24 Butterworth Filters

Solutions to Recommended Problems

S24.1

(a) For $N = 5$ and $\omega_c = (2\pi)1$ kHz, $|B(j\omega)|^2$ is given by

$$|B(j\omega)|^2 = \frac{1}{1 + \left(\dfrac{j\omega}{j2000\pi}\right)^{10}}$$

(b) The denominator of $B(s)B(-s)$ is set to zero. Thus

$$0 = 1 + \left(\frac{s}{j2000\pi}\right)^{10}, \qquad \text{or} \qquad s = (-1)^{1/10}j2000\pi$$

Expressing -1 as $e^{j\pi}$ and j as $e^{j\pi/2}$, we find that the poles of $B(s)B(-s)$ are

$$s = 2000\pi e^{j[(\pi/10)+(\pi/2)+(\pi k/5)]},$$

as shown in Figure S24.1-1.

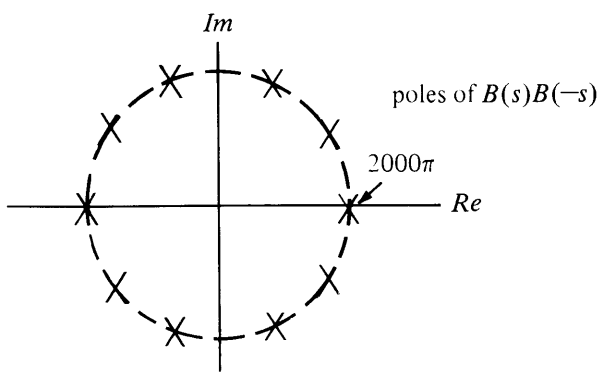

Figure S24.1-1

(c) For $B(s)$ to be stable and causal, its poles must be in the left half-plane, as shown in Figure S24.1-2.

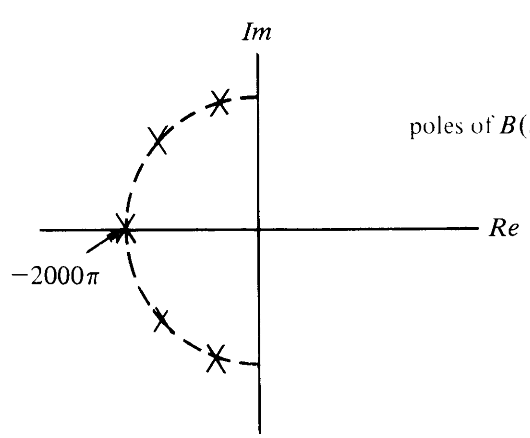

Figure S24.1-2

(d) Since the total number of poles must be as shown in part (b), the poles of $B(-s)$ must be given as in Figure S24.1-3.

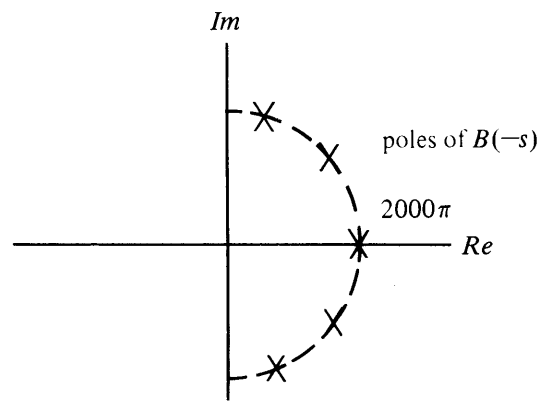

Figure S24.1-3

S24.2

(a) When there is no aliasing, the relation in the frequency domain between the continuous-time filter and the discrete-time filter corresponding to impulse invariance is

$$H(e^{j\Omega}) = \frac{1}{T} H_a\left(j\frac{\Omega}{T}\right), \qquad |\Omega| \leq \pi$$

Thus, there is an amplitude scaling of T and a frequency scaling given by

$$\Omega = \omega T, \qquad |\Omega| \leq \pi, \qquad |\omega| \leq \pi T$$

The required transfer function can be found by reflecting $TH(e^{j\Omega})$ through the preceding transformation, as shown in Figure S24.2-1.

