More Root Locus 6

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 6.1 (P4.6)

For the first transfer function $a(s)$, the root locus is shown in Figure S6.1*a*.

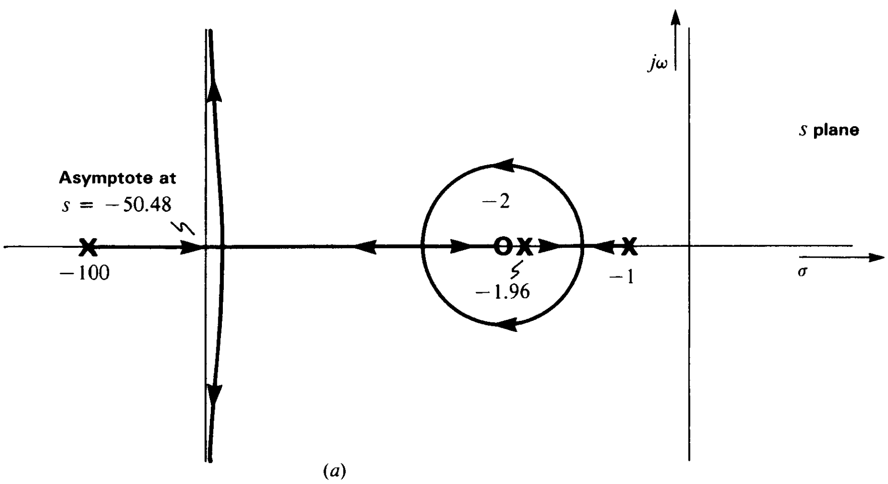


Figure S6.1 Root loci for Problem 6.1 (P4.6). (*a*) Root locus for $a(s) = \dfrac{a_o(0.5s + 1)}{(s + 1)(0.01s + 1)(0.51s + 1)}$.

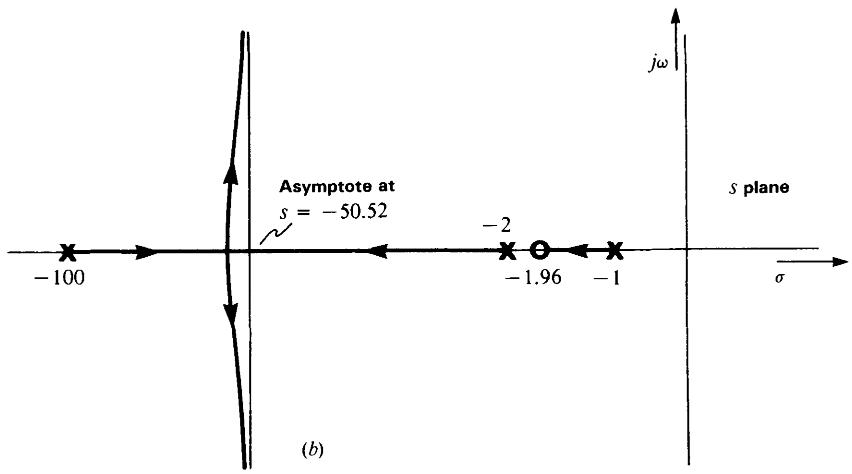
For moderate values of a_o, we evaluate the root locus by ignoring the pole at $s = -100$. Then, the locus is similar to the locus of Figure 4.8 in the textbook. The exact locations of the breakaway and reentry points are not necessary for this problem, and we do not solve for them. By Rule 5, two of the poles approach asymptotes of $\pm 90°$ from the real axis. These asymptotes intersect the real axis at

$$s = \frac{-100 - 1.96 - 1 + 2}{2} = -50.48 \qquad \textbf{(S6.1)}$$

Considering that the third pole is moving to the right towards the zero at $s = -2$, the two poles must break away slightly to the right of the asymptotes, in order to satisfy Rule 4.

For the second transfer function $a'(s)$, the root locus is shown in Figure S6.1*b*.

