For Section 8.3:

Prob. 8.3.1 A pair of electrodes is constructed from thin sheets separated by a thin sheet of insulator. This dielectric "sandwich" is dipped into an insulating liquid having the polarization constitutive law

$$\vec{D} = \frac{\vec{E}}{\alpha_1\sqrt{\alpha_2^2 + E^2}} + \varepsilon_o\vec{E}$$

where α_1 and α_2 are constant parameters. The objective here is to describe the rise of the dielectric liquid around the outside edges of the electrodes, where there is a strong surrounding fringing field. Assume that the applied voltage is alternating at a sufficiently high frequency so that free charge effects are absent and effects of the time-varying part of the electric stress are "ironed out" by the fluid viscosity and inertia.

(a) Determine the electric field in the neighborhood of one of the edges under the assumption that the dielectric rises in an axisymmetric fashion ($\xi = \xi(r)$, with r as defined in Fig. P8.3.1). The right and left edges of the electrodes (see the side view in the figure) are sufficiently far apart so that they can be considered not to influence each other.

(b) Find $\xi(r)$.

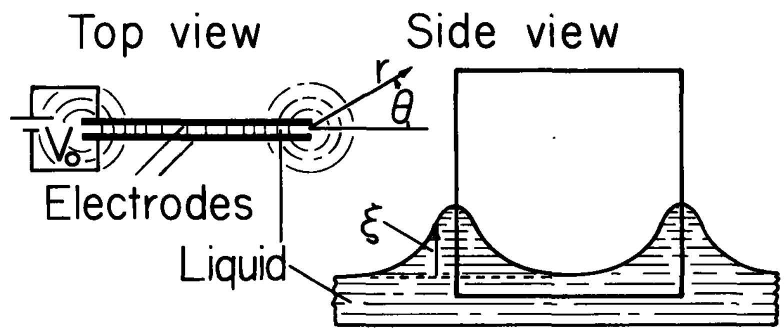

Fig. P8.3.1

Prob. 8.3.2 An insulating liquid is represented by the constitutive law

$$|\vec{D}| = \varepsilon_o|E| + \alpha_1 \tanh \alpha_2|E|$$

where $\vec{D}$ and $\vec{E}$ are collinear and α_1 and α_2 are properties of the fluid. The liquid is placed in a dish as shown in Fig. P8.3.2. Shaped electrodes are dipped into the liquid and held at a potential difference V_o. The variable spacing s(z) between the electrodes is small compared to the electrode dimensions in the x and z directions, so the electric field can be taken as essentially in the y direction. With the application of the field, the liquid reaches a static equilibrium of profile $\xi(z)$. Find an expression for $\xi(z) - \xi_o$.

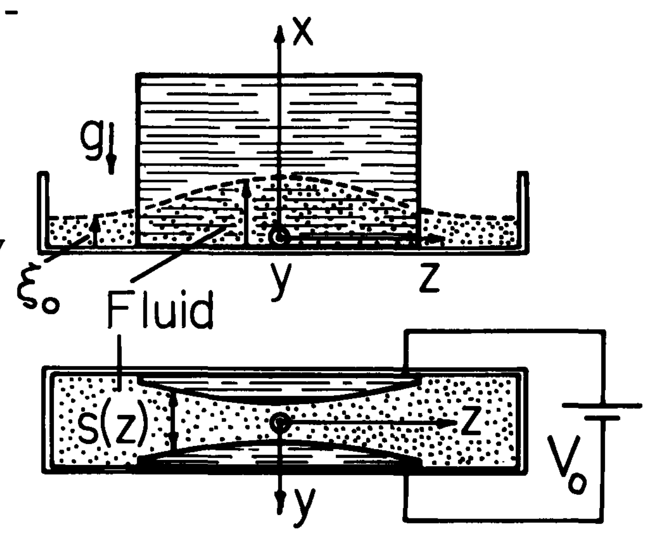

Fig. P8.3.2

For Section 8.4:

Prob. 8.4.1 The configuration of Fig. 8.4.4 is altered by replacing the magnet with a periodic distribution of magnets. These constrain the normal magnetic flux density in the plane x = d to be B_o cos ky. As in the example treated, ignore effects of the self fields and of surface tension. Assume that $\xi = \xi_1$ at y = 0.

(a) Show that an implicit expression for $\xi(y)$ is

$$k(\xi-\xi_o)\, e^{-k(\xi-\xi_o)} = e^{k(d-\xi_o)} \frac{J_oB_o}{g(\rho_b-\rho_a)} \sin ky$$

(b) Make sketches of the left side of this expression (as a function of (ξ=ξ_o) and the right side of the expression (as a function of ky) and describe in graphical terms how you would find ($\xi-\xi_o$) as a function of y. What is the significance of there being two solutions for $\xi-\xi_o$ or none at all? For what value of J_oB_o would you expect the static equilibrium to be unstable?

