18 Discrete-Time Processing of Continuous-Time Signals

Solutions to Recommended Problems

S18.1

(a) Since $x_p(t) = x_c(t)p(t)$, then $X_p(\omega)$ is just a replication of $X_c(\omega)$ centered at multiples of the sampling frequency, namely 8 kHz or $2\pi 8 \times 10^3$ rad/s. The sampling period is $T = 1/8000$.

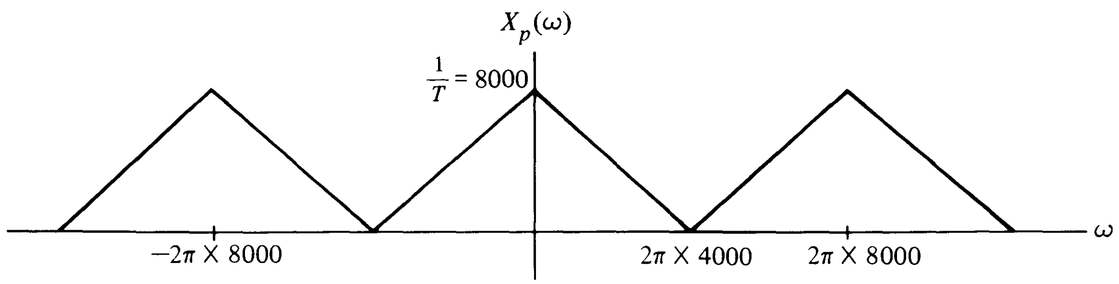

Figure S18.1-1

(b) $X(\Omega)$ is just a rescaling of the frequency axis, where $2\pi 8 \times 10^3$ becomes 2π. $X(\Omega)$ is shown in Figure S18.1-2.

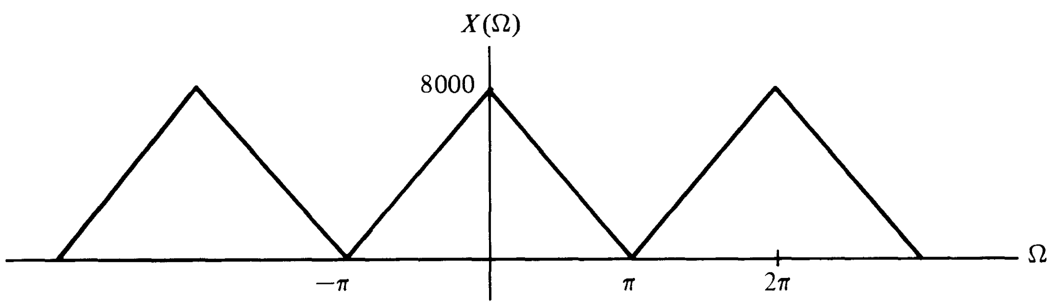

Figure S18.1-2

(c) $Y(\Omega)$ is the product $G(\Omega)X(\Omega)$. Therefore, $Y(\Omega)$ appears as in Figure S18.1-3.

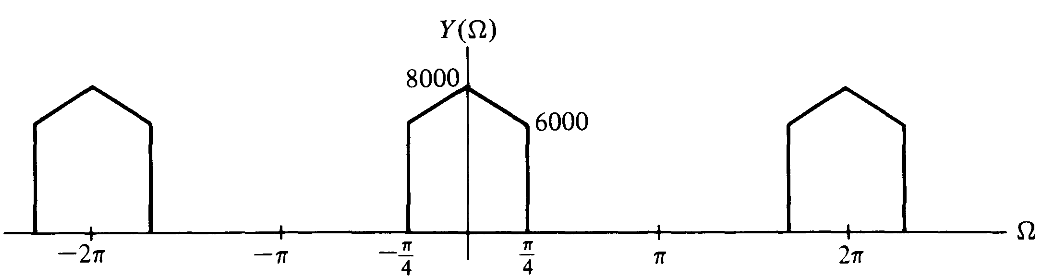

Figure S18.1-3

(d) $Y_c(\omega)$ is a frequency-scaled version of $Y(\omega)$ but only in the range $\Omega = -\pi$ to π, as shown in Figure S18.1-4. Also note the gain of T.

