as shown here, to obtain this tensor. And it is this one that would go then into all our Program 2 computations.

If, in Program 2, we use this Cauchy stress, defined by the second Piola-Kirchhoff stress, as shown here. And this tangent material relation, you find by this tensor, which we know, then Program 2 will give identically the same results as Program 1. The kinematic relations that we talked earlier about are buried in the proper implementation of the updated Lagrangian formulation in Program 2.

Conversely, if we say that the material relationship for Program 2 is given, say, in other words, the Cauchy stress is given and also the tangent material relationship is given from laboratory test results. Then you can surely show that Program 1 with the total Lagrangian formulation will give identically the same results, provided these transformations are performed in the program.

Now, the Cauchy stress is given, the tangent material relationship is given, and what we have to do in the program, in every solution step, in every iteration, is to transform this Cauchy stress to the second Piola-Kirchhoff stress that's going to be used in Program 1. And we have to transform this tensor here, as shown here, to this tensor. And this tensor here, 0Cijs, is going to be used in Program 1. Therefore we can really conclude that's the choice of whether to use a total Lagrangian formulation or the updated Lagrangian formulation is based only merely on how effective computationally one formulation is over the other.

And then we note now that the B matrix, the strain displacement matrix in the U.L. formulation contains less entries than the strain displacement matrix in the T.L. formulation. This means, of course, that the product, B transpose CB, which is used in the stiffness matrix, is cheaper to evaluate, less expensive to evaluate, in the updated Lagrangian formulation. Across So there's a plus here for the updated Lagrangian formulation.

However, if the stress-strain law is available in terms of the second Piola-Kirchhoff stress, then it is most natural to use a total Lagrangian formulations. And this is the case when we analyze rubber type materials, for example we use the Mooney-Rivlin material law, which we discussed very briefly in the previous lecture, and, in particular, when we want to analyze inelastic response and we want to allow for large displacements and large rotations. but we have small strain conditions.

I'd like to refer you now back to the previous lecture, where we made a big point out of the fact that the second Piola-Kirchhoff stress, the Green-Lagrange strains, are the components of these two tensors are invariant under rigid body rotation, and that that means that we can directly use material relationships that are applicable to infinitesimal displacement and strains, in other words, the engineering stress and engineering strain variables.

You can use these material relations directly in a large displacement, large rotation analysis, provided we use the total Lagrangian formulation. And I explained that, I spent a bit of time on that, in the previous lecture. Please refer back to that information.

Let us now look at a very special case, namely, the case of elasticity. We discussed in the previous lecture already that one way to proceed is to use this relationship here in the total Lagrangian formulation, where we have the second Piola-Kirchhoff stress on the left hand side, the Green-Lagrange strain on the right hand side, multiplying the constitutive relation, corresponding to the total Lagrangian formulation.

Now if we use this equation in the relation that we just established, or that we talked about, namely giving us of a Cauchy stress, in terms of the second Piola-Kirchhoff stress, we obtain this equation here. And of course, we also have to use in the updated Lagrangian formulation, this transformation, as we just discussed. Now notice that when we compare these two right hand sides, we see the deformation gradient entering twice here, whereas four times here. We would like to recast, now, this equation on the Cauchy stress, so as to have the deformation gradient here entering in the same form as we have it entering here. And that is achieved by the definition of the Almansi strain tensor.

Here, now, we have the Cauchy stress in terms of the Almansi strain tensor, multiplying a new constitutive tensor. And that new constitutive tensor is given down here. Notice the right hand side here contains the constitutive tensor of the total Lagrangian formulation, t0 here. And we are multiplying this tensor four times by the deformation gradient components. This is exactly what we try to achieve.

And we achieve it by using the Almansi strain tensor, which is defined as follows. Here we have the definition of the Almansi strain tensor, in terms of the Green-Lagrange strain tensor. And here, we have the inverse deformation gradient entering twice. The same definition is also given here, but in matrix form.

And if we substitute for the inverse deformation gradient, in terms of displacements, and multiply components out, we obtain directly this relationship here for the Almansi strain tensor, where, notice our notation is this t and that t are those two t's here. We are taking the displacement from time 0 to time t, and we differentiating those displacements with respect to the current coordinates. And that's important.

There's also a minus here, which is a bit of a surprise, because in the Green-Lagrange strain definition, of course, we have a plus here, but different terms, of course, different terms. Anyway, this is the definition of the Almansi strain tensor. And this tensor is very useful in the way I just described.

The Almansi strain tensor is a symmetric tensor. The ij components are equal to the ji components. The components are not invariant under rigid body rotation of the material. This is a property, an important property, of the Green-Lagrange strain tensor, but it is not a property of the Almansi strain tensor. The Almansi strain tensor is really, overall, not a very useful measure, strain measure. But we wanted to introduce it here briefly. And it is quite useful in the one particular case, namely, in the analysis of isotropic materials undergoing large displacements and large rotations, but generally small strains.

