S O L U T I O N S

Linearized Analysis of Nonlinear Systems 14

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 14.1 (P6.1)

When the elements are cascaded in the order *ab*, the overall output is zero for all inputs, as shown in Figure S14.1*a*. When they are cascaded in the order *ba*, the transfer characteristic shown in Figure S14.1*b* results.

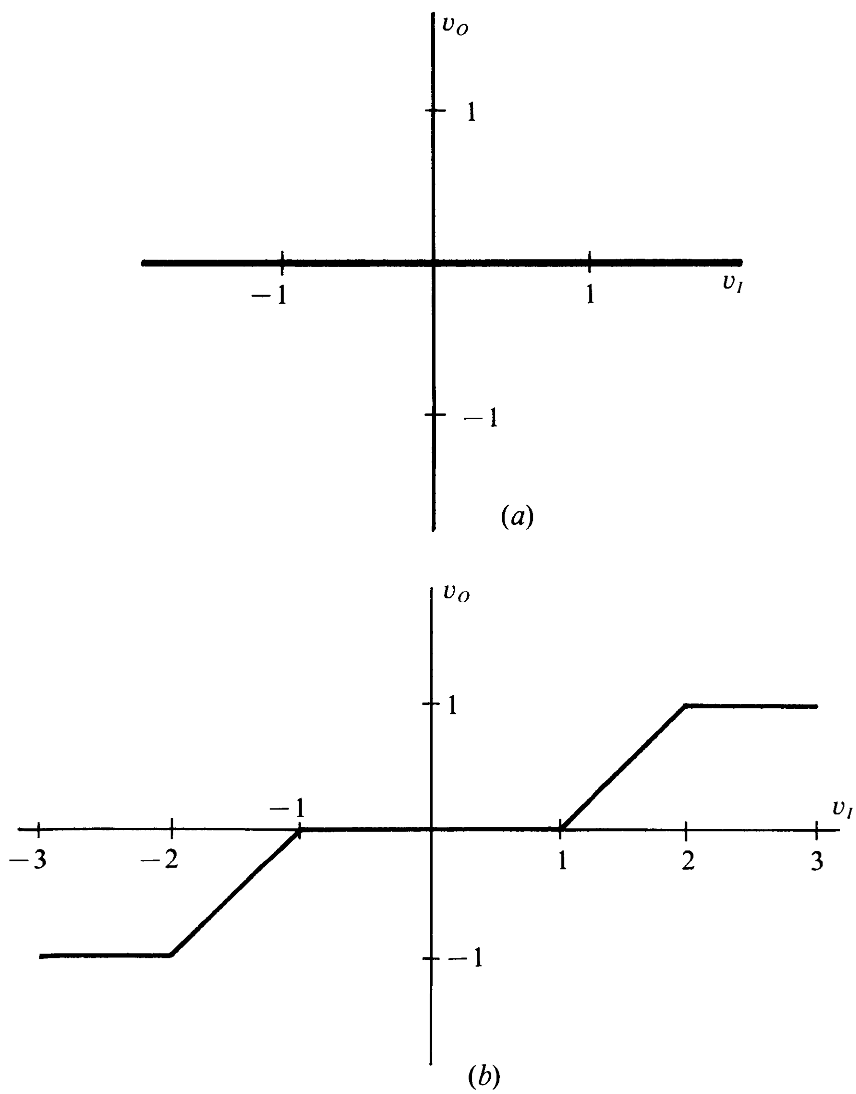

Figure S14.1 Transfer characteristics for system of Problem 14.1 (P6.1). (*a*) Cascaded in order *ab*. (*b*) Cascaded in order *ba*.

Solution 14.2 (P6.2)

(a) At each of the equilibria, $\theta_E = n\pi$. The incremental gain at each equilibrium is $\frac{dv_E}{d\theta_E}$, and is given by

$$\frac{dv_E}{d\theta_E} = \begin{cases} 1 & \theta_E = 2n\pi \\ -1 & \theta_E = (2n+1)\pi \end{cases} \qquad n = 0, 1, \ldots \tag{S14.1}$$

At equilibrium, then, the linearized loop transmission is

$$\text{L.T.} = \begin{cases} \dfrac{-10}{s(0.1s+1)}, & \theta_E = 2n\pi \\ \dfrac{10}{s(0.1s+1)}, & \theta_E = (2n+1)\pi \end{cases} \qquad n = 0, 1, \ldots \tag{S14.2}$$