Prob. 8.4.2 In the configuration of Fig. 8.4.1, the lower fluid is a perfectly conducting liquid while the upper one is an insulating gas ($\rho_a << \rho_b$). Surface deformations have a very long characteristic length in the y direction compared to $d-\xi$, so that the electric field normal to the interface in

Prob. 8.4.2 (continued)

the gas can be approximated as the voltage divided by the spacing $d-\xi$.

(a) Show that for a given V(y) static deformations of the interface are described by

$$\gamma \frac{d}{dy}\left\{[1+(\frac{d\xi}{dy})^2]^{-\frac{1}{2}}\frac{d\xi}{dy}\right\} + \frac{1}{2}\varepsilon_o \frac{V^2(y)}{(d-\xi)^2} - \rho g(\xi-b) = 0$$

where $\xi = b$ at a location where V=0.

(b) Now consider the application of this equation to the special case shown in Fig. P8.4.2. The plane horizontal electrode is of uniform potential V. An infinite pool of liquid to the left communicates liquid to the region below the electrode. In the fringing region, the interface is covered by a flat electrode. At y=0 the sharp edge of the electrode constrains the interface to have depth ξ=b. The field elevates the interface to the height ξ_o as $y\to\infty$. For small amplitudes ξ-b, determine $\xi(y)$.

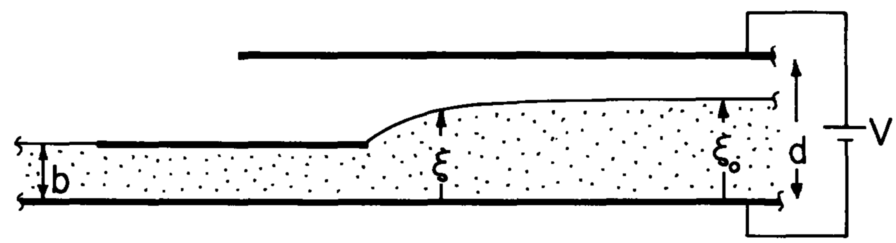

Fig. P8.4.2

(c) Show that for arbitrary deformations, the interfacial position is given implicitly by the integral

$$y = \int_b^{\xi} \frac{d\xi}{\sqrt{[1+P(\xi_o)-P(\xi)]^2-1}} \quad ; \quad P(\xi) \equiv \frac{1}{2}\frac{\varepsilon_o V^2}{\gamma(d-\xi)} - \rho\frac{g}{\gamma}(\xi-b)^2$$

For Section 8.6:

Prob. 8.6.1 In Prob. 7.9.2, the transfer relations are found for an annular region of fluid that is perturbed from an equilibrium in which it suffers a rigid-body rotation of angular velocity Ω about the z axis. Based upon those results, consider now the dynamics of fluid completely filling a container having radius R (there is no inner cylindrical region).

(a) Find the eigenfrequencies of the temporal modes having wavenumber k but m = 0.

(b) Rigid walls cap the cylinder at z = 0 and z = ℓ. What are the natural frequencies of the temporal modes m = 0 for this enclosed system?

For Section 8.7:

Prob. 8.7.1 Show that in the limit where times of interest are long compared to the relaxation time ε/σ, Eq. 8.7.6 reduces to the linearized form of $D\Phi/Dt = 0$.

Prob. 8.7.2 A magnetoquasistatic continuum conserves the free current linking any surface of fixed identity

$$\frac{d}{dt}\int_s \vec{J}_f \cdot \vec{n}da = 0$$

Show that the appropriate equations for an incompressible fluid are

$$\nabla \cdot \vec{v} = 0$$

$$\frac{\partial \vec{J}_f}{\partial t} - \nabla\times(\vec{v} \times \vec{J}_f) = 0$$

Prob. 8.7.2 (continued)

$$\rho\frac{D\vec{v}}{Dt} + \nabla p = \vec{J}_f \times \vec{B} + \eta\nabla^2\vec{v}$$

$$\nabla \times \vec{H} = \vec{J}_f \quad ; \quad \nabla\cdot\vec{J}_f = 0$$

where Faraday's law is used only if the electric field is required.

Prob. 8.7.3 As a particular example of the current-conserving continua from Prob. 8.7.2, the configuration shown in Fig. P8.7.3 consists of a layer of fluid having essentially zero conductivity in the y and z directions compared to that in the x direction. The walls are composed of segments, each constrained to constant current. Thus, in static equilibrium, there is a uniform current density $J_o\vec{i}_x$ throughout and an imposed magnetic field $B_o\vec{i}_x$. Assume that the magnetic field induced by $\vec{J}_f$ is negligible compared to B_o. As the fluid moves, the current through any given open surface of fixed identity remains constant.