Let's look at one simple example here that gives us a bit of insight, what's the Almansi strain tensor looks, what the strain components look like. Here we have a simple four node element that is being pulled out into the red configuration. Notice the original length is 0L. The pullout is t delta. Of course, t delta could also be negative, in which case we are pushing in here. And the current length is tL. Notice the Green-Lagrange strain is given here, where we have a plus sign here and where we have 0L down here.

The Almansi strain 1, 1 component, of course, in this particular case, has this relationship here. Notice t [? sigma ?] over tL, versus 0L in the Green-Lagrange strain. And a minus sign here, whereas we have a plus sign there. If we plot these strain tensor components as a function of t delta over the original lengths of the element, we find this blue curve for the Green-Lagrange strain. That is very easy to see that this is the Green-Lagrange strain. Notice here we have 1/2 minus 1/2. That minus 1/2 comes from this relationship. This is minus one at this point. And that is plus 1/2. So we get a minus 1/2 here. And at this point here, when delta over L0 is one, we have 3/2, verified by that formula.

The engineering strain, of course, is just a straight line. The Almansi strain looks as shown here by the red curve. Notice that when delta is compressive, going into this direction in the picture, in other words, when we are compressing that element, of course, tL becomes smaller and smaller, meaning this variable down here becomes smaller and smaller, and our Almansi strain component very rapidly becomes very large negative.

It turns out that the use of the Almansi strain tensor, corresponding with this material relationship, t t Caijrs, which I defined on the earlier view graph, is quite effective when we want to analyze with the U.L. Formulation situations that involve a linear isotropic material and large displacements, large rotations, but small strains. In this case, we can use directly that t t Caijrs is given by the right hand side here. Notice here we have the Lame constants, lambda and mu, which we already used in the previous lecture, and the chronica deltas. The t 0 Cijrs tensor is given as here on the right hand side. Again, Lame constants and chronica delta entries.

We would use here the same Lame constants in both of these relationships. And we also use that the incremental, the tangent material relationship, is the same as the total stress-strain relationship in both cases. If we use reformulating, if we use this right hand side for that tensor and make this tensor equal to that tensor, and if we use the same right hand side as up there, down here, for the t 0 Cijrs and the 0Cijrs, same numbers, same Young's modulus, same Poisson ratio, then for large displacement, large rotation, but small strain analysis, we virtually obtain identically the same results.

I'd like to just demonstrate to you with a very simple problem analysis what I mean here. Let us turn to just one slide in which we show a solution response of an arch that was analyzed using the updated and total Lagrangian formulations. Here you now see the slide that shows the arch. It is subjected to a point load at it's apex. And we measure the displacement at the apex. The displacement is w0, half of the arch was modeled with 12 eight-node elements. And as you can see, the TL and UL solutions for this arch are practically, identically the same. We can't see any difference to the accuracy that we have plotted the response.

Notice that in these TL and UL solutions, we used exactly the same Young's modulus and Poisson ratio. The reason that practically the same response is calculated using the updated Lagrangian formulation and the total Lagrangian formulation lies in that the constitutive transformations that would have to be applied to obtain the exact response, exactly the same response with a two formations, really reduce here to mere rotations. And that is the fact because, for this type of problem, large displacement, large rotation, but small strain problem, the mass density remains constant and the deformation gradient, which can always be written as a product all of a rotation matrix, an orthogonal matrix, times another symmetric matrix.

This is, of course, here is a polar decomposition of the deformation gradient, that for this particular case, large displacement, large rotation, but small strains, this deformation gradient is simply almost equal to the rotation matrix only. In other words, the stretch matrix is almost equal to the identity matrix.

However, when we look at large strain problems, large strain analysis, and we were to use this relationship here for the Cauchy stress, in terms of the Almansi strain, and this relationship here, giving us a second Piola-Kirchhoff stress in terms of the Green-Lagrange strain, with these two constitutive tensors, given as shown here by the same Lame constants. In other words, Poisson ratio and Young's modulus, being identically the same, entering here.

More specifically, you would put in here a Young's modulus, say, of 30 million pounds per square inch and the Poisson ratio of 0.3. These are the exact numbers that would go into here, with these, of course, being the chronica deltas. If we were to use this relationship in the total Lagrangian formulation and that relationship in the updated Lagrangian formulation for large strains, then we would get very different results.

Let us look at the simple problem that we already solved earlier ones, or that we considered earlier ones in the previous lecture, using the total Lagrangian formulation. I'd like to now consider the same problem using the updated Lagrangian formulation with this material relationship. Cauchy stress given in terms of the Almansi strain, and this E curl, which we used already before in the last lecture in this total Lagrangian formulation. Notice this E curl here is given on the right hand side here in terms of Young's modulus and Poisson's ratio.