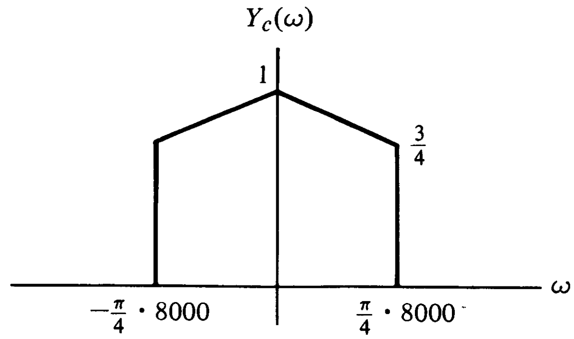

Figure S18.1-4

S18.2

(a) The maximum nonzero frequency component of $H(\omega)$ is 500π. Therefore, this frequency can correspond to, at most, the maximum digital frequency before folding, i.e., $\Omega = \pi$. From the relation $\omega T = \Omega$, we get

$$T_{\max} = \frac{\pi}{500\pi} = 2 \text{ ms}$$

(b) Since $\omega = 500\pi$ maps to $\Omega = \pi$, the discrete-time filter $G(\Omega)$ is as shown in Figure S18.2-1.

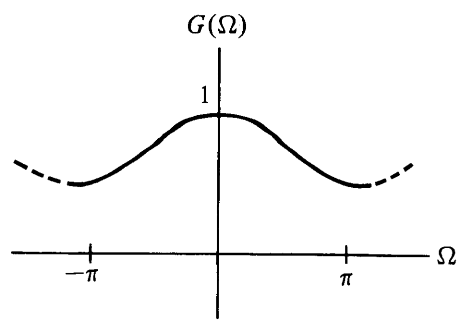

Figure S18.2-1

(c) The complete system is given by Figure S18.2-2. Note the need for an anti-aliasing filter.

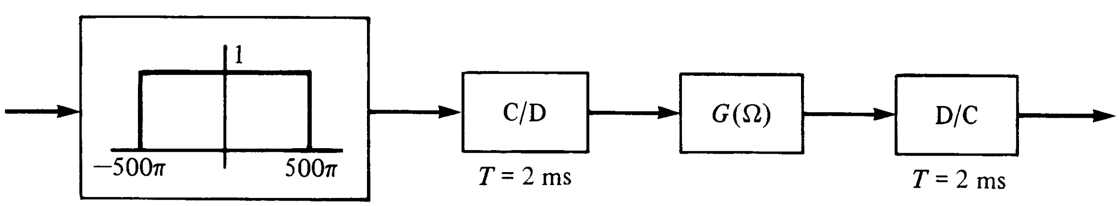

Figure S18.2-2

S18.3

(a) Recall that $X_c(\omega)$ is as given by Figure S18.3-1.

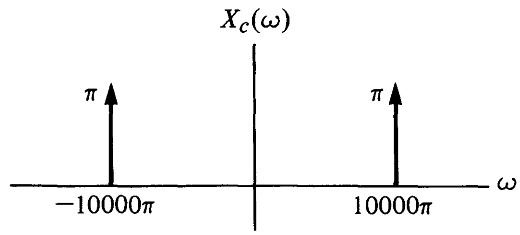

Figure S18.3-1

$X(\Omega)$ is given by eq. (S18.3-1) and Figure S18.3-2.

$$X(\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\left(\frac{\Omega}{T} - \frac{2\pi n}{T}\right) = 20000 \sum_{n=-\infty}^{\infty} X_c[20000(\Omega - 2\pi n)] \quad \text{(S18.3-1)}$$

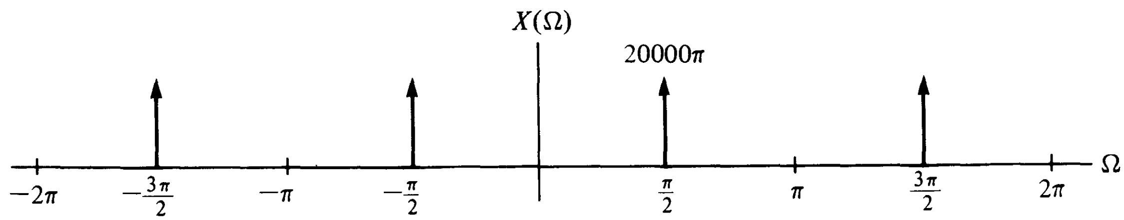

Figure S18.3-2

$Y_c(\omega)$ is given by eq. (S18.3-2) and Figure S18.3-3.

$$Y_c(\omega) = \begin{cases} TX(\omega T), & |\omega| < \frac{\pi}{T} \\ 0, & \text{elsewhere} \end{cases} \quad \text{(S18.3-2)}$$

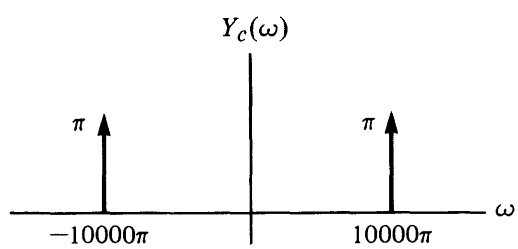

Figure S18.3-3

Thus $x(t) = y(t)$ in this case.

