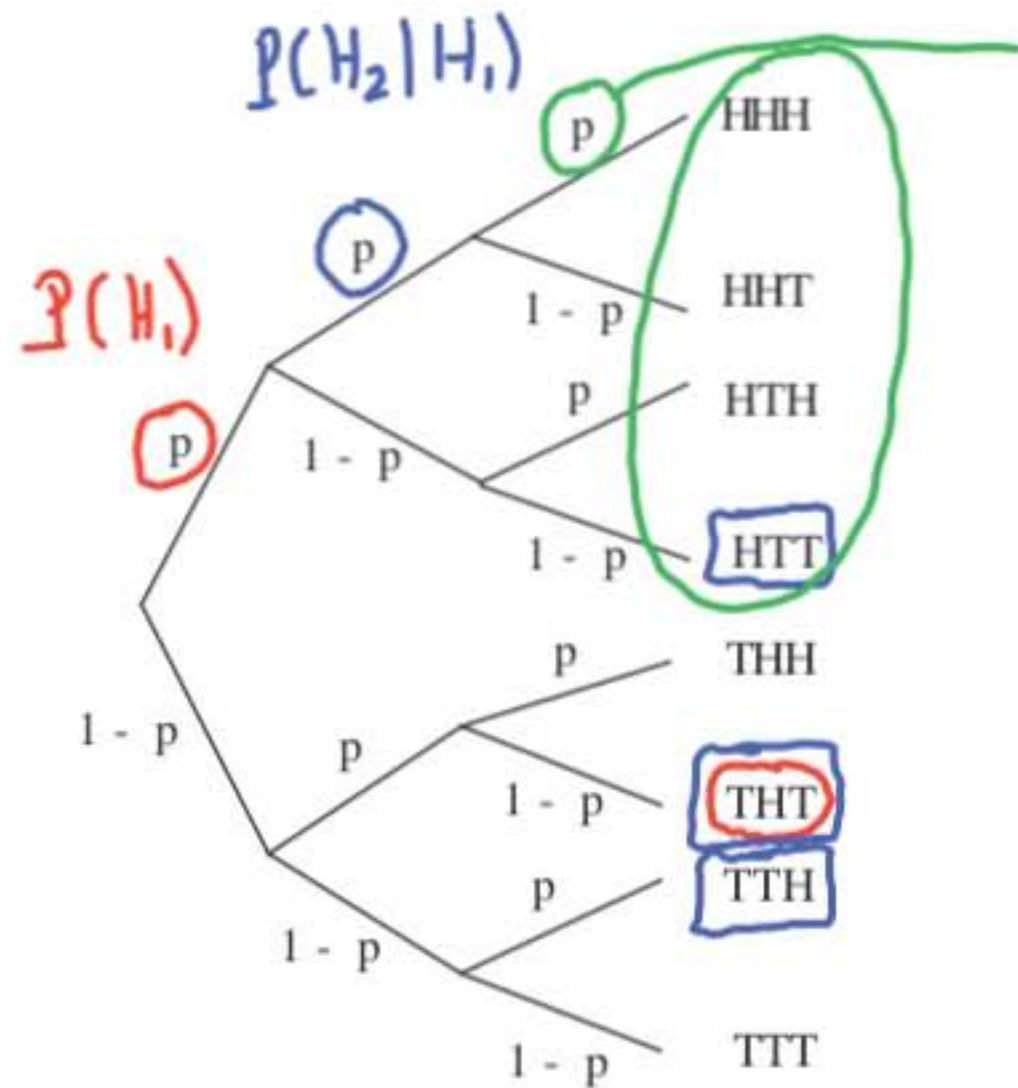


LECTURE 3: Independence

- Independence of two events
- Conditional independence
- Independence of a collection of events
- Pairwise independence
- Reliability
- The king's sibling puzzle