Figure S6.1 (*b*) Root locus for $a'(s) = \dfrac{a_o(0.51s + 1)}{(s + 1)(0.01s + 1)(0.5s + 1)}$.



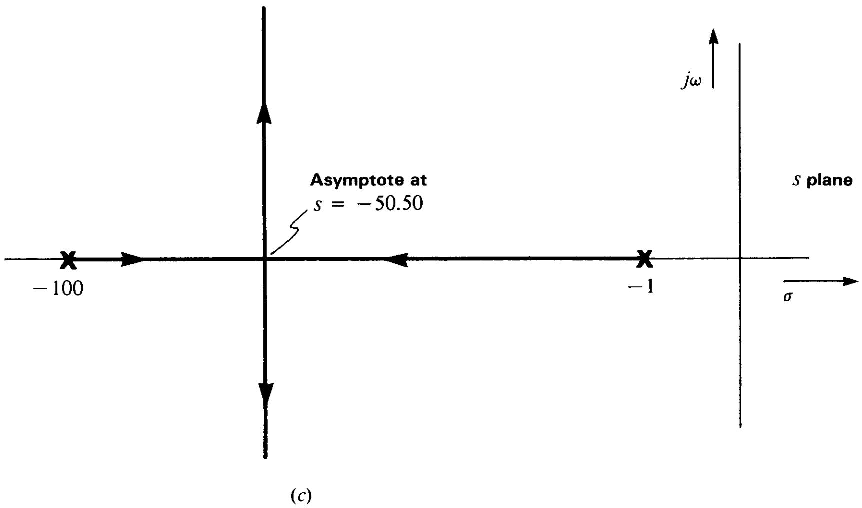
By Rule 5, two of the poles will approach asymptotes of $\pm 90°$ from the real axis. These asymptotes intersect the real axis at

$$s = \frac{-100 - 2 - 1 + 1.96}{2} = -50.52 \qquad \textbf{(S6.2)}$$

Because the third pole is moving to the left towards the zero at $s = -1.96$, the two poles must break away from the real axis slightly to the left of the asymptotes, in order to satisfy Rule 4.

The root locus for the third transfer function $a''(s)$ is shown in Figure S6.1*c*.

Figure S6.1 (c) Root locus for
$$a''(s) = \frac{a_o}{(s+1)(0.01s+1)}.$$



Again, the asymptotes are at $\pm 90°$, and intersect the real axis at

$$s = \frac{-100 - 1}{2} = -50.50 \qquad \textbf{(S6.3)}$$

To satisfy Rule 4, the poles break away from the axis exactly on the asymptotes.

Inspection of the three root-locus diagrams indicates that the three systems will have very similar behavior for moderate to large values of a_o. This is true because the complex pairs approach nearly identical asymptotes in all three cases. Further, the low-frequency pole-zero doublets effectively cancel out. Thus, intuition is verified by the root-locus behavior.

Solution 6.2 (P4.8)

The unity-gain inverter connection is shown in Figure S6.2.

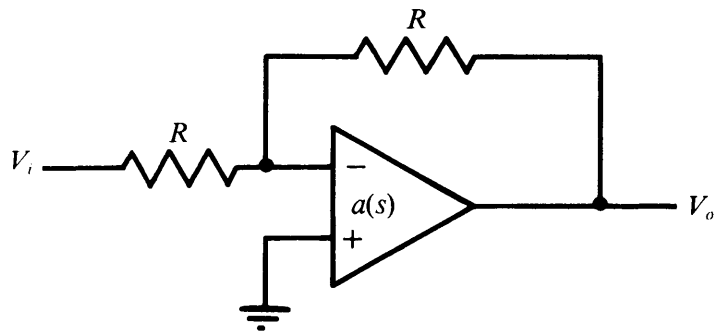


Figure S6.2 Unity-gain inverter.