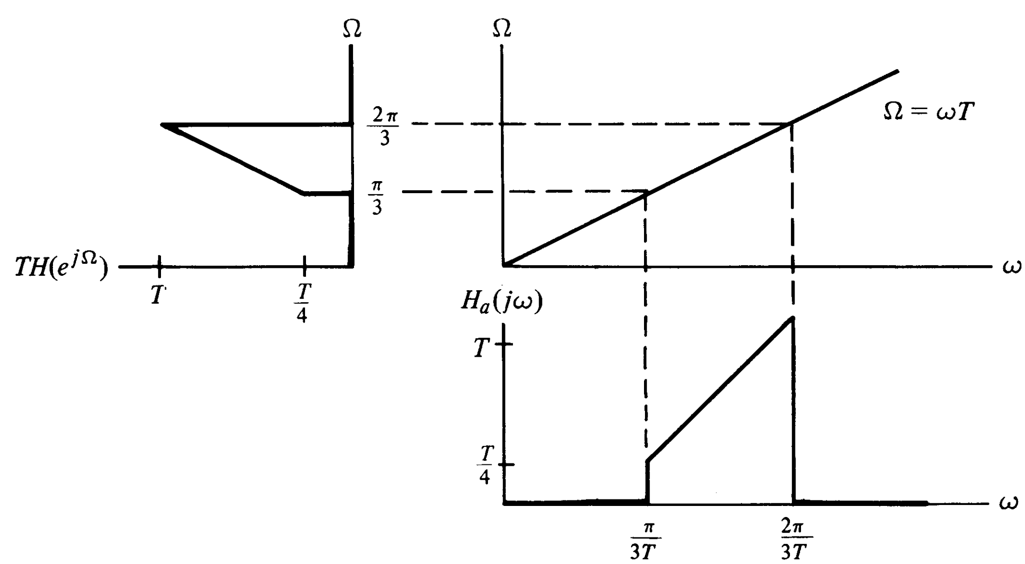

Figure S24.2-1

Since the relation between Ω and ω is linear, the shape of the frequency response is preserved.

(b) For the bilinear transformation, there is no amplitude scaling of the frequency response; however, there is the following frequency transformation:

$$\Omega = 2 \arctan\left(\frac{\omega T}{2}\right)$$

As in part (a), we can find $H_a(j\omega)$ by reflecting $H(e^{j\Omega})$ through the preceding frequency transformation, shown in Figure S24.2-2.