We see that the loop transmission is positive for odd multiples of π, which leads one to suspect that the loop will be unstable in this case. From a root locus perspective, for any positive d-c loop-transmission magnitude, the open-loop pole at $s = 0$ will move into the right-half plane, resulting in an unstable system. Solving for the closed-loop poles as the roots of $1 -$ L.T. confirms this, as there is a right-half-plane pole at $s = 6.2\ \text{sec}^{-1}$ for the equilibria at $\theta_E = (2n+1)\pi$, $n = 0, 1, \ldots$, whereas both poles are in the left-half plane for $\theta_E = n\pi$, $n = 0, 1, \ldots$. Thus, the equilibria are unstable when $\theta_E = (2n+1)\pi$ and stable when $\theta_E = n\pi$, for all integers n.

(b) In the steady state, the output will also ramp at 7 rad/sec. That is, we will have $\dot{\theta}_O = 7$ rad/sec. From the block diagram, $\dot{\theta}_O$ is related to v_E as

$$\dot{\theta}_O = \frac{10}{0.1s+1} v_E \tag{S14.3}$$

Steady-state conditions are evaluated by setting $s = 0$, to find $v_E = 0.1\,\dot{\theta}_O = 0.7$. Then, the steady-state θ_E is $\theta_E = \sin^{-1} v_E = 0.78$ radians. Note that this is an exact value for the operating point, but because θ_E is small, it is close to the error predicted by the model linearized about zero, which would be 0.7 radians. However, the slope of the sine function at 0.78 radians is quite different from unity, so we should linearize about $\theta_E = 0.78$ radians to maintain accuracy in the incremental analysis.

(c) The incremental gain of the resolver at this operating point is $\left.\frac{dv_E}{d\theta_E}\right|_{0.78} = 0.71$. Using this incremental gain, we solve for the closed-loop poles, which are located at $s = -5.0 \pm j6.7$. Thus, the angle error θ_E will take a positive step of about 0.01 radians, and the output angle will settle down to a ramp of 7.1 rad/sec. Both of these changes occur with an initial second-order transient characterized by $\omega_n = 8.4$ rad/sec, and $\zeta = 0.6$.

Solution 14.3 (P6.3)

Applying Equation 6.10 from the textbook, the closed-loop poles are located at the roots of $1 + V_B a(s)/20$. Here we are given $a(s) = \frac{3 \times 10^5}{(s+1)(10^{-5}s+1)^2}$, and look for the range of V_B for which the closed-loop poles remain in the left-half plane. A Routh analysis indicates that two closed-loop poles lie in the right-half plane for $V_B > 13.3$, and a single pole lies in the right-half plane for $V_B < -6.7 \times 10^{-5} \approx 0$. Between these values, all poles are in the left-half plane. Thus, the loop *is* stable for the specified input ranges.

For the square-root circuit, the ideal input–output relationship is found by applying the virtual ground method, as in Section 6.2.2. This yields

$$v_I + v_B = 0 \tag{S14.4}$$

and

$$v_B = \frac{v_A^2}{10} = \frac{v_O^2}{10} \tag{S14.5}$$

Solving Equations S14.4 and S14.5 for v_O in terms of v_I yields the ideal relationship

$$v_O = \sqrt{-10v_I}, \quad v_I < 0 \tag{S14.6}$$

Note that v_I must be negative for a real solution to exist. Applying Equation 6.3 to Equation S14.5 shows that

$$V_B + v_b = \frac{V_O^2}{10} + \frac{V_O}{5} v_o \tag{S14.7}$$

The incremental portion of this equation is

$$v_b = \frac{V_O}{5} v_o \tag{S14.8}$$

Then, the incremental dependence of V_o on V_i is given by

$$\frac{V_o(s)}{V_i(s)} = \frac{-\frac{1}{2}a(s)}{1 + \dfrac{V_O a(s)}{10}} \qquad \textbf{(S14.9)}$$

With the given $a(s)$, a Routh analysis indicates that the poles of this expression are in the left-half plane for $0 < V_O < 6.67$. Thus, because by Equation S14.6, $v_I = -\frac{v_O^2}{10}$, the system will be stable for $-4.44 < V_I < 0$.