The loop transmission for this connection is $-\frac{1}{2}a(s)$. Thus the characteristic equation of 1 minus the loop transmission is

$$1 + \tfrac{1}{2}a(s) = 0 \qquad \textbf{(S6.4)}$$

or, substituting in for $a(s)$:

$$1 + \frac{5 \times 10^4}{(\tau s + 1)(10^{-6}s + 1)} = 0 \qquad \textbf{(S6.5)}$$

Clearing fractions and multiplying terms gives

$$10^{-6}\tau s^2 + (\tau + 10^{-6})s + 1 + 5 \times 10^4 = 0 \qquad \textbf{(S6.6)}$$

Following Equation 4.75 in the textbook, we make the associations

$$p'(s) = s(10^{-6}s + 1) \qquad \textbf{(S6.7a)}$$

and

$$q'(s) \simeq 10^{-6}s + 5 \times 10^4 \qquad \textbf{(S6.7b)}$$

Then, the root contour method indicates that we should form a root locus with poles at the zeros of $q'(s)$ and zeros at the zeros of $p'(s)$. Thus, we have a pole at $s = -5 \times 10^{10}$, and zeros at $s = 0$, and $s = -10^6$. This configuration of singularities may seem strange, because it represents a physically impossible system having more zeros than poles in the finite s plane. Remember, however, that the zeros associated with the root contour technique are *not* the closed-loop zeros for the system under study. (The inverting connection in question here certainly doesn't have a zero at the origin.) The root contour does, however, accurately represent the location of the closed-loop poles as τ is varied.

We construct the root contour by recognizing that there is a pole at infinity that will move in from the left. Thus, the contour is as shown in Figure S6.3.

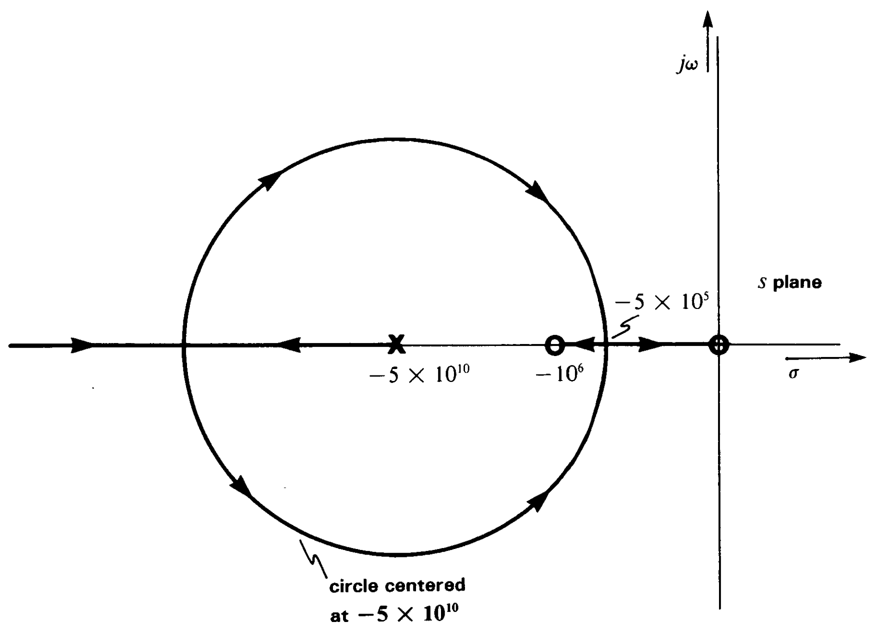


Figure S6.3 Root contour for Problem 6.2 (P4.8).