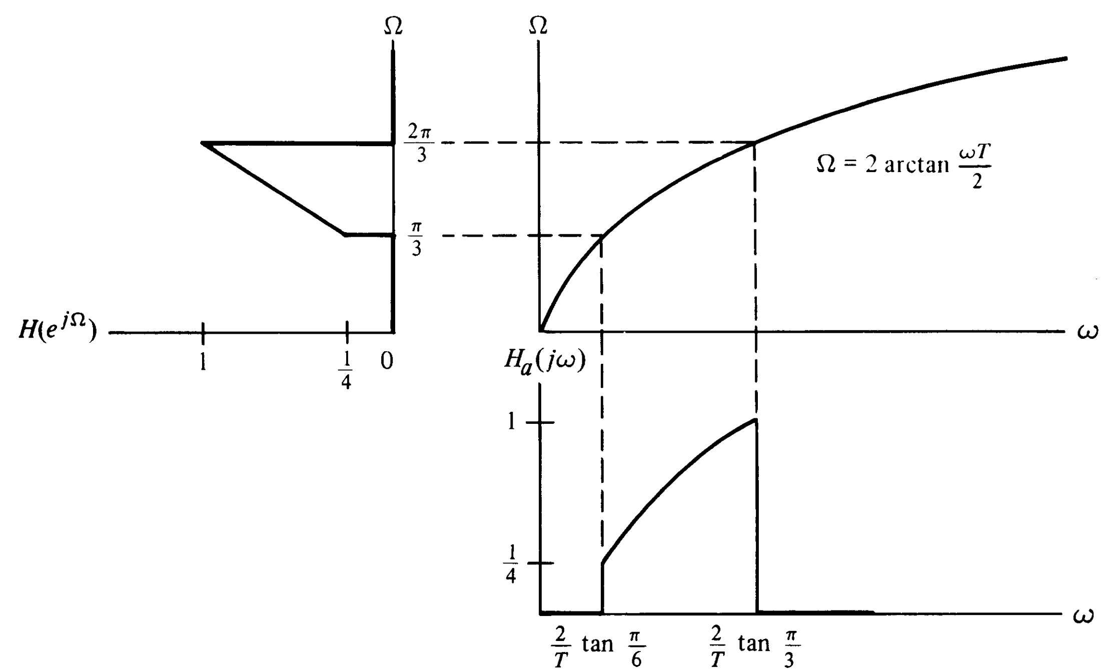

Figure S24.2-2

Because of the nonlinear relation between Ω and ω, $H_a(j\omega)$ does not exhibit a linear slope as $H(e^{j\Omega})$ does.

(c) We redraw the transformation of part (a) for the new $H(e^{j\Omega})$ in Figure S24.2-3. As in part (a), the shape of the frequency response is preserved.

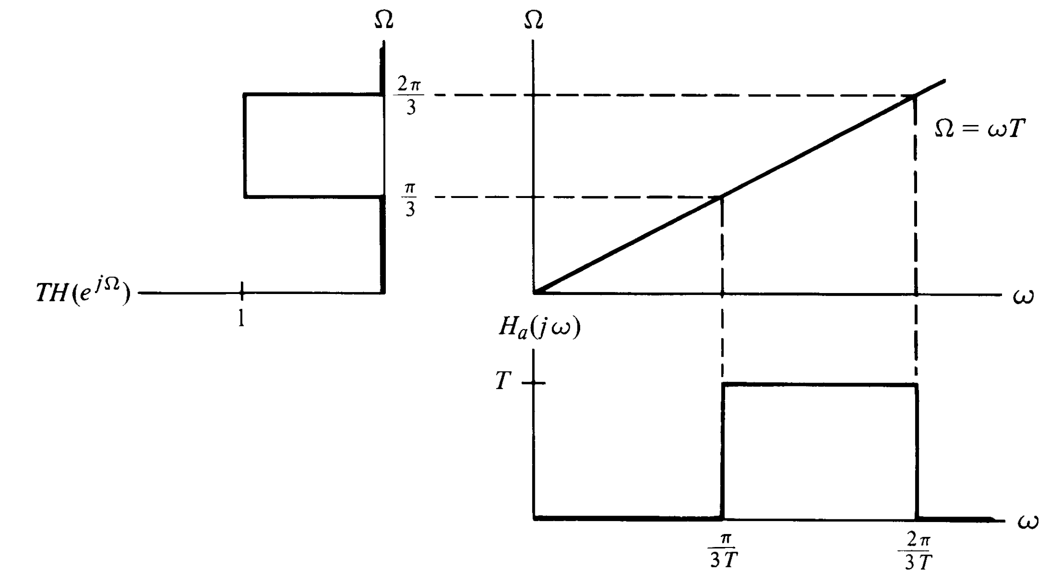

Figure S24.2-3

We redraw the transformation of part (b) for the new $H(e^{j\Omega})$ in Figure S24.2-4. Unlike part (b), the general shape of $H(e^{j\Omega})$ is preserved because of the piecewise-constant nature of $H(e^{j\Omega})$.