A model based on conditional probabilities

- 3 tosses of a biased coin: $P(H) = p$, $P(T) = 1 - p$



$$P(H_3 | H_1 \cap H_2)$$

$$P(H_2 | H_1) = p = P(H_2 | T_1)$$

$$P(H_2) = P(H_1) P(H_2 | H_1) + P(T_1) P(H_2 | T_1)$$

$$= p$$

- Multiplication rule: $P(\underline{THT}) = (1-p)p(1-p)$

- Total probability:

$$P(1 \text{ head}) = 3 p(1-p)^2$$

- Bayes rule:

$$P(\underline{\text{first toss is H}} \mid \underline{1 \text{ head}}) = \frac{P(H_1 \cap 1 \text{ head})}{P(1 \text{ head})}$$

$$= \frac{p(1-p)^2}{3 p(1-p)^2} = \frac{1}{3}$$

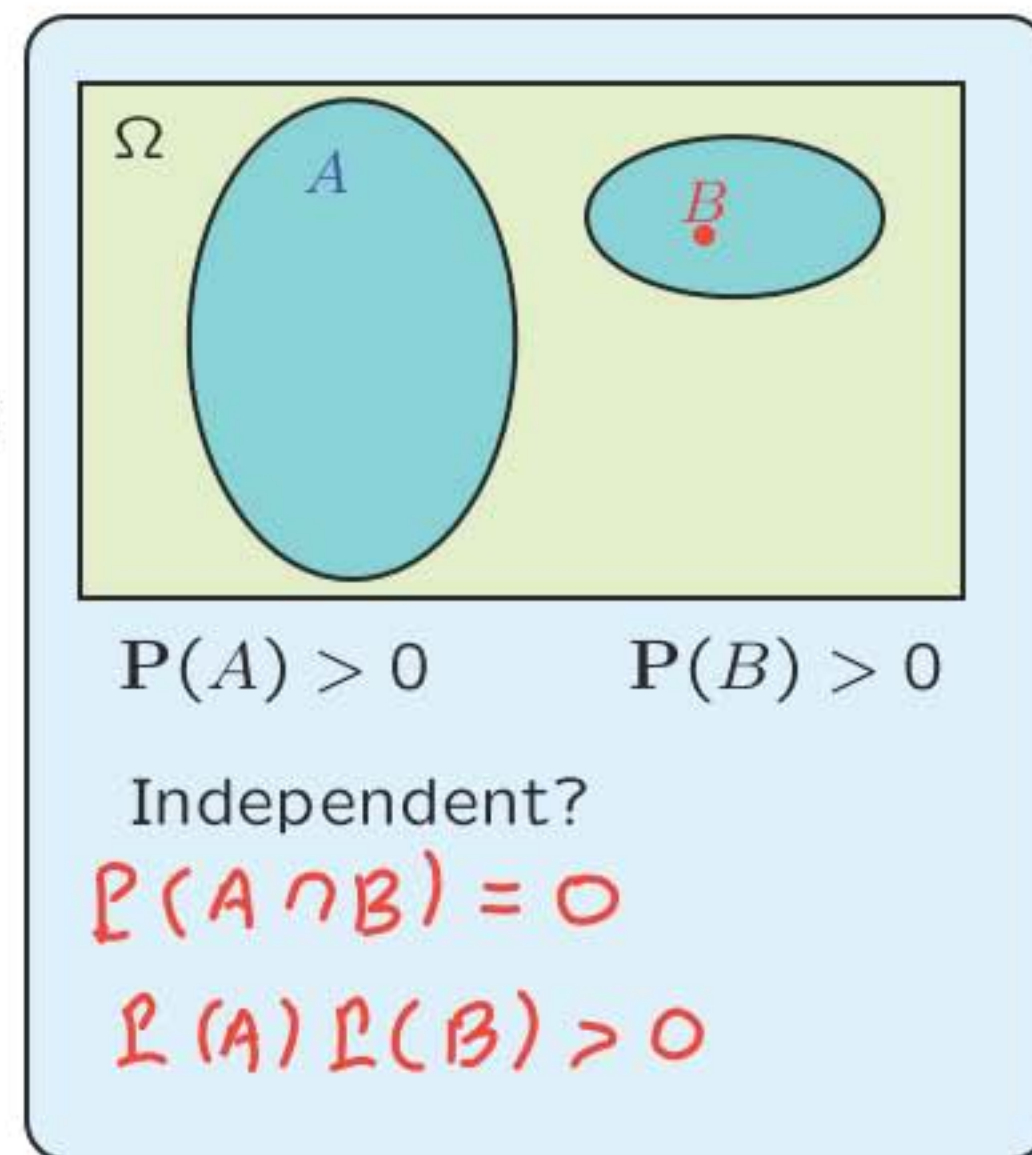
Independence of two events

- Intuitive “definition”: $P(B | A) = P(B)$
 - occurrence of A provides no new information about B