(b) $X_c(\omega)$ is as given in Figure S18.3-4.

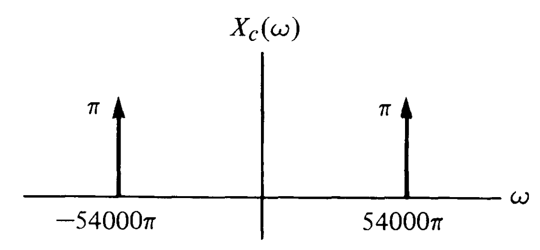

Figure S18.3-4

We now use eq. (S18.3-1), shown in Figure S18.3-5.

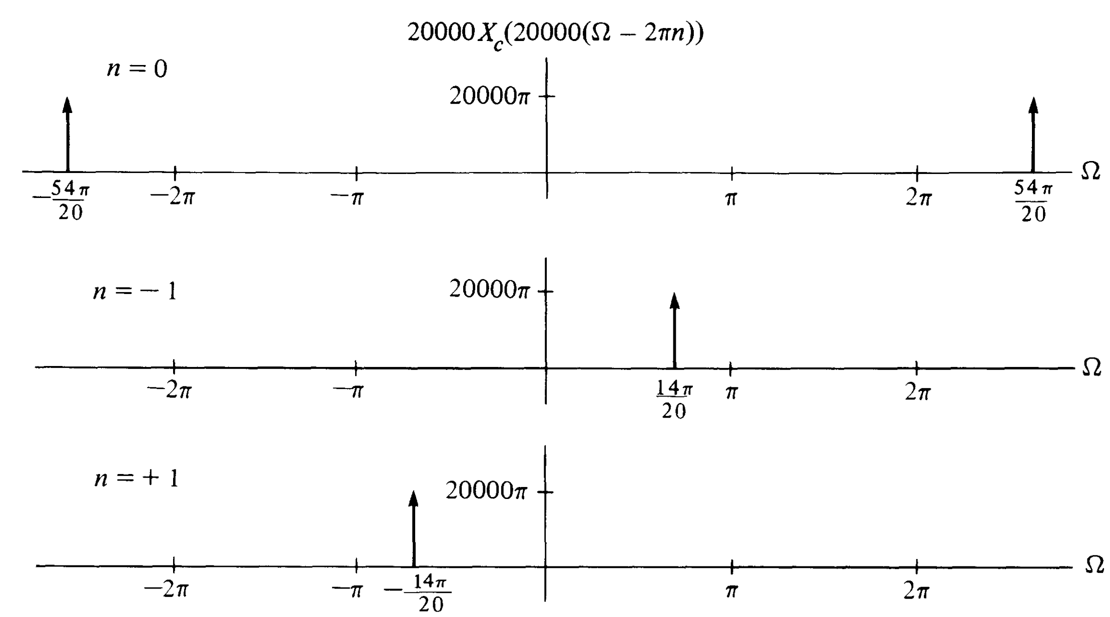

Figure S18.3-5

Thus, in the range $\pm\pi$, $X(\Omega) = 20000 \sum_{n=-\infty}^{\infty} X_c[20000(\Omega - 2\pi n)]$ is given as in Figure S18.3-6.

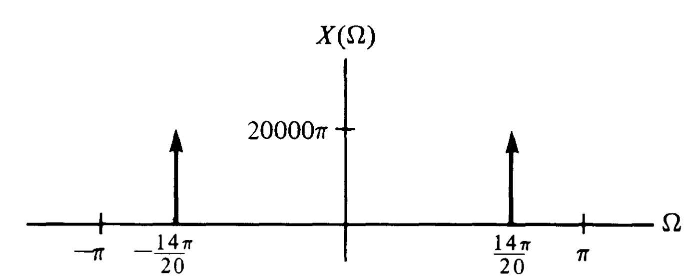

Figure S18.3-6

Using eq. (S18.3-2), we find $Y_c(\omega)$ as in Figure S18.3-7.

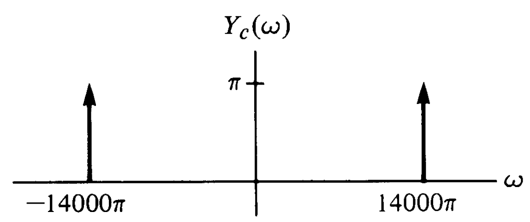

Figure S18.3-7

Note aliasing since 27000 Hz is above half the sampling rate of 20000 Hz.

(c) $X_c(\omega)$ is as given in Figure S18.3-8.