The fluid has the electrical nature of conducting "wires" insulated from each other and stretched in the x direction. The "wires" deform with the fluid, and might actually consist of conducting fluid columns in an insulating fluid having the same mechanical properties.[1]

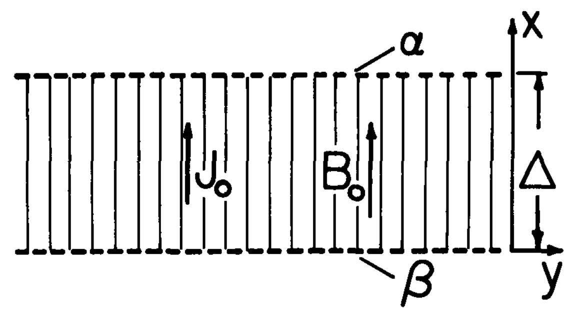

Fig. P8.7.3

(a) Assume that motions and field depend only on (x,t) and show that the equations formed in Prob. 8.7.2 are satisfied by solutions of the form

$$\vec{v} = v_y(x,t)\vec{i}_y + v_z(x,t)\vec{i}_z \text{ and } \vec{J} = J_o\vec{i}_x + J_y(x,t)\vec{i}_y + J_z(x,t)\vec{i}_z$$

where

$$\frac{\partial J_y}{\partial t} - J_o\frac{\partial v_y}{\partial x} = 0$$

$$\frac{\partial J_z}{\partial t} - J_o\frac{\partial v_z}{\partial x} = 0$$

$$\rho\frac{\partial v_y}{\partial t} = B_oJ_z + \eta\frac{\partial^2 v_y}{\partial x^2}$$

$$\rho\frac{\partial v_z}{\partial t} = -B_oJ_y + \eta\frac{\partial^2 v_z}{\partial x^2}$$

(b) Describe how you would establish transfer relations for the layer, given that the surface variables are the velocities and the shear stresses. Show that in the limit where there is no electromechanical coupling, $B_o = 0$, there is no coupling between the y directed motions and the z directed motions.

(c) As a specific example, rigid boundaries are imposed at x = 0 and x = ℓ. Find the eigenfrequencies of the resulting temporal modes.

Prob. 8.7.4 A spherical particle is impact-charged to saturation so that its mobility is given by Eq. (a) of Table 5.2.1. It is pulled through a fluid by the same electric field used to achieve this saturation charging. Show that the electroviscous time based on this field and the fluid viscosity is the time required for the particle to move a distance equal to its own diameter.

---

1. For discussion of the related dynamics of a current conserving "string" in a similar configuration, see H. H. Woodson and J. R. Melcher, Electromechanical Dynamics, Part II, John Wiley & Sons, New York, 1968, p. 627.

For Section 8.10:

Prob. 8.10.1 A planar layer of insulating liquid having a mass density ρ_s has the equilibrium thickness d. The layer separates infinite half-spaces of perfectly conducting liquid, each half-space having the same mass density ρ. The interfaces between insulating and conducting liquids each have a surface tension γ, but ρ_s is sufficiently close to ρ so that gravity effects can be ignored. Voltage applied between the conducting fluids results in an electric field in the insulating layer. In static equilibrium, this field is E_o. Determine the dispersion equations for kinking and sausage modes on the interfaces. Show that in the long-wave limit kd << 1, the effect of the field on the kinking motions is described by a voltage-dependent surface tension. In this long-wave limit, what is the condition for incipient instability?

For Section 8.11:

Prob. 8.11.1 A vertical wire carries a current I so that there is a surrounding magnetic field

$$\vec{H} = \vec{i}_\theta H_o(R/r), \quad H_o \equiv I/2\pi R$$

(a) In the absence of gravity, a static equilibrium exists in which a ferrofluid having permeability μ forms a column of radius R coaxial with the wire. (The equilibrium shown in Fig. 8.3.2b approaches this circular cylindrical geometry.) Show that conditions for a static equilibrium are satisfied.

(b) Assume that the wire is so thin that its presence has a negligible effect on the fluid mechanics and on the magnetic field. The ferrofluid has a surface tension γ and a mass density much greater than that of the surrounding medium. Find the dispersion equation for perturbations from this equilibrium.

(c) Show that the equilibrium is stable provided the magnetic field is large enough to prevent capillary instability. How large must H_o be made for the equilibrium to be stable?

(d) To generate a significant magnetic field using an isolated wire requires a substantial current. A configuration that makes it easy to demonstrate the electromechanics takes advantage of the magnet from a conventional loudspeaker. A cross section of such a magnet is shown in Fig. P8.11.1. In the region above the magnet, the fringing field has the form H_oR/r. Ferrofluid placed over the gap will form an equilibrium figure that is roughly hemispherical with radius R. Viewed from the top, each half-cylindrical segment of the hemisphere closes on itself with a total length ℓ. For present purposes, the curvature introduced by this closure is ignored so that the axial distance is approximated by z with the understanding that z = 0 and z = ℓ are the same position. Effects of surface tension and gravity are ignored. Argue that the m = 0 mode represented by the dispersion equation from (b) is mechanically and magnetically consistent with this revised configuration.