The problem, very briefly reviewed, is that we are looking at a bar with constant cross sectional area A bar. And we will pull this bar out and also compress it in and look at the force required to pull the bar out or compress it. And we want to now look at this problem using this material relationship here, Cauchy stress in terms of Almansi strain with E curl constant.

Here we have, for this one dimensional problem, the Almansi strain, given as shown up here. Notice this differentiation here of tU1 with respect to the coordinate 1 at time t. It's nothing else than tL minus 0L over tL. Here we have that same quantity squared. Of course, there's a minus 1/2 in front. And the result is this. The Cauchy stress is directly obtained in terms of the physical force applied to the bar divided by the cross sectional area, which is constant.

If we use that current length is given in terms of the original length plus the extension and this material relationship here, we obtain directly this force displacement response. Force plotted vertically up, displacement plotted along here, t sigma is the displacement. And notice that the force displacement response is highly nonlinear-- here is, by the way, the actual expression-- highly nonlinear, and looks quite different from what we calculated earlier using the total Lagrangian formulation with the same E curl, same E curl used earlier in the total Lagrangian formulation, in which case the response really looked something like this. So we get a totally different description of the material response using these two formulation when the material is subjected to large strains.

Let us now also look at one example that demonstrates this feature, the features that I just discussed a bit more. Here we have a frame subjected to a tip load over at this end. A is the tip load. The frame has a thickness h, L length here, L length there, thickness h here, as well. The width of the frame structure is b. And the geometric data are given here. And the material data are given here. Notice that h/L is 1/50.

We modeled this structural using 51 two-dimensional eight node elements, plane strain elements. Notice here is a typical element shown, an eight node element. There are 25 elements along here. There's one element there for the corner and another 25 elements in the column.

Now we want to ask the question what happens in the elastic analysis when we subject this structure, in other words, to the load r and we used once a TL and once a UL formulation, but with the same material constants. In other words, the material constants E and nu that I gave on the earlier view graph are the same for the TL solution and the UL solution. And we do not make any transformation, therefore, on the material relationships.

We simply use the total Lagrangian formulation with E and nu given, these two constants plugged in. And these constants are the same throughout the TL solution. We perform this way the TL solution. And then, afterwards, we do the UL solution, again, simply putting E and nu into the analysis and keeping it constant throughout that analysis. For large displacement, large rotation, but small strain conditions, the TL and UL formulation will give similar results, similar because we don't make these transformations that I discussed earlier in the lecture in order to obtain exactly the same results.

But for large displacement, large rotation, small strain conditions, similar results are obtained. Of course, we have shown that already, by the solution that I showed you on the slide, the analysis of the arch subjected to an apex load. But we will see that now again here in this example. For large displacement, large rotation, and large strain conditions, the TL and UL formulations will gave quite different results.

Well, let's look now at the results that we obtained for this problem. Here we have plotted the force vertically, meganewtons, and the vertical displacement of the tip in terms of meter. The 2-D elements, TL and UL formulations, give this result here, the black curve, solid curve. And we also wanted to solve this problem once using beam elements using the total Lagrangian formulation. In fact, we used four node isoparametric beam elements. We have not discussed these yet. We will discuss the formulation of these elements in a later lecture.

And you can see that the beam elements give a slightly different response solution. Of course, the assumptions, the kinematic assumptions, in the beam elements are different. And that explains the difference in the response calculated. But the TL and UL formulations, using the eight node isoparametric plane, the strain elements, we obtain basically the same response.

The formations are very large for this frame. Notice here we have the undeformed frame. There's the load. We are pressing down here. These are the deformations at a load of one meganewton. And this is how the frame looks at full load, five meganewton-- very large deformations. In fact, the deformation are so large for the frame, that in an actual practical problem, probably the frame, of course, would have undergone inelastic deformations. However, we look at this problem as a numerical experiment just to demonstrate what is happening and what we have been discussing in the lecture.

If you look at the maximum deformations once more, a little bit closer, we find that the vertical tip displacement in the TL formulation for this problem is given as 15.289 meters, and in the UL formulation, as 15.282 meters. Notice that there is only a change in the fifth digit. The displacements and rotations have certainly been very large in the analysis of this frame, but the strains are still quite small. If you look at the strain at the base of the frame, maybe we have the maximum strain, maximal moments there, we find that the moment is approximately as shown here. And the strain, using strength of material formulas, is approximately given by 3%.

Now up to 2%, you certainly would consider it a small strain problem. So here we are just at the limit of going over into a lot strain region near the base of the column. This is all I wanted to present to you, discuss with you, in this lecture. Thank you very much for your attention.