$$P(A \cap B) = P(A) P(B|A) = P(A) P(B)$$

Definition of independence: $P(A \cap B) = P(A) \cdot P(B)$

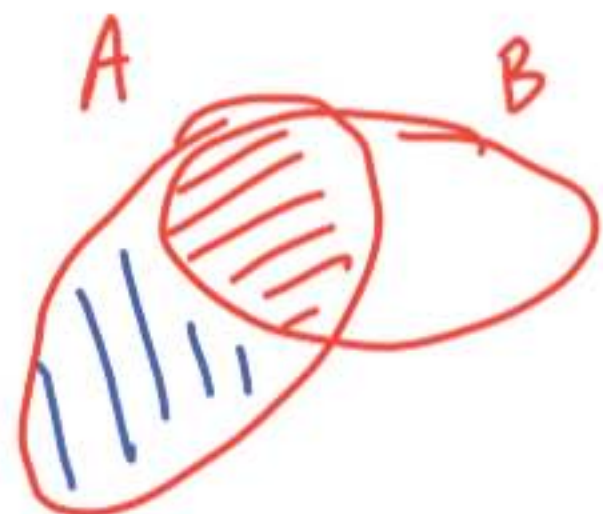
- Symmetric with respect to A and B
- implies $P(A | B) = P(A)$
- applies even if $P(A) = 0$



Independence of event complements

Definition of independence: $P(A \cap B) = P(A) \cdot P(B)$

- If A and B are independent, then A and B^c are independent.
 - Intuitive argument
 - Formal proof



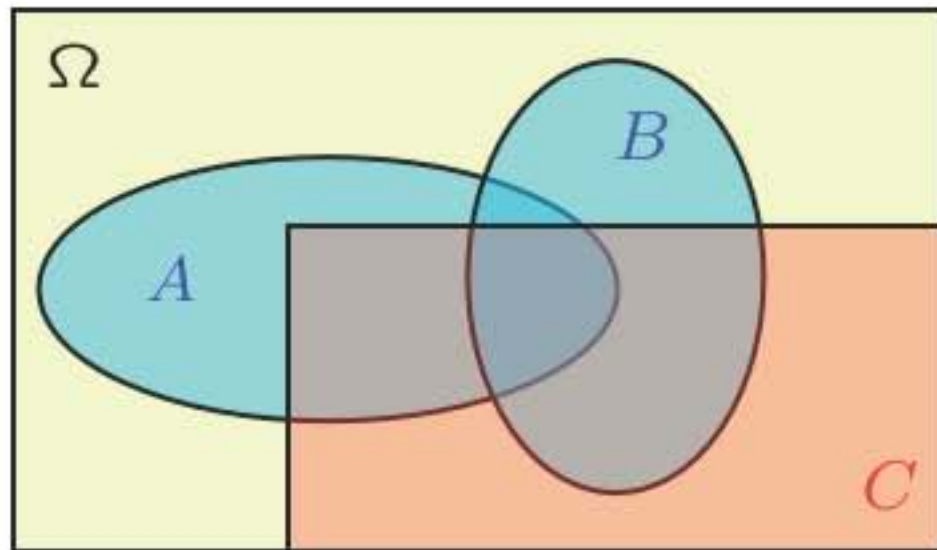
$$A = (A \cap B) \cup (A \cap B^c)$$

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B^c) \\ &= P(A)P(B) + P(A \cap B^c) \end{aligned}$$

$$\begin{aligned} P(A \cap B^c) &= P(A) - P(A)P(B) = P(A)(1 - P(B)) \\ &= P(A)P(B^c) \end{aligned}$$