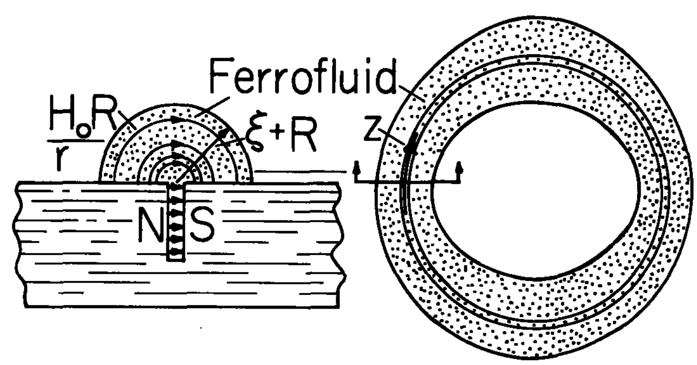

Fig. P8.11.1

(e) Show that, in the long-wave limit kR << 1, the m = 0 waves that propagate in the z direction (around the closed loop of ferrofluid) do so without dispersion. What is the dispersion equation?

(f) One way to observe these waves exploits the fact that the fluid is closed in the z direction, and therefore displays resonances. Again using the long-wave approximations, what are the resonant frequencies? How would you excite these modes?

For Section 8.12:

Prob. 8.12.1 The planar analog of the axial pinch is the sheet pinch shown in Fig. P8.12.1. A layer of perfectly conducting fluid (which models a plasma as an incompressible inviscid fluid), is in equilibrium with planar interfaces at x = $\pm$ d/2. At distances a to the left and right of the interfaces are perfectly conducting electrodes that provide a return path for surface currents which pass vertically through the fluid interfaces. The equilibrium magnetic field intensity to right and left is H_o, directed as shown. Regions a and b are occupied by fluids having negligible density.

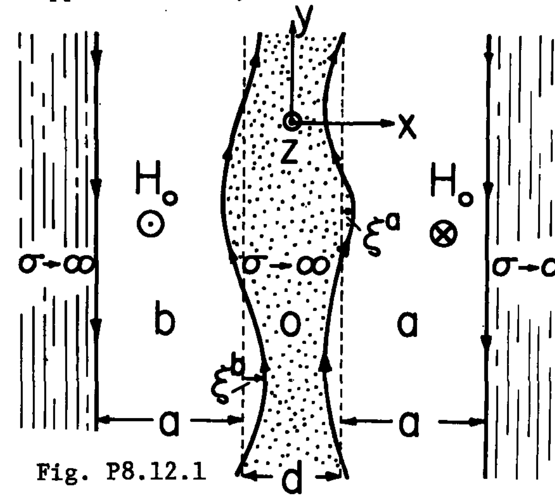

Fig. P8.12.1

Prob. 8.12.1 (continued)

(a) Determine the equilibrium difference in pressure between the regions a and b and the fluid o.

(b) Show that deflections of the interfaces can be divided into kink modes $[\xi^a(y,z,t) = \xi^b(y,z,t)]$, and sausage modes $[\xi^a(y,z,t) = -\xi^b(y,z,t)]$.

(c) Show that the dispersion equation for the kink modes is*, with $k \equiv \sqrt{k_y^2 + k_z^2}$,

$$\frac{\rho\omega^2}{k}\tanh\left(\frac{kd}{2}\right) = \mu_o H_o^2 \frac{k_z^2}{k}\coth(ka)$$

while the dispersion equation for the sausage modes is

$$\frac{\rho\omega^2}{k}\coth\left(\frac{kd}{2}\right) = \mu_o H_o^2 \frac{k_z^2}{k}\coth(ka)$$

(d) Is the equilibrium, as modeled, stable? The same conclusion should follow from both the analytical results and intuitive arguments.

Prob. 8.12.2 At equilibrium, a perfectly conducting fluid (plasma) occupies the annular region $R < r < a$ (Fig. P8.12.2.) It is bounded on the outside by a rigid wall at $r = a$ and on the inside by free space. Coaxial with the annulus is a "perfectly" conducting rod of radius b. Current passing in the z direction on this inner rod is returned on the plasma interface in the -z direction. Hence, so long as the interface is in equilibrium, the magnetic field in the free-space annulus $b < r < R$ is

$$\vec{H} = H_o \frac{R}{r}\vec{i}_\theta$$

(a) Define the pressure in the region occupied by the magnetic field as zero. What is the equilibrium pressure Π in the plasma?

(b) Find the dispersion equation for small-amplitude perturbations of the fluid interface. (Write the equation in terms of the functions $F(\alpha,\beta)$ and $G(\alpha,\beta)$.)

(c) Show that the equilibrium is stable.