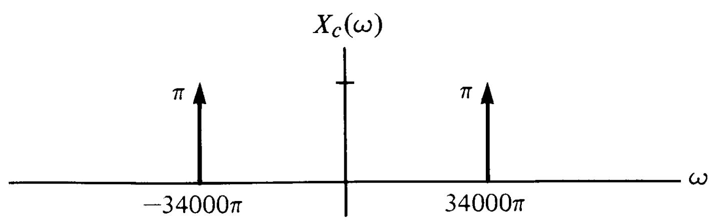

Figure S18.3-8

Again we use eq. (S18.3-1), shown in Figure S18.3-9.

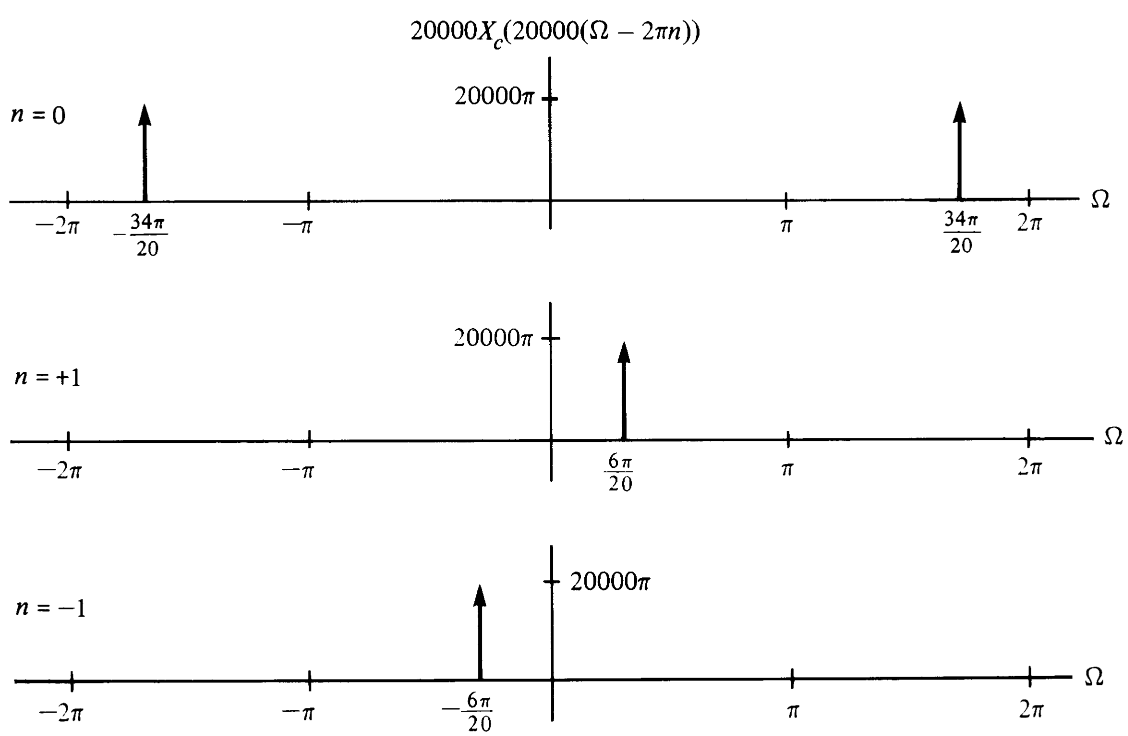

Figure S18.3-9

Thus $X(\Omega)$ is given as in Figure S18.3-10.

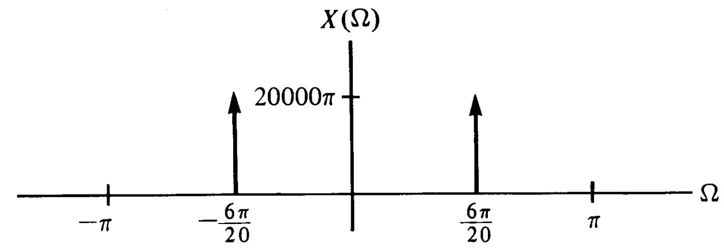

Figure S18.3-10

Finally, from eq. (S18.3-2) we have $Y_c(\omega)$ shown in Figure S18.3-11.

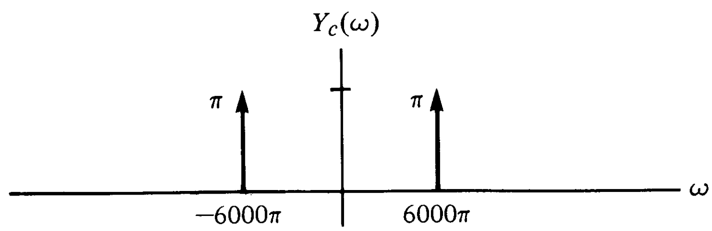

Figure S18.3-11

S18.4

It is required that we sample at a rate such that the discrete-time frequency $\pi/2$ will correspond to ω_c. The relation between Ω_c and ω_c is $\Omega_c = \omega_c T_0$. Thus, we require