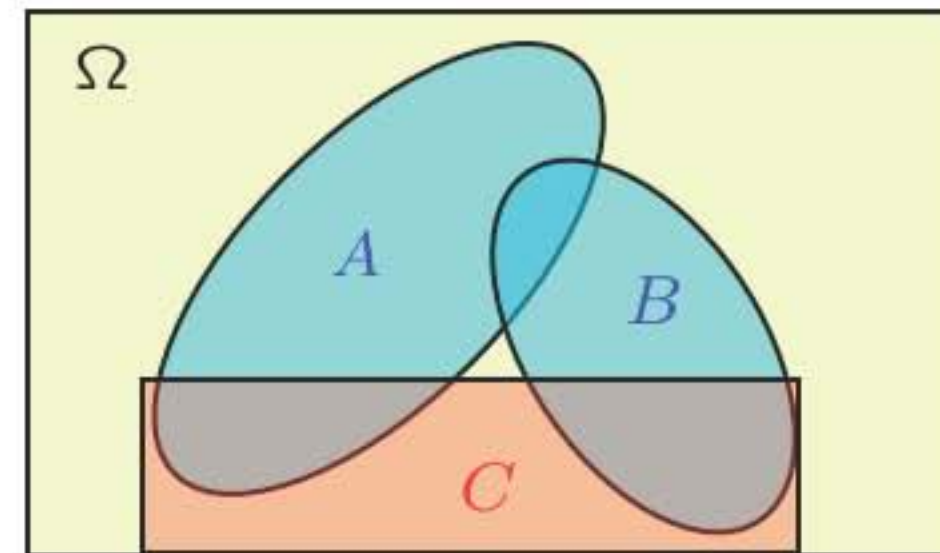
Conditional independence

- Conditional independence, given C , is defined as independence under the probability law $\mathbf{P}(\cdot | C)$