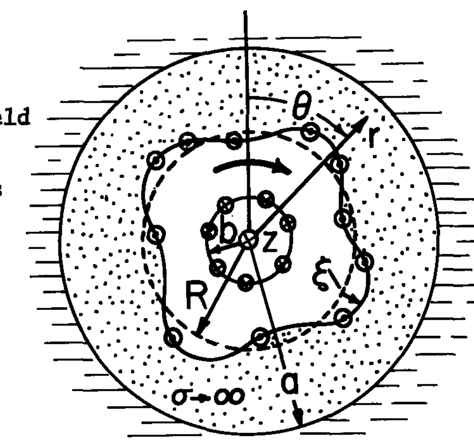

Fig. P8.12.2

Prob. 8.12.3 A "perfectly" conducting incompressible inviscid liquid layer rests on a rigid support at $x = -b$ and has a free surface at $x = \xi$. At a distance a above the equilibrium interface $\xi=0$ is a thin conducting sheet having surface conductivity σ_s. This sheet is backed by "infinitely" permeable material. The sheet and backing move in the y direction with the imposed velocity U. With the liquid in static equilibrium, there is a surface current $K_z = -H_o$ in the conducting sheet that is returned on the interface of the liquid. Thus, there is an equilibrium magnetic field intensity $\vec{H} = H_o\vec{i}_y$ in the gap between liquid and sheet. Include in the model gravity acting in the -x direction and surface tension. Determine the dispersion equation for temporal or spatial modes.

Prob. 8.12.4 In the pinch configuration of Fig. 8.12.1, the wall at r=a consists of a thin conducting shell of surface conductivity σ_s (as described in Sec. 6.3) surrounded by free space.

(a) Find the dispersion equation for the plasma column coupled to this lossy wall.

(b) Suppose that the frequencies of modes have been found under the assumption that the wall is perfectly conducting. Under what condition would these frequencies be valid for the wall of finite conductivity?

(c) Now suppose that the wall is very lossy. Show that the dispersion equation reduces to a quadratic expression in $(j\omega)$ and show that the wall tends to induce damping.

For Section 8.13:

Prob. 8.13.1 A cylindrical column of liquid, perhaps water, of equilibrium radius R, moves with uniform equilibrium velocity U in the z direction, as shown in Fig. P8.13.1. A coaxial cylindrical electrode is used to impose a radially symmetric electric field intensity

---

* $\coth kd - \frac{1}{\sinh kd} \equiv \tanh\left(\frac{kd}{2}\right)$; $\coth kd + \frac{1}{\sinh kd} \equiv \coth\left(\frac{kd}{2}\right)$

Prob. 8.13.1 (continued)

$$\vec{E} = E_o \frac{R}{r} \vec{i}_r$$

in the region between the electrode and liquid.

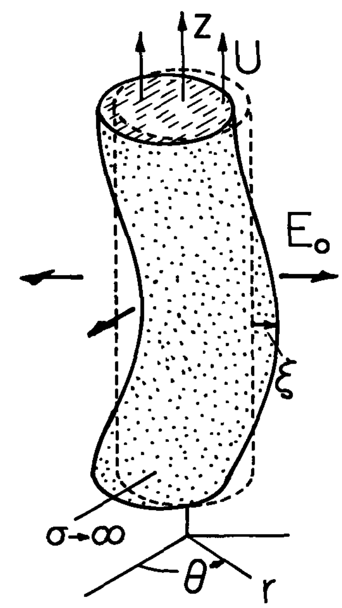

Fig. P8.13.1

Assume that the density of the liquid is large compared to that of the surrounding gas. Moreover, consider the liquid to have a relaxation time short compared to any other times of interest, and assume that the cylindrical electrode is well removed from the surface of the liquid.

(a) Determine the equilibrium pressure jump at the interface.

(b) Show that the dispersion equation is

$$(\omega - kU)^2 = \frac{\gamma}{\rho R^3}[-Rf_m(0,R)]\left\{m^2 - 1 + (kR)^2 + \frac{\varepsilon_o E^2 R}{\gamma}[1 - Rf(\infty,R)]\right\}$$

by using the transfer relations of Tables 2.16.2 and 7.9.1.

Prob. 8.13.2 A spherical drop of insulating liquid is of radius R and permittivity ε. At its center is a metallic, spherical particle of radius $b < R$ supporting the charge q. Hence, in equilibrium, the drop is stressed by a radial electric field.

(a) What is the equilibrium $\vec{E}$ in the drop ($b < r < R$) and in the surrounding gas, where the mass density is considered negligible and $\varepsilon \simeq \varepsilon_o$?

(b) Determine the dispersion equation for perturbations from the equilibrium.

(c) What is the maximum q consistent with stability for $b \ll R$?

For Section 8.14:

Prob. 8.14.1 For a conducting drop, such as water in air, the model of Sec. 8.13, where the drop is pictured as perfectly conducting, is appropriate. Here, the drop is pictured as perfectly insulating with charge distributed uniformly over its volume. The goal is to find the limit on the net drop charge consistent with stability; i.e., the analogue of Rayleigh's limit. This model is of historical interest because it was used as a starting point in the formulation of the liquid drop model of the nucleus.[2] In fact, the term in that model from nuclear physics that accounts for fission is motivated by the effect of a uniform charge density. Assume that the drop is uniformly charged, has a net charge Q but has permittivity equal to that of free space. Find the maximum charge consistent with stability.

Prob. 8.14.2 Consider the same configuration as developed in this section with the following generalization. The fluids in the upper and lower regions have permittivities ε_a and ε_b respectively.