The angle condition imposes the geometric constraint that the branches circle the pole at -5×10^{10}. The entrance point on the real axis at $s = -5 \times 10^5$ is solved for by applying Rule 7 to the group of singularities near the origin (i.e., the zeros at $s = 0$ and $s = -10^6$). That is, the breakaway point found by considering only the zeros is at the point

$$s = \frac{-10^6 + 0}{2} = -5 \times 10^5 \qquad \textbf{(S6.8)}$$

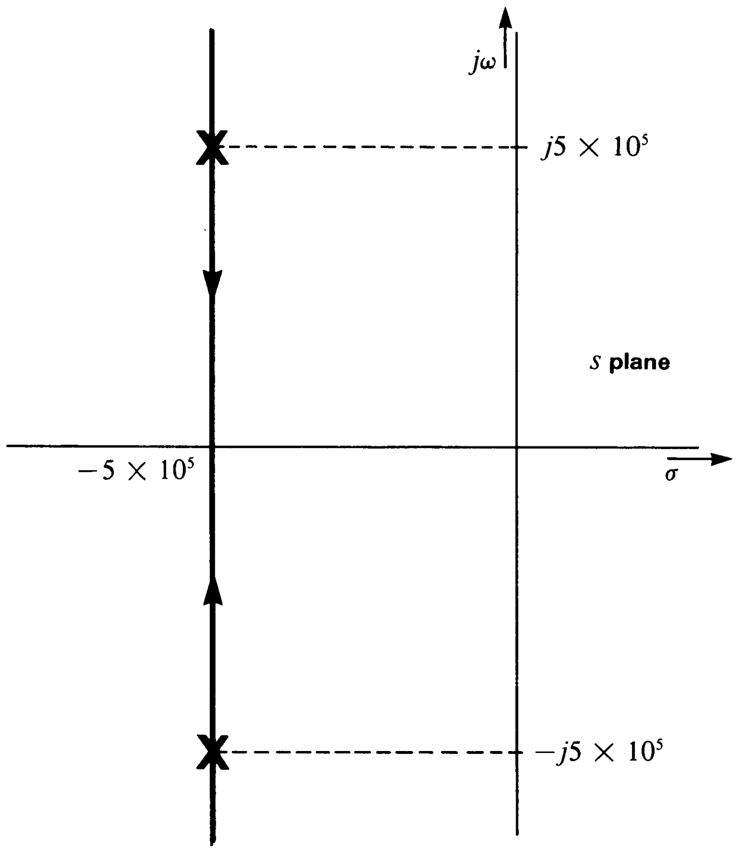
For some people (including the author of this solution), this root contour will still seem contrary to common sense. What is perhaps a more intuitive solution may be found by making the substitution $\alpha = 1/\tau$. Then, the characteristic equation (after clearing fractions) is

$$s(10^{-6}s + 1) + \alpha(10^{-6}s + 5 \times 10^4) = 0 \qquad \textbf{(S6.9)}$$

This root contour has poles where the contour of Figure S6.3 has zeros and vice versa, which will look more natural to many readers. The location of the contour is identical in both cases.

Returning to the contour of Figure S6.3, we solve for the value of τ required to set $\zeta = 0.707$. In the vicinity of the origin, where the closed-loop poles have a damping ratio of 0.707, the root contour is well approximated by a vertical line through the point $s = -5 \times 10^5$, as shown in Figure S6.4.

Figure S6.4 Root contour for Problem 6.2 (P4.8) in the vicinity of the origin.



Poles on this contour with a damping ratio of 0.707 will be at $s = -5 \times 10^5 \pm j5 \times 10^5$ as shown. Then, Rule 8 is used to solve for the value of τ which will result in this damping ratio. From Equation 4.56, the required value is

$$\begin{aligned}
\tau &= \left|\frac{q'(s)}{p'(s)}\right|_{s=-5\times10^5(1+j)} \\
&= \left|\frac{10^{-6}s + 5\times10^4}{s(10^{-6}s+1)}\right|_{s=-5\times10^5(1+j)} \qquad \textbf{(S6.10)} \\
&= 0.1
\end{aligned}$$

In closing, we note that this problem could be solved quite directly by putting Equation S6.6 into the standard form $\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n}s + 1 = 0$. Then, simply set $\zeta = 0.707$, and solve for τ. Such an approach verifies the results we have obtained via the root contour.