$$P(A \cap B | C) = P(A | C) P(B | C)$$

Assume A and B are independent

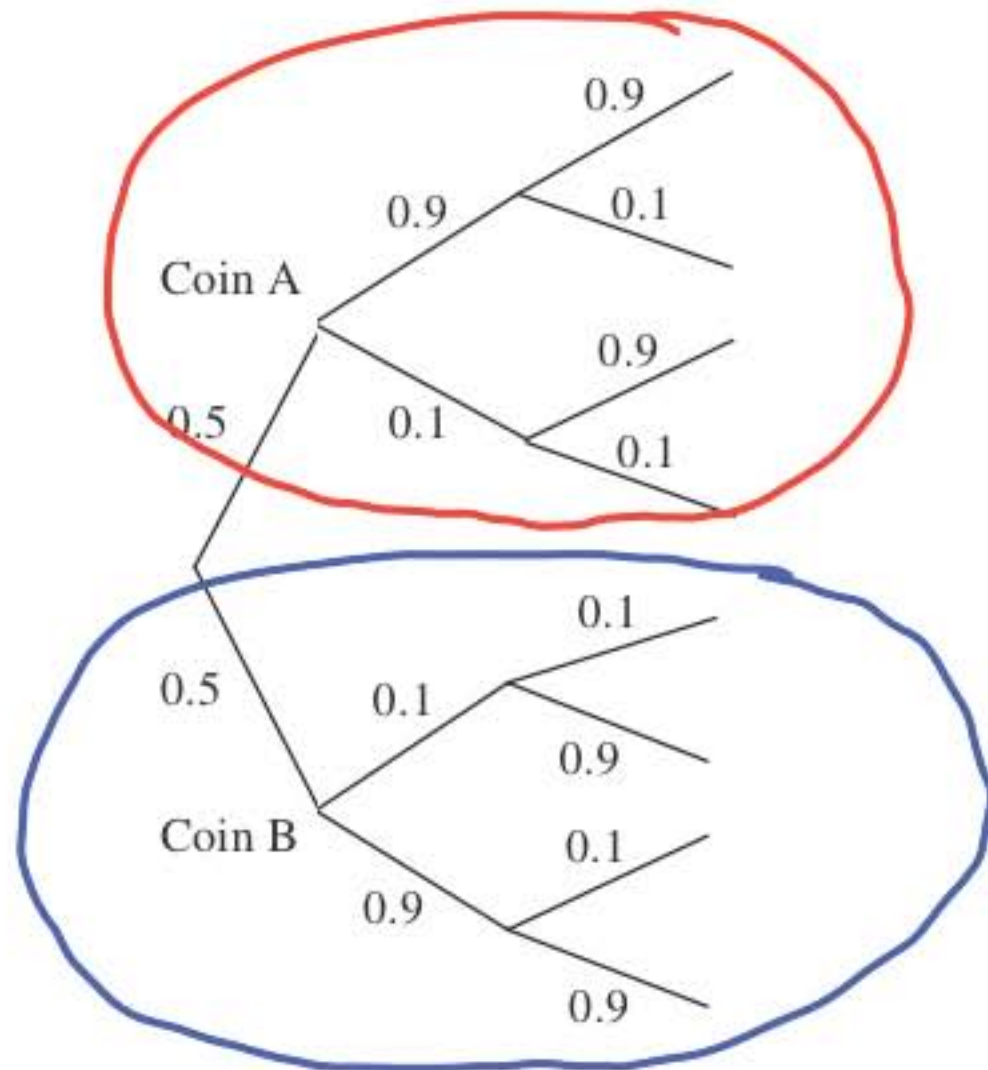


- If we are told that C occurred, are A and B independent? **No**

Conditioning may affect independence

- Two unfair coins, A and B :
 $P(H | \text{coin } A) = 0.9$, $P(H | \text{coin } B) = 0.1$
- choose either coin with equal probability

given a coin:
independent tosses



- Are coin tosses independent?

No!

– Compare:

$$\begin{aligned} P(\text{toss } 11 = H) &= P(A)P(H_{11}|A) + P(B)P(H_{11}|B) \\ &= 0.5 \times 0.9 + 0.5 \times 0.1 = 0.5 \end{aligned}$$

$P(\text{toss } 11 = H | \text{first } 10 \text{ tosses are heads})$

$$\approx P(H_{11} | A) = 0.9$$

Independence of a collection of events

- **Intuitive “definition”:** Information on some of the events does not change probabilities related to the remaining events

$$A_1, A_2, \dots, \text{indep} \Rightarrow \mathbb{P}(A_3 \cap A_4^c) = \mathbb{P}(A_3 \cap A_4^c \mid A_1 \cup (A_2 \cap A_5^c)) \quad \bullet$$
$$\mathbb{P}(A_3) = \mathbb{P}(A_3 \mid A_1 \cap A_2) = \mathbb{P}(A_3 \mid A_1 \cap A_2^c) = \mathbb{P}(A_3 \mid A_1^c \cap A_2)$$

Definition: Events A_1, A_2, \dots, A_n are called **independent** if:

$$\mathbb{P}(A_i \cap A_j \cap \dots \cap A_m) = \mathbb{P}(A_i)\mathbb{P}(A_j) \dots \mathbb{P}(A_m) \quad \text{for any distinct indices } i, j, \dots, m$$

$n = 3$:

$$\left. \begin{aligned} \mathbb{P}(A_1 \cap A_2) &= \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \\ \mathbb{P}(A_1 \cap A_3) &= \mathbb{P}(A_1) \cdot \mathbb{P}(A_3) \\ \mathbb{P}(A_2 \cap A_3) &= \mathbb{P}(A_2) \cdot \mathbb{P}(A_3) \end{aligned} \right\} \text{ pairwise independence}$$

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3)$$

Independence vs. pairwise independence

- Two independent fair coin tosses

- H_1 : First toss is H
- H_2 : Second toss is H

$$P(H_1) = P(H_2) = 1/2$$

H_1	HH $1/4$	HT $1/4$
H_2	TH $1/4$	TT $1/4$

- C : the two tosses had the same result = $\{HH, TT\}$

$$P(H_1 \cap C) = P(H_1 \cap H_2) = 1/4 \quad P(H_1)P(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

H_1, C : indep.
 H_2, C : indep.