(a) Write the equilibrium and perturbation bulk and boundary conditions.

(b) Find the dispersion equation and discuss the implications of the terms.

For Section 8.15:

Prob. 8.15.1 This problem is similar to that treated in the section. However, the magnetic field is imposed and the motions are two-dimensional, so that it is possible to represent the magnetic force density as the gradient of a scalar. This makes the analysis much simpler. A column of liquid-metal carries the uniform current density J_o in the z direction but suffers deformations that are independent of z. A wire at the center of the column also carries a net current I along the z axis. The field associated with this current is presumed much greater than that due to J_o. Thus, self fields due to J_o are ignored. Assume that the wire provides a negligible mechanical constraint on the motion and that the mass density of the gas surrounding the column is much less than that of the column.

(a) Show that the magnetic force density is of the form $-\nabla\mathcal{E}$, where

$$\mathcal{E} = -\frac{\mu_o I}{2\pi} \ln\left(\frac{r}{R}\right)$$

---

2. I. Kaplan, Nuclear Physics, Addison-Wesley Publishing Company, Reading, Mass., 1955, p. 425.

(b) The column has an equilibrium radius R and surface tension γ. Find the dispersion equation for perturbations $\xi = \xi(\theta,t)$.

(c) Show that the column is unstable in the m = 1 mode if $J_oI < 0$, and is stable in all modes if $J_oI > 0$. Use physical arguments to explain this result.

For Section 8.16:

Prob. 8.16.1 The fluid of Fig. 8.16.1 is perfectly conducting rather than perfectly insulating. Show that the dispersion equation is

$$j\omega\eta \frac{[k(\gamma_v-k)^2 - \gamma_v(\gamma_v+k)^2]}{k(\gamma_v+k)} = \rho g + \gamma k^2 - \varepsilon_o k E_o^2$$

Show that in the limit of low viscosity the dispersion equation is Eq. 8.16.15, and that in the opposite extreme, where $\gamma_v \approx k + j\omega\rho/2\eta k$, the dispersion equation is

$$\frac{3}{2}\frac{\omega^2\rho}{k} = 2j\omega\eta k + \rho g + \gamma k^2 - \varepsilon_o k E_o^2$$

Discuss effects of viscosity on incipience and rates of growth of instability in these two limits.

Prob. 8.16.2 The magnetohydrodynamic counterpart of the interaction studied in this section might be taken as that shown in Fig. P8.16.2. The interface between a perfectly insulating liquid in the lower half space and the air above is covered by a layer of perfectly conducting liquid. In static equilibrium, a uniform magnetic field H_o is imposed in the x direction. Instead of space-charge electroviscous oscillations caused by conservation of charge and stress equilibrium, there are now magnetoviscous oscillations within the plane of the interface caused by conservation of flux for any loop of fixed identity in the conducting layer. Assume that the layer has the same mechanical properties as the fluid below.

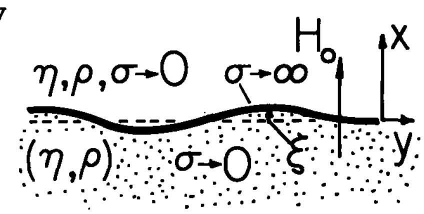

Fig. P8.16.2

Show that the thin perfectly conducting layer can be represented by the boundary condition

$$\frac{\partial H_x}{\partial t} = -H_o \frac{\partial v_y}{\partial y} \quad \text{at } x = \xi$$

Determine the dispersion equation for perturbations of the interface. Show that in the low-viscosity limit there are shearing modes of oscillation similar to those described by Eq. 8.16.16, except that

$$\omega_o = \left[\frac{2\mu_o H_o^2 k}{\sqrt{\eta\rho}}\right]^{2/3}$$

and that there are transverse modes of oscillation. Discuss the effect of viscosity on the latter in the limit where the transverse modes have a frequency that is high and that is low compared to ω_o.

Prob. 8.16.3 In the configuration of Fig. 8.16.1, the liquid layer has equilibrium thickness b, and uniform viscosity η, mass density ρ, permittivity ε and electrical conductivity σ. The upper electrode, at a distance a from the interface, has a potential -V relative to the rigid electrode at x = -b. Because the region between electrode and interface is highly insulating relative to the liquid, the equilibrium electric field is $V/a = E_o$ between the interface and the electrode and zero in the liquid layer. Effects due to the depth b and of the width a of the air gap are to be included.

(a) Write the perturbation boundary conditions and bulk conditions in terms of complex amplitudes.