$$\frac{\pi/2}{\omega_c} = T_0$$

As ω_c increases, demanding a wider filter, T_0 decreases, and consequently the sampling frequency must be increased. There are two ways to calculate ω_a. First, since we are sampling at a rate of

$$\frac{2\pi}{T_0} \quad \text{or} \quad \frac{2\pi}{(\pi/2)/\omega_c} = 4\omega_c,$$

we need an anti-aliasing filter that will remove power at frequencies higher than half the sampling rate; therefore $\omega_a = 2\omega_c$. Alternatively, we note that the "folding frequency," or the frequency at which aliasing begins, is $\Omega = \pi$. Since $\Omega = \pi/2$ corresponds to ω_c, then π must correspond to $2\omega_c$.

S18.5

(a) We sketch $X(\Omega)$ by stretching the frequency axis so that 2π corresponds to the sampling frequency with a gain of $1/T_0$. We then repeat the spectrum, as shown in Figure S18.5-1.

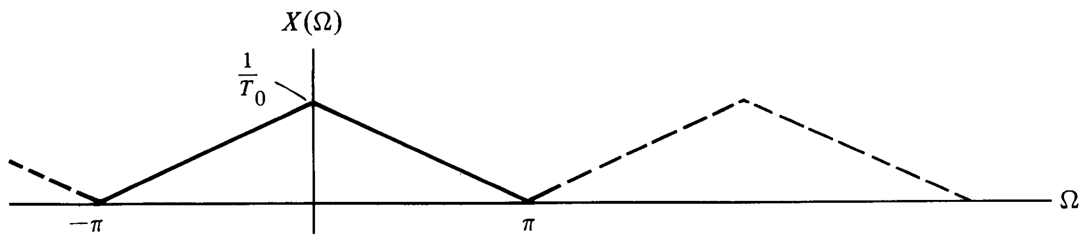

Figure S18.5-1

After filtering, $Y(\Omega)$ is given as in Figure S18.5-2.

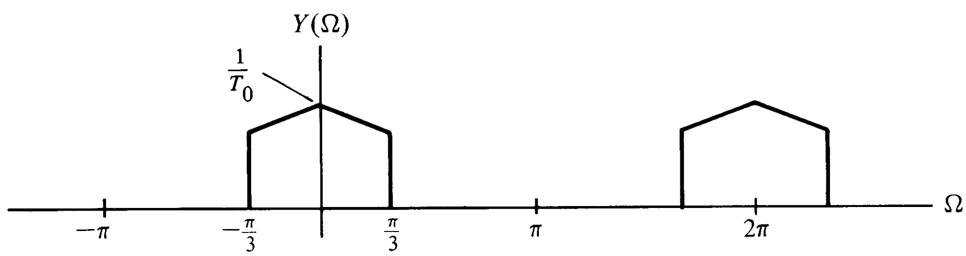

Figure S18.5-2

(b) We see that $Y(\Omega)$ looks like $X(\omega)$ filtered and then sampled. The discrete-time frequency is $\pi/3$. Again, 2π corresponds to $2\pi/T_0$, so $\pi/3$ corresponds to $\pi/3T_0$. Thus, if $x(t)$ is filtered by $G(\omega)$ as given in Figure S18.5-3, then $y[n] = z[n]$.

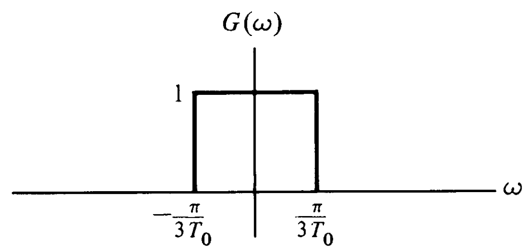

Figure S18.5-3

Solutions to Optional Problems

S18.6

(a) Since we are allowing all frequencies less than 100π through the anti-aliasing filter, we need to sample at least twice 100π, or 200π. Thus, $200\pi = 2\pi/T_0$ or $T_0 = 10$ ms. To find K, recall that impulse sampling introduces a gain of $1/T_0$. To account for this, K must equal T_0, or $K = 0.01$.

(b) (i) Since $X(\omega)$ is bandlimited to 100π, the anti-aliasing filter has no effect. The Fourier transform of $x_p(t)$, the modulated pulse train, is given in Figure S18.6-1.