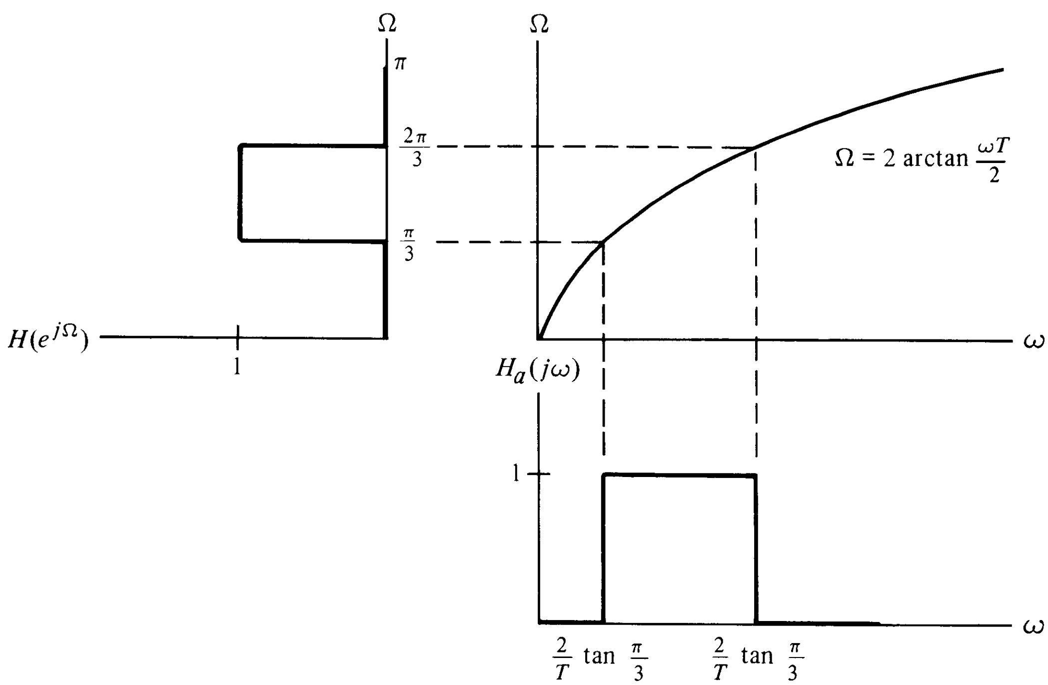

Figure S24.2-4

S24.3

(a) Using the bilinear transformation, we get

$$H(z) = \frac{1}{a + \dfrac{1 - z^{-1}}{1 + z^{-1}}} = \frac{1 + z^{-1}}{a + 1 + z^{-1}(a - 1)} = \frac{\dfrac{1 + z^{-1}}{1 + a}}{1 - \left(\dfrac{1 - a}{1 + a}\right) z^{-1}}$$

(b) Since $H(s)$ has a pole at $-a$, we need $a > 0$ for $H(s)$ to be stable and causal.

(c) Figure S24.3 contains a plot of $(1 - a)/(1 + a)$, the pole location of $H(z)$, versus a.

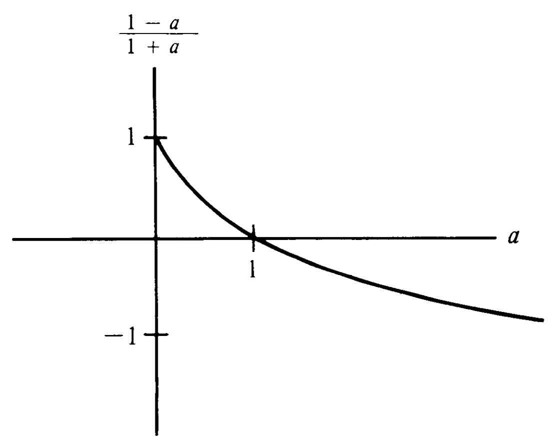

Figure S24.3

We see that for $a > 0$, $(1 - a)/(1 + a)$ is between -1 and 1. Since the only pole of $H(z)$ occurs at $z = (1 - a)/(1 + a)$, $H(z)$ must be stable whenever $H(s)$ is stable, assuming that $H(z)$ represents a causal $h[n]$.

S24.4

(a) For $T = 1$ and the impulse invariance method, $B(j\omega)$ must satisfy

$$1 \geq |B(j\omega)| \geq 0.8 \qquad \text{for } 0 \leq \omega \leq \frac{\pi}{4},$$

$$0.2 \geq |B(j\omega)| \geq 0 \qquad \text{for } \frac{3\pi}{4} \leq \omega$$

Therefore, if we ignore aliasing,

$$\left|B\left(j\frac{\pi}{4}\right)\right|^2 = \frac{1}{1 + \left(\dfrac{j\pi/4}{j\omega_c}\right)^{2N}} = (0.8)^2,$$

$$\left|B\left(j\frac{3\pi}{4}\right)\right|^2 = \frac{1}{1 + \left(\dfrac{j3\pi/4}{j\omega_c}\right)^{2N}} = (0.2)^2$$