(b) Show that the normalized dispersion equation is

$$M_{11}M_{22} - M_{12}M_{21} = 0$$

where in terms of normalized variables

$$M_{11} = -P_{11}j\omega - \rho - k^2 + \frac{kURS(j\omega r+1)}{j\omega rC + R}$$

Prob. 8.16.3 (continued)

$$M_{12} = -P_{13} + j\frac{\varepsilon_o}{\varepsilon}\frac{rUkS}{j\omega r C + R}$$

$$M_{21} = -P_{31}j\omega - jUk + j\frac{kU(j\omega r+1)R}{j\omega rC + R}$$

$$M_{22} = -P_{33} - \frac{kU\frac{\varepsilon_o}{\varepsilon}r}{j\omega rC + R}$$

The normalizations are

$\underline{\omega} = \omega b\eta/\gamma$, $\underline{\rho} = \rho g b^2/\gamma$, $\underline{k} = kb$, $\underline{a} = a/b$, $U = b\varepsilon_o E_o^2/\gamma$, $r = (\gamma/b\eta)(\varepsilon/\sigma)$, $\underline{P}_{ij} = bP_{ij}$ (defined by Eq. 7.19.13 or 7.33.6), $C = (\varepsilon_o/\varepsilon)\coth \underline{ka} + \coth \underline{k}$, $S = \coth \underline{ka}$

(c) Interpret the characteristic time used to normalize ω and form the dimensionless numbers $\underline{\rho}$, r and U.

(d) In the limit of complete viscous diffusion ($\omega\rho b^2/\eta \ll 1$) and instantaneous charge relaxation ($\omega\varepsilon/\sigma \ll 1$), show that this expression reduces to simply

$$j\omega = (kUS-\rho-k^2)P_{33}/(P_{11}P_{33}+P_{13}^2)$$

(e) Again, viscous diffusion is complete but the liquid is sufficiently insulating that charge relaxation is negligible (r>>1). Show that the dispersion equation becomes

$$a(j\omega)^2 + b(j\omega) + c = 0$$

where

$$a \equiv P_{11}P_{33}+P_{13}^2 \; ; \; b = [(\rho+k^2)P_{33}+Uk(\frac{P_{11}}{C}\frac{\varepsilon_o}{\varepsilon} - \frac{RS}{C}P_{33})-2j\frac{kUP_{13}}{C}\frac{S\varepsilon_o}{\varepsilon}] \; ; \; c = \frac{kU}{C}\frac{\varepsilon_o}{\varepsilon}(\rho+k^2-UkS)$$

Prob. 8.16.4 In the configuration of Fig. 8.16.1, the liquid is replaced by a perfectly elastic incompressible solid that can be regarded as perfectly conducting (perhaps Jello). The interface, like that in the case of the viscous fluid, must be described by a balance of both normal and shear stresses. Directly applicable transfer relations are deduced in Prob. 7.19, and in the limit $\omega \to 0$ in Prob. 7.20. The solid layer, which has a thickness b, is rigidly attached to the lower solid plate. The mass density and viscosity of the gas make negligible contributions to the dynamics.

(a) Determine the dispersion equation for deformations of the solid.

(b) Under the assumption that the principle of exchange of stabilities holds (that instability is incipient with ω=0) and that perturbation wavelengths are very short compared to b, determine the voltage threshold for instability.

For Section 8.18:

Prob. 8.18.1 An important connection between smoothly inhomogeneous systems and the piece-wise uniform ones considered in Sec. 8.14 is made by considering the temporal modes from another point of view. As shown in Fig. P8.18.1, the distribution of charge and mass density is approximated by two layers, each uniform in its properties.

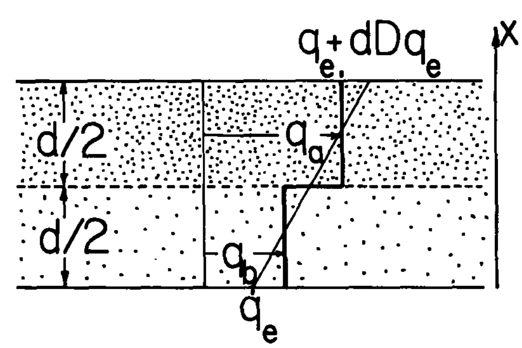

Fig. P8.18.1

(a) Show that for layers of equal thickness,

$$q_a = q_e + \frac{3}{4}Dq_e d \; : \; q_b = q_e + \frac{1}{4}Dq_e d \; : \; E_o = \frac{V_o}{d} - \frac{d^2 Dq_e}{16\varepsilon_o}$$

Prob. 8.18.1 (continued)

where, consistent with the usage in Section 8.14, E_o is the equilibrium electric field evaluated at the interface between layers.

(b) Show that the dispersion equation for the layer model, based on the results of Section 8.14, takes the normalized form

$$\frac{\underline{\omega}^2\coth(\frac{\underline{k}}{2})}{\underline{k}}\left(2+\frac{D\rho_m}{\rho_m}\right)=\frac{1}{2}\left[\frac{\frac{V_o}{d}Dq_e-gD\rho_m}{\frac{|V_o|}{d}\,|Dq_e|}\right]+S\left(\frac{1}{8\underline{k}}-\frac{1}{16}\right)$$

(c) Using $\underline{k}=1$, $D\rho_m=0$, $V_o/|V_o|=1$, $Dq_e/|Dq_e|=1$ and $S=1$, compare the prediction of the first eigenfrequency to the first resonance frequency predicted in the weak-gradient approximation and to the "exact" result shown in Fig. 8.18.2a. Compare the analytical expression to that for the weak-gradient imposed field approximation in the long-wave limit. Should it be expected that the layer approximation would agree with numerical results for very short wavelengths?