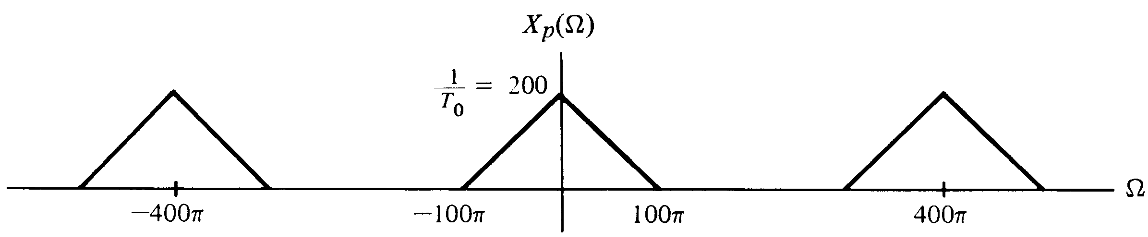

Figure S18.6-1

Since $T_0 = 0.005$, the sampling frequency is 400π. After conversion to a discrete-time signal, $X(\Omega)$ appears as in Figure S18.6-2.

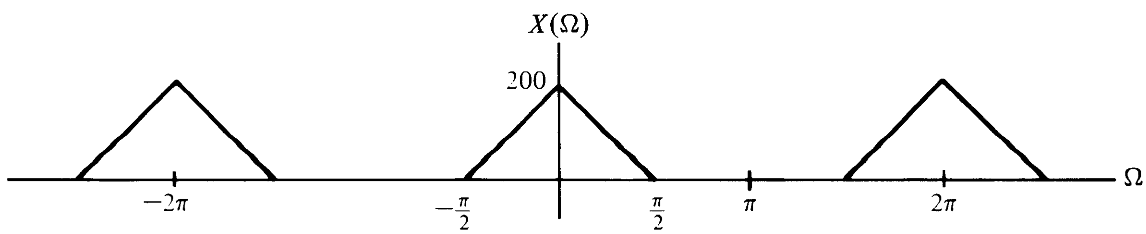

Figure S18.6-2

After filtering, $Y(\Omega)$ is given by Figure S18.6-3.

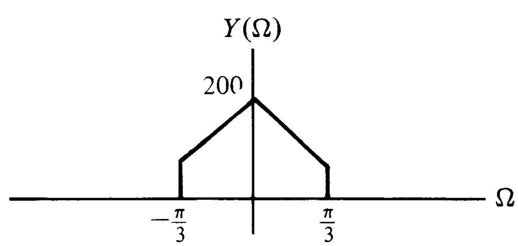

Figure S18.6-3

(ii) There are three effects to note in D/C conversion: (1) a gain of T_0, (2) a frequency scaling by a factor of T_0, and (3) the removal of repeated spectra. Thus, $Y(\omega)$ is as shown in Figure S18.6-4.

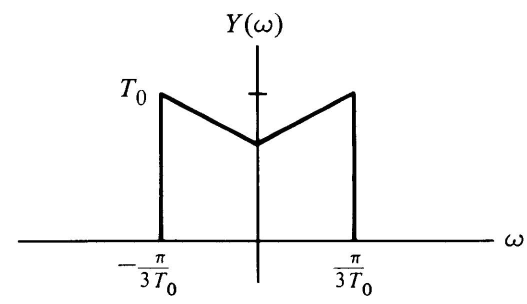

Figure S18.6-4

S18.7

After the initial shock, you should realize that this problem is not as difficult as it seems. If instead of $h[n]$ we had been given the frequency response $H(\Omega)$, then $H_c(\omega)$ would be just a scaled version of $H(\Omega)$ bandlimited to π/T. Let us find, then, $H(\Omega)$. Using properties of the Fourier transform, we have

$$Y(\Omega) = \frac{1}{2}e^{-j\Omega}Y(\Omega) + X(\Omega)$$

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}}$$

Thus,

$$|H(\Omega)| = \frac{1}{\sqrt{\frac{5}{4} - \cos\Omega}}$$

$$\sphericalangle H(\Omega) = -\tan^{-1}\left(\frac{\frac{1}{2}\sin\Omega}{1 - \frac{1}{2}\cos\Omega}\right)$$

Therefore, the magnitude and phase of $H_c(\omega)$ are as shown in Figure S18.7.

$$|H_c(\omega)| = \begin{cases} \dfrac{1}{\sqrt{\frac{5}{4} - \cos\omega T}}, & |\omega| < \dfrac{\pi}{T}, \\ 0, & \text{elsewhere} \end{cases}$$

$$\sphericalangle H_c(\omega) = \begin{cases} -\tan^{-1}\left(\dfrac{\frac{1}{2}\sin\omega T}{1 - \frac{1}{2}\cos\omega T}\right), & |\omega| < \dfrac{\pi}{T}, \\ 0, & \text{elsewhere} \end{cases}$$