(b) For $T = 1$ and the bilinear transformation, $B(j\omega)$ must satisfy

$$1 \geq |B(j\omega)| \geq 0.8, \qquad 0 \leq \omega \leq 2 \tan \frac{\pi}{8},$$

$$0.2 \geq |B(j\omega)| \geq 0, \qquad 2 \tan \frac{3\pi}{8} \leq \omega$$

Therefore,

$$\frac{1}{1 + \left[\dfrac{j2 \tan (\pi/8)}{j\omega_c}\right]^{2N}} = (0.8)^2,$$

$$\frac{1}{1 + \left[\dfrac{j2 \tan (3\pi/8)}{j\omega_c}\right]^{2N}} = (0.2)^2$$

S24.5

(a) The relation between Ω and ω is given by $\Omega = \omega T$, where $T = 1/15000$. Thus,

$$1 \geq |H(e^{j\Omega})| \geq 0.9 \qquad \text{for } 0 \leq \Omega \leq \frac{2\pi}{5},$$

$$0.1 \geq |H(e^{j\Omega})| \geq 0 \qquad \text{for } \frac{3\pi}{5} \leq \Omega \leq \pi$$

Note that while $H_d(j\omega)$ was restricted to be between 0.1 and 0 for all ω larger than $2\pi(4500)$, we can specify $H(e^{j\Omega})$ only up to $\Omega = \pi$. For values higher than π, we rely on some anti-aliasing filter to do the attenuation for us.

(b) Assuming no aliasing,

$$H(e^{j\Omega}) = \frac{1}{T} G\left(j \frac{\Omega}{T}\right)$$

Therefore,

$$3 \geq |G(j\omega)| \geq 2.7, \qquad 0 \leq \omega \leq \frac{2\pi}{15},$$

$$0.3 \geq |G(j\omega)| \geq 0, \qquad \frac{\pi}{5} \leq \omega < \frac{\pi}{3}$$

(c) The relation between ω and Ω is given by $\Omega = 2 \arctan (\omega)$. Thus,

$$1 \geq |G(j\omega)| \geq 0.9, \qquad 0 \leq \omega \leq \tan \frac{\pi}{5},$$

$$0.1 \geq |G(j\omega)| \geq 0, \qquad \tan \frac{3\pi}{10} \leq \omega < \infty$$

(d) If T changes, then the specifications for $G(j\omega)$ will change for either the impulse variance method or the bilinear transformation. However, they will change in such a way that the resulting discrete-time filter $H(e^{j\Omega})$ will *not* change. Thus, $H_c(j\omega)$ will also not change.

Solutions to Optional Problems

S24.6

(a) We first assume that a $B(s)$ exists such that the filter specifications are met exactly. Since

$$|B(j\omega)|^2 = \frac{1}{1 + \left(\dfrac{j\omega}{j\omega_c}\right)^{2N}},$$

we require that

$$|B(j2\pi)|^2 = \frac{1}{1 + \left(\dfrac{j2\pi}{j\omega_c}\right)^{2N}} = (10^{-0.05})^2 = 10^{-0.1},$$

$$|B(j3\pi)|^2 = \frac{1}{1 + \left(\dfrac{j3\pi}{j\omega_c}\right)^{2N}} = 10^{-1.5}$$

Substituting $N = 5.88$ and $\omega_c = 7.047$, we see that the preceding equations are satisfied.

(b) Since we know that $N = 6$, we use the first equation to solve for ω_c:

$$10^{-0.1} = \frac{1}{1 + \left(\dfrac{j2\pi}{j\omega_c}\right)^{12}}$$

Solving for ω_c, we find that $\omega_c = 7.032$. The frequency response at $\omega = 0.3\pi$ is given by

$$|B(j3\pi)|^2 = \frac{1}{1 + \left(\dfrac{j3\pi}{j7.032}\right)^{12}} = 0.02890,$$

$$20 \log_{10}|B(j3\pi)| = -15.4 \text{ dB}$$

(c) If we picked $N = 5$, there would be *no* value of ω_c that would lead to a Butterworth filter that would meet the filter specifications.