$$P(H_1 \cap H_2 \cap C) = P(HH) = 1/4$$

$$P(H_1)P(H_2)P(C) = 1/8 \quad \left. \vphantom{P(H_1)P(H_2)P(C)} \right\} \text{diff.}$$

$$P(C | H_1) = P(H_2 | H_1) = P(H_2) = 1/2 = P(C)$$

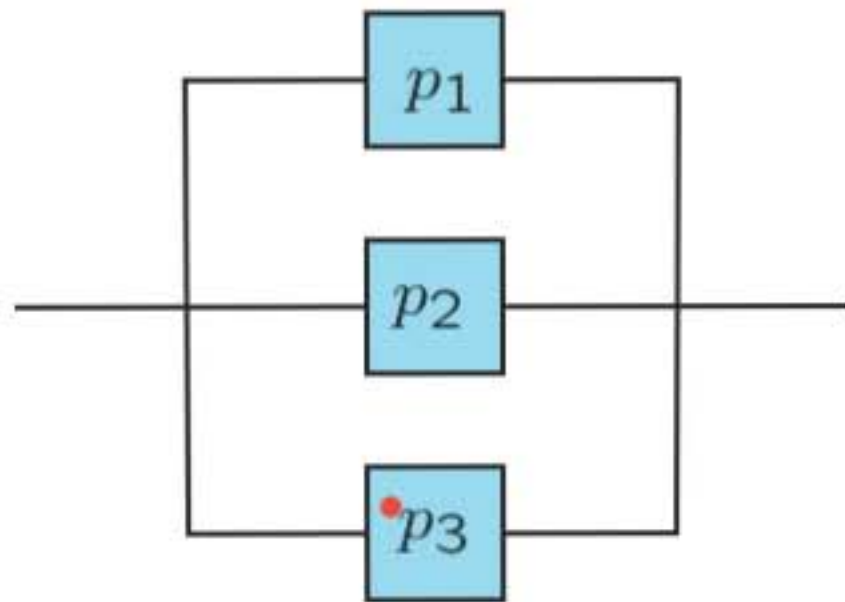
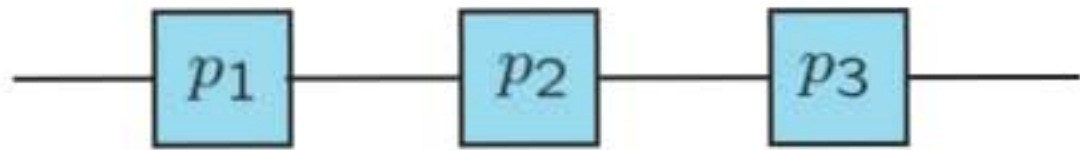
$$P(C | H_1 \cap H_2) = 1 \neq P(C) = 1/2$$

$H_1, H_2,$ and C are pairwise independent, but not independent

Reliability

p_i : probability that unit i is "up"

independent units



U_i : i th unit up

U_1, U_2, \dots, U_n independent

F_i : i th unit down

$\Rightarrow F_i$ independent

probability that system is "up"?

$$P(\text{system up}) = P(U_1 \cap U_2 \cap U_3)$$

$$= P(U_1) P(U_2) P(U_3) = p_1 p_2 p_3$$

$$P(\text{system is up}) = P(U_1 \cup U_2 \cup U_3)$$

$$= 1 - P(F_1 \cap F_2 \cap F_3)$$

$$= 1 - P(F_1) P(F_2) P(F_3)$$

$$= 1 - (1 - p_1)(1 - p_2)(1 - p_3)$$

The king's sibling

- The king comes from a family of two children.
What is the probability that his sibling is female?

~~$1/2$~~ ?

boys have precedence

$P(\text{boy}) = P(\text{girl}) = 1/2$
independent

BB $1/4$	BG $1/4$
GB $1/4$	GG $1/4$

$2/3$

- till 1 boy $\Rightarrow P(G) = 1$
- till 2 boys $\Rightarrow P(G) = 0$

MIT OpenCourseWare

<https://ocw.mit.edu>

Resource: Introduction to Probability

John Tsitsiklis and Patrick Jaillet

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