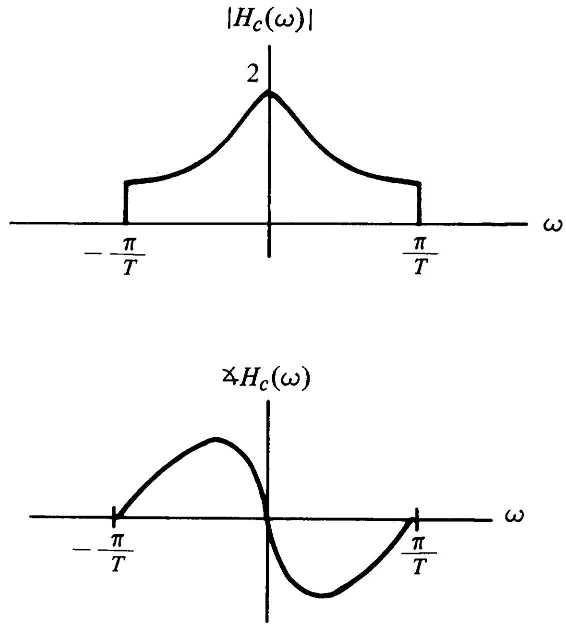

Figure S18.7

S18.8

The system under study is shown in Figure S18.8.

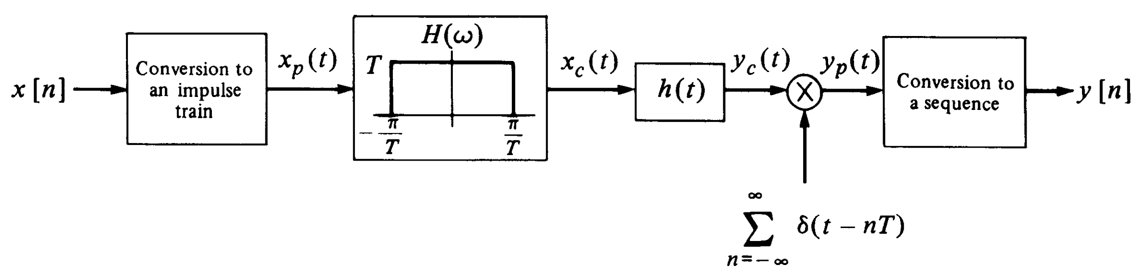

Figure S18.8

From our previous study, we know that $X_c(\omega)$ in the range $\pm\pi/T$ looks just like $X(\Omega)$ in the range $\pm\pi$. Similarly, $Y_c(\omega)$ between $-\pi/T$ and $+\pi/T$ looks like $Y(\Omega)$ in the range $-\pi$ to π. Although there is a factor of T, we can disregard it in analyzing this system because it is accounted for in the $H(\omega)$ filter. The transformation of $x_c(t)$ to $y_c(t)$ will correspond to filtering $x[n]$, yielding $y[n]$. In fact, the equivalent system will have a system function $H(\Omega)$ given by

$$H(\Omega) = H_c\left(\frac{\Omega}{T}\right), \qquad |\Omega| < \pi,$$

where $H_c(\omega)$ is the Fourier transform of $h(t)$. Thus, we need to find $H_c(\omega)$. The relation between $y_c(t)$ and $x_c(t)$ is governed by the following differential equation:

$$\frac{d^2y_c(t)}{dt^2} + 4\frac{dy_c(t)}{dt^2} + 3y_c(t) = x_c(t)$$

Using the properties of the Fourier transform, we have

$$(j\omega)^2 Y_c(\omega) + 4(j\omega)Y_c(\omega) + 3Y_c(\omega) = X_c(\omega),$$
$$H_c(\omega) = \frac{1}{(j\omega)^2 + 4j\omega + 3}$$

Therefore,

$$H(\Omega) = \frac{1}{\left(j\frac{\Omega}{T}\right)^2 + 4j\frac{\Omega}{T} + 3}, \qquad |\Omega| < \pi$$

S18.9

(a) It is instructive to sketch a typical $y_p(t)$, which we have done in Figure S18.9-1.

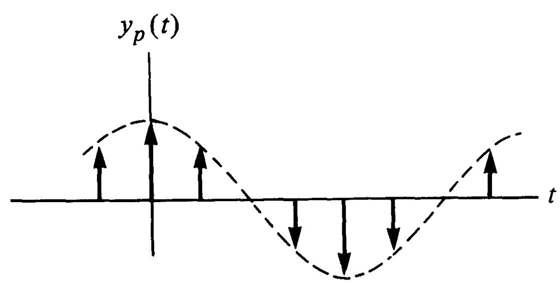

Figure S18.9-1

Let us suppose that T is changed by being reduced. Then the envelope of $y_p(t)$ seems to correspond to a higher-frequency cosine. At time kt,

$$y_p(t) = \cos\frac{2\pi k}{N}\,\delta(t - kT) = \cos\frac{2\pi(kT)}{NT}\,\delta(t - kT) = \cos\frac{2\pi t}{NT}\,\delta(t - kT),$$

where we use the sampling property of the impulse function. Thus,

$$y_p(t) = \sum_{k=-\infty}^{\infty} \cos\frac{2\pi k}{N}\,\delta(t - kT) = \cos\omega_0 t \sum_{k=-\infty}^{\infty} \delta(t - kT),$$

where $\omega_0 = 2\pi/NT$.