S24.7

We require an $H_d(z)$ such that

$$0 \geq 20 \log_{10}|H_d(e^{j\Omega})| \geq -0.75, \qquad 0 \leq \Omega \leq 0.2613\pi,$$
$$-20 \text{ dB} \geq 20 \log_{10}|H_d(e^{j\Omega})|, \qquad 0.4018\pi \leq \Omega \leq \pi$$

We will for the moment assume that the specifications can be met exactly. Let Ω_p be the frequency where

$$20 \log_{10}|H_d(e^{j\Omega_p})| = -0.75, \qquad \text{or} \qquad |H_d(e^{j\Omega_p})|^2 = 10^{-0.075}$$

Similarly, we define Ω_s as the frequency where

$$20 \log_{10}|H_d(e^{j\Omega_s})| = -20, \qquad \text{or} \qquad |H_d(e^{j\Omega_s})|^2 = 10^{-2}$$

Using $T = 1$, we find the specifications for the continuous-time filter $H_a(j\omega)$ as

$$|H_a(j\omega_p)|^2 = 10^{-0.075}, \qquad |H_a(j\omega_s)|^2 = 10^{-2},$$

where

$$\omega_p = 2 \tan \frac{\Omega_p}{2} = 2 \tan \left(\frac{0.2613\pi}{2}\right) = 0.8703,$$

$$\omega_s = 2 \tan \frac{\Omega_s}{2} = 2 \tan \left(\frac{0.4018\pi}{2}\right) = 1.4617$$

For the specification to be met exactly, we need N and ω_c such that

$$1 + \left(\frac{j0.8703}{j\omega_c}\right)^{2N} = 10^{0.075} \quad \text{and} \quad 1 + \left(\frac{j1.4617}{j\omega_c}\right)^{2N} = 10^2$$

Solving for N, we find that $N = 6.04$. Since N is so close to 6 we may relax the specifications slightly and choose $N = 6$. Alternatively, we pick $N = 7$. Meeting the passband specification exactly, we choose ω_c such that

$$1 + \left(\frac{j0.8703}{j\omega_c}\right)^{14} = 10^{0.075}, \qquad \text{or} \qquad \omega_c = 0.9805$$

The continuous-time filter $H_a(s)$ is then specified by

$$H_a(s)H_a(-s) = \frac{1}{1 + \left(\dfrac{s}{j0.9805}\right)^{14}}$$

The poles are drawn in Figure S24.7.

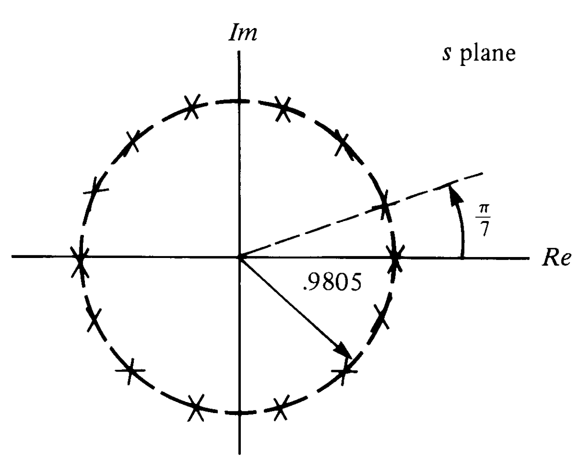

Figure S24.7

We associate with $H_a(s)$ the poles that are on the left half-plane, as follows:

$$s_1 = -0.9805, \qquad s_2 = 0.9805e^{j8\pi/14}, \qquad s_3 = s_2^*,$$
$$s_4 = 0.9805e^{j10\pi/14}, \qquad s_5 = s_4^*, \qquad s_6 = 0.9805e^{j12\pi/14}, \qquad s_7 = s_6^*$$

$H_a(s)$ is given by

$$H_a(s) = \frac{(0.9805)^7}{\prod_{i=1}^{7} (s - s_i)}$$

$H_d(z)$ can be obtained by the substitution

$$H_d(z) = H_a(s)|_{s=2[(1-z^{-1})/(1+z^{-1})]}$$

S24.8

(a) Assuming no aliasing, $H_d(e^{j\Omega})$ is related to $\hat{H}_b(j\omega)$ by

$$H_d(e^{j\Omega}) = \frac{1}{T}\hat{H}_b\left(j\frac{\Omega}{T}\right), \qquad T = 2$$