(d) How should the model be refined to include the second mode in the prediction?

Prob. 8.18.2 A layer of magnetizable liquid is in static equilibrium, with mass density and permeability having vertical distributions $\rho_s(x)$ and $\mu_s(x)$ (Fig. P18.8.2). The equilibrium magnetic field $H_s(x)$ is assumed to also have a weak gradient in the x direction, even though such a field is not irrotational. (For example, this gradient represents fields in the cylindrical annulus between concentric pole faces, where the poles have radii large compared to the annulus depth ℓ. The gradient in H_s is a quasi-one-dimensional model for the circular geometry.) Assume that the fluid is perfectly insulating and inviscid.

(a) Show that the perturbation equations can be reduced to

$$D(\mu_s D\hat{h}_z)-k^2\mu_s\hat{h}_z-j\frac{k_z^2}{\omega}H_sD\mu_s\hat{v}_x=0$$

$$D(\rho_s D\hat{v}_x)-k^2\left(\rho_s-\frac{\mathcal{N}}{\omega^2}\right)\hat{v}_x+j\frac{k^2H_sD\mu_s}{\omega}\hat{h}_z=0$$

where $k^2=k_y^2+k_z^2$, $\vec{H}=H_s\vec{i}_z+\vec{h}$ and $\mathcal{N}=-g\,D\rho_s+\frac{1}{2}D\mu_s DH_s^2$

(b) As an example, assume that the profiles are $\rho_s=\rho_m\exp\beta x$, $\mu_s=\mu_m\exp\beta x$, $H_s=$ constant. Show that solutions are a linear combination of $\exp\gamma x$, where

$$\gamma=-\frac{\beta}{2}\pm c_\pm;\quad c_\pm=\left[\left(\frac{\beta}{2}\right)^2+k^2+a\pm b\right]^{1/2};\quad b=\left[\left(\frac{g\beta k^2}{2\omega^2}\right)^2+\frac{k^2k_z^2}{\omega^2}\frac{H_s^2\mu_m\beta^2}{\rho_m}\right]^{1/2}$$

$$a=g\beta k^2/2\omega^2$$

(c) Assume that boundary conditions are $\hat{v}_x(^\ell_0)=0$, $\hat{h}_z(^\ell_0)=0$, and show that the eigenvalue equation is

$$\frac{2b}{a^2-b^2}\sinh c_+\ell\,\sinh c_-\ell=0$$

and that eigenfrequencies are

$$\omega_n^2=\frac{k^2k_z^2H_s^2\mu_m\beta^2}{K_n^4\,\rho_m}-\frac{g\beta k^2}{K_n^2};\qquad K_n^2=\left(\frac{n\pi}{\ell}\right)^2+\left(\frac{\beta}{2}\right)^2+k^2$$

(d) Discuss the stabilizing effect of the magnetic field on the bulk Rayleigh-Taylor instability.

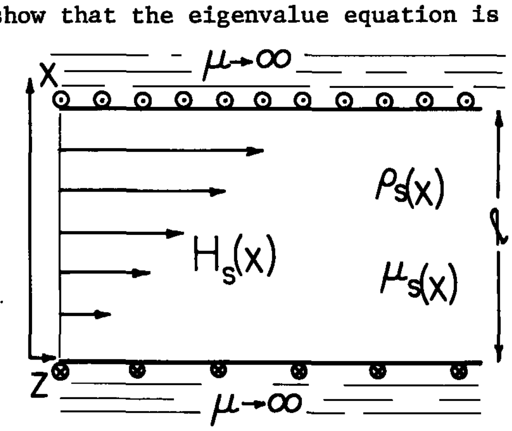

Fig. P8.18.2

Fig. 8.18.2 (continued)

(e) Discuss the analogous electric coupling with $\mu_s \rightarrow \varepsilon_s$ and $H_s \rightarrow E_s$ and describe the analogous physical configuration.

Prob. 8.18.3 As a continuation of Problem 8.18.2, prove that the principle of exchange of stabilities holds, and specifically that the eigenfrequencies are given by

m3 5/14/97

$$\omega^2 = \frac{k^2 k_z^2 |I_4|^2 + I_1 I_3}{I_1 I_2}$$

where

$$I_1 = \int_0^{\ell} (\mu_s |D\hat{h}_z|^2 + k^2 \mu_s |\hat{h}_z|^2) dx \; ; \qquad I_2 = \int_0^{\ell} (\rho_s |D\hat{v}_x|^2 + k^2 \rho_s |\hat{v}_x|^2) dx$$

$$I_3 = \int_0^{\ell} k^2 N |\hat{v}_x|^2 dx \; ; \qquad I_4 = \int_0^{\ell} H_s D\mu_s \hat{v}_x^* \hat{h}_z dx$$