We have seen that the binomial distribution plays an important role in the study of the Bernoulli process.

And the reason is that the binomial distribution describes the number of arrivals during a fixed number of slots.

We will now develop an approximation to the binomial distribution that applies to one particular regime, and that regime is when we have a very large number of slots, but we have a small probability of success in each slot.

And this is in a way so that the product of these two numbers, which is the expected number of successes, is a moderate number.

One example where such a situation might arise is the following.

Suppose you're interested in earthquakes in your city, and you divide time into slots of one hour.

During each hour, the probability of having a noticeable earthquake in your city would be a very small number.

On the other hand, if you're interested in a time frame of five years, there's going to be many hours during that time frame, so that n would be quite large.

But the expected number of earthquakes over a period of five years should be a moderate number.

And one can think of other situations where this regime might arise.

The one particular situation that will be very interesting for us is going to be when we try to take a continuous time approximation of the Bernoulli process by dividing time into very small slots, so that we have many slots, but a small probability of success during each one of those slots.

So to start, let us look at the form of the binomial PMF.

And let us just try to develop an approximation to this PMF, when we fix k to be particular constant number, and then take the limit as n goes to infinity and p goes to 0, but in a way that lambda remains constant.

And in particular, because of this relation here, we will have p equal to lambda over n.

So let us take this expression and start rewriting it.

Let us look at the ratio of n factorial divided by this.

The denominator has the product of all numbers going up to n minus k.

So by dividing by this number, what is left out of the n factorial is only the terms that go up to n minus k plus 1.

Then we have, in the denominator, the factor of k factorial.

Now p is equal to lambda over n, so this term becomes lambda to the k divided by n to the k.

And similarly, for the last term, we have 1 minus lambda over n to the power n minus k.

Now let us rearrange terms.

Here, we have a product of k terms in the numerator.

Here, we have n multiplying itself k times.

So we can take a factor of n and place it underneath each one of those terms to obtain n over n times n minus 1 over n times-- we continue this way all the way to n minus k plus 1 divided by n.

Take this term, k factorial, move it underneath the lambda to the k term, and then let us split this last term into 2 pieces in this manner.

And now let us start taking limits as n goes to infinity.

The first term that we have here is equal to 1.

How about the second term?

n divided by n is equal to 1, 1 over n goes to 0, so this term also converges to 1.

And by a similar argument, all of the terms in this product, including this one, converge to 1.

The term lambda k over k factorial remains exactly as is.

And now, let us look at this term.

This is probably familiar.

There is a basic fact which tells us that if we take this expression and raise it to the nth power, what we get is e to the minus lambda in the limit as n goes to infinity.

So using this basic result, this term becomes e to the minus lambda.

And finally, let's look at the last term.

Remember that k is fixed, is a constant.

1 minus lambda over n converges to 1, and when we raise that number to the k-th power, we still get a 1 in the limit.

So the only terms that are left are here, and essentially, what we have just established is that in the limit, the probability of k arrivals in a Bernoulli process or the binomial probability evaluated at k, in the limit, as n goes to infinity and p goes to 0, is given by this formula, here.

This is the formula for the Poisson PMF.

And so what we have established is that the binomial PMF converges to a Poisson PMF when we take the limit in this particular way.