If the minimum ω_0 is ω_1, and since $T = 2\pi/N\omega_0$,

$$T_{\max} = \frac{2\pi}{N\omega_1}$$

Similarly,

$$T_{\min} = \frac{2\pi}{N\omega_2}$$

(b) Recall that sampling with an impulse train repeats the spectrum with a period of $2\pi/T$ and a gain factor of $1/T$. Since $\mathcal{F}[\cos(2\pi t/NT)]$ is as given by Figure S18.9-2, $Y_p(\omega)$ is then given by Figure S18.9-3.

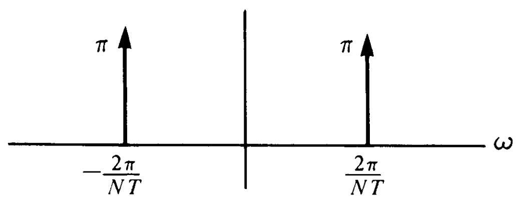

Figure S18.9-2

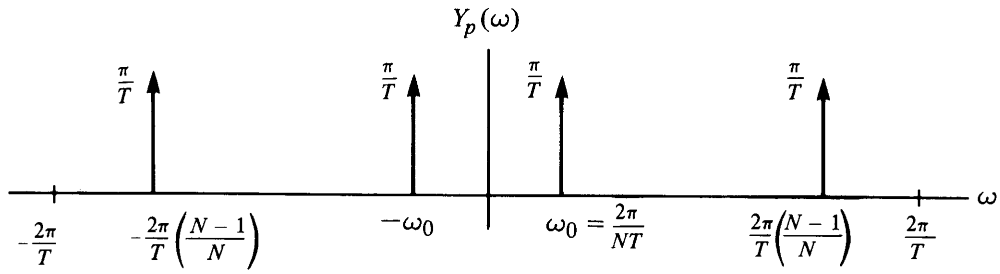

Figure S18.9-3

(c) The minimum value of N is 2, corresponding to the impulses at ω_0 and $(2\pi/T - \omega_0)$ being superimposed at π/T. The lowpass filter cutoff frequency must be such that the (superimposed) impulses at π/T are in the passband and those at $3\pi/T$ are outside the passband. Consequently,

$$\frac{\pi}{T} < \omega_c < \frac{3\pi}{T}$$

(d) Comparing $Y(\omega)$ and $Y_p(\omega)$ in Figures S18.9-2 and S18.9-3 respectively, we see that for $N > 2$ the cosine output will have an amplitude of $1/T = \omega/2\pi$. If $N = 2$, then the output amplitude will be $2/T = \omega/\pi$.

S18.10

(a) By sampling $s_c(t)$, we get

$$s[n] = s_c(nT) = x(nT) + \alpha x(nT - T_0) = x(nT) + \alpha x[(n-1)T]$$

since $T = T_0$. Let $x[n] = x(nT)$. Then

$$s[n] = x[n] + \alpha x[n-1]$$

Therefore

$$x[n] = -\alpha x[n-1] + s[n]$$

This is a first-order difference equation, so given $s[n]$, we can find $x[n]$. Since $x(t)$ is appropriately bandlimited, we can then set

$$y[n] = -\alpha y[n-1] + s[n]$$

which will make

$$y_c(t) = \frac{A}{T}x(t)$$

(b) From part (a) we see that $T = A$ will make $y(t) = x(t)$.

(c) Since we do not want to alias, we still need $T < \pi/\omega_M$. Now

$$s(t) = x(t) + \alpha x(t - T_0)$$

Taking the *continuous* Fourier transform, we see that

$$S(\omega) = X(\omega) + \alpha e^{-j\omega T_0}X(\omega)$$

Thus, the continuous-time inverse system has frequency response

$$H_c(\omega) = \frac{1}{1 + \alpha e^{-j\omega T_0}}$$

We want to implement this in discrete time. Therefore, using the relation, we obtain

$$H(\Omega) = H_c\left(\frac{\Omega}{T}\right) = \frac{1}{1 + \alpha e^{-j\Omega(T_0/T)}}, \qquad \frac{-\pi}{\omega_M} < \Omega < \frac{\pi}{\omega_M}$$

Again, the filter should be $A = T$.