Thus, the specifications for $\hat{H}_b(j\omega)$ are given by

$$\begin{aligned} 2 \geq |\hat{H}_b(j\omega)| &\geq 2a, &\quad 0 \leq \omega \leq 0.2\pi/2, \\ 2b \geq |\hat{H}_b(j\omega)| &\geq 0, &\quad 0.3\pi/2 \leq \omega \end{aligned}$$

(b) Substituting

$$\hat{H}_s(j\omega) = \frac{2}{3}H_s\left(j\frac{2\omega}{3}\right)$$

for $\omega = 0.2\pi/2$, we have

$$\left|\hat{H}_s\left(j\frac{0.2\pi}{2}\right)\right| = \frac{2}{3}\left|H_s\left(j\frac{2}{3}\frac{0.2\pi}{2}\right)\right| = \frac{2}{3}\left|H_s\left(j\frac{0.2\pi}{3}\right)\right|$$

But

$$\left|H_s\left(j\frac{0.2\pi}{3}\right)\right| = 3a$$

Thus

$$\left|\hat{H}_s\left(j\frac{0.2\pi}{2}\right)\right| = \frac{2}{3}3a = 2a$$

Similarly,

$$\left|\hat{H}_s\left(j\frac{0.3\pi}{2}\right)\right| = 2b$$

Thus, $\hat{H}_s(s)$ satisfies the filter specifications for $H_b(j\omega)$ exactly.

(c) $\hat{H}(e^{j\Omega})$ is given by

$$\hat{H}(e^{j\Omega}) = \frac{1}{2}\sum_{k=-\infty}^{\infty}\hat{H}_s\left[j\left(\frac{\Omega}{2} - \frac{2\pi k}{2}\right)\right]$$

But $\hat{H}_s(j\omega) = \frac{2}{3}H_s\left(j\frac{2}{3}\omega\right)$. Therefore,

$$\begin{aligned} \hat{H}(e^{j\Omega}) &= \frac{1}{2}\sum_{k=-\infty}^{\infty}\frac{2}{3}H_s\left[j\frac{2}{3}\left(\frac{\Omega}{2} - \frac{2\pi k}{2}\right)\right] \\ &= \frac{1}{3}\sum_{k=-\infty}^{\infty}H_s\left[j\left(\frac{\Omega}{3} - \frac{2\pi k}{3}\right)\right] = H(e^{j\Omega}) \end{aligned}$$

S24.9

(a) Using properties of the Laplace transform, we have

$$sY(s) = X(s), \qquad \text{or} \qquad H(s) = \frac{1}{s}$$

(b) Here h is given by T, a is given by $x[(n-1)T]$, and b is given by $x(nT)$. Therefore, the area is given by

$$\left(\frac{a+b}{2}\right)h = \frac{T}{2}[x((n-1)T) + x(nT)] = A_n$$

(c) From the definition of $\hat{y}[n]$, we find that

$$\hat{y}[n-1] = \sum_{k=-\infty}^{n-1} A_k$$

Subtracting $\hat{y}[n-1]$ from $\hat{y}[n]$, we find

$$\hat{y}[n] - \hat{y}[n-1] = \sum_{k=-\infty}^{n} A_k - \sum_{k=-\infty}^{n-1} A_k = A_n$$

Therefore,

$$\hat{y}[n] = \hat{y}[n-1] + A_n.$$

(d) From the answer to part (a), we substitute for A_n, yielding

$$\begin{aligned}\hat{y}[n] &= \hat{y}[n-1] + \frac{T}{2}[x((n-1)T) + x(nT)] \\ &= \hat{y}[n-1] + \frac{T}{2}\{\hat{x}[n-1] + \hat{x}[n]\}\end{aligned}$$

(e) Using z-transforms, we find

$$\hat{Y}(z) = z^{-1}\hat{Y}(z) + \frac{T}{2}[z^{-1}\hat{X}(z) + \hat{X}(z)],$$

$$H(z) = \frac{\hat{Y}(z)}{\hat{X}(z)} = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right) = H(s)\Big|_{s=(2/T)[(1-z^{-1})/(1+z^{-